ORIGINAL PAPER

$\mathbb{BC}$-DISTRIBUTION FUNCTION AND $\mathbb{BC}$-REARRANGEMENT OF $\mathbb{BC}$-MEASURABLE FUNCTIONS WITH HYPERBOLIC NORM

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Abstract. *This paper investigates the distribution function and nonincreasing rearrangement of* $\mathbb{BC}$*-valued functions equipped with the hyperbolic norm. The concept of the distribution function for* $\mathbb{BC}$*-valued functions is introduced, which characterizes the cumulative distribution of values taken by the function. Next, it is delved into the nonincreasing rearrangement of* $\mathbb{BC}$*-valued functions with the hyperbolic norm. By exploring the nonincreasing rearrangement of* $\mathbb{BC}$*-valued functions, we determined how the hyperbolic norm influences the rearrangement process and its impact on the function's behavior and properties.*

Keywords: *Bicomplex numbers;* $\mathbb{BC}$*-valued functions;* $\mathbb{BC}$*-distribution function;* $\mathbb{BC}$*-rearrangement; hyperbolic norm.*

1. INTRODUCTION AND PRELIMINARIES ON $\mathbb{BC}$

Bicomplex ($\mathbb{BC}$)-valued functions arise naturally in various mathematical fields, including probability theory, mathematical analysis, and functional analysis, and understanding their properties is crucial for advancing these areas of study. Indeed, the study of modules with bicomplex scalars in the context of functional analysis has gained significant attention in recent years. The book likely presents groundbreaking results and insights related to this topic. Functional analysis traditionally deals with vector spaces over a field, such as the complex numbers or the real numbers. However, by considering modules with bicomplex scalars, where the scalars are elements of the bicomplex numbers, a broader framework is introduced. This extension allows for the exploration of new mathematical structures and the investigation of properties beyond the classical setting. The book by Alpay et al., referenced as [1], is a valuable resource for researchers and enthusiasts interested in this area. It presents notable results, techniques, and applications pertaining to the study of modules with bicomplex scalars in the context of functional analysis. These results may encompass various aspects of functional analysis, such as operator theory, function spaces, and spectral theory, among others. The remarkable results shed light on the behavior of modules with bicomplex scalars, reveal connections to other areas of mathematics, and potentially find applications in physics, engineering, or other disciplines.

The series of articles undermentioned highlight the systematic study of topological bicomplex modules and various fundamental theorems related to them. Here is a breakdown of the articles and their contributions:

The article [2] focuses on the study of topological bicomplex modules, exploring their topological properties and investigating concepts such as convergence, continuity, and

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compactness in this context. The authors presented fundamental theorems, including the principle of uniform boundedness, open mapping theorem, interior mapping theorem and closed graph theorem for bicomplex modules to establish important results regarding the continuity and invertibility of linear operators between topological bicomplex modules in [3].

In [4], in collaboration with [3], the authors extend the study of fundamental theorems to the setting of topological bicomplex modules. Also, the authors delve further into the study of topological hyperbolic modules, topological bicomplex modules, exploring the properties of linear operators, continuity, and related topological concepts specific to these settings.

The Hahn-Banach theorem for bicomplex modules and hyperbolic modules are investigated in [5]. Its extension to hyperbolic and bicomplex modules explores the uniqueness and extensions of such functionals in these contexts. The work [6], focuses on bicomplex C*-algebras. It covers topics such as bicomplex operator algebras, spectral theory, and other aspects related to the algebraic and topological properties of C*-algebras defined on bicomplex vector space.

Bicomplex linear operators on $\mathbb{BC}$-Hilbert spaces are examined in [7, 8]. They explored properties of these operators, functional analysis techniques in the bicomplex setting, and potentially touch upon topics like Littlewood’s subordination theorem. They also provided a detailed study of operators acting on bicomplex modules and explore the construction and properties of functional calculi specific to the bicomplex framework.

The book [9] provides an in-depth exploration of bicomplex analysis and geometry. It covers a wide range of topics, including holomorphic functions, integration, differential equations, and geometric properties specific to the bicomplex domain. The bicomplex spectral decomposition theorem when infinite-dimension is introduced in [10].

Bicomplex versions of some crucial inequalities with respect to the hyperbolic-valued norm and some topological and geometric concepts such as completeness, convexity, strict convexity and uniform convexity in the bicomplex setting with respect to the hyperbolic norm are given in [11-13].

These references collectively represent significant contributions to the study of bicomplex modules, functional analysis, and related areas. They showcase the exploration of properties, the development of new theorems, and the application of functional analysis techniques in the context of bicomplex numbers. Researchers and readers interested in these topics can refer to these articles and the books for detailed insights into the respective areas of study.

Now, we will give a small summary of bicomplex numbers with some basic properties. The set bicomplex numbers $\mathbb{BC}$ which is a two-dimensional extension of the complex numbers is defined as

$$\mathbb{BC} := \{W = w_1 + jw_2 \mid w_1, w_2 \in \mathbb{C}(i)\}$$

where i and j are imaginary units satisfying $ij = ji$, $i^2 = j^2 = -1$ and $\mathbb{C}(i)$ is the field of complex numbers with the imaginary unit i. According to ring structure, for any $Z = z_1 + jz_2$, $W = w_1 + jw_2$ in $\mathbb{BC}$ usual addition and multiplication are defined as

$$Z + W = (z_1 + w_1) + j(z_2 + w_2)$$

$$ZW = (z_1w_1 - z_2w_2) + j(z_2w_1 + z_1w_2).$$

The set $\mathbb{BC}$ forms a commutative ring under the usual addition and multiplication of bicomplex numbers. The bicomplex numbers have a unit element denoted as $1_{\mathbb{BC}} := 1$ and it acts as the identity for multiplication, such that for any bicomplex number W, $1 \cdot W = W \cdot 1 = W$. In the sense of module structure, the set $\mathbb{BC}$ is a module over itself.

The product of the imaginary units i and j bring out a hyperbolic unit k, such that $k^2 = 1$. This implies that k is a square root of 1 and is distinct from i and j. The product operation of all units $i, j,$ and k in the bicomplex numbers is commutative. Specifically, the following relations hold:

$$ij = k,\ jk = -i \text{ and } ik = -j.$$

These properties summarize the basic characteristics of bicomplex numbers and their algebraic structure. Hyperbolic numbers $\mathbb{D}$ are a two-dimensional extension of the real numbers that form a number system known as the hyperbolic plane or hyperbolic plane algebra. They can be represented in the form $\alpha = \beta_1 + k\beta_2$, where β_1 and β_2 are real numbers, and k is the hyperbolic unit. In the hyperbolic number system, for any two hyperbolic numbers $\alpha = \beta_1 + k\beta_2$ and $\gamma = \gamma_1 + k\gamma_2$, addition and multiplication are defined as follows:

$$\alpha + \gamma = (\beta_1 + \gamma_1) + k(\beta_2 + \gamma_2)$$

$$\alpha\gamma = (\beta_1\gamma_1 + \beta_2\gamma_2) + k(\beta_1\gamma_2 + \beta_2\gamma_1).$$

The hyperbolic numbers form a ring, however, unlike the complex numbers, the hyperbolic numbers do not have a multiplicative inverse for all nonzero elements. The nonzero hyperbolic numbers that have multiplicative inverses are called units. The hyperbolic numbers can also be considered a significant subset of the bicomplex numbers $\mathbb{BC}$.

Let $W = w_1 + jw_2 \in \mathbb{BC}$ where $w_1, w_2 \in \mathbb{C}(i)$. By the notation of W with imaginary units i and j, the conjugations are formed for bicomplex numbers in [1, 9, 14] as follows:

(I) The first conjugation $\overline{W} = \overline{w}_1 + j\overline{w}_2$ is defined by taking the complex conjugate of w_1 and w_2, which are elements of $\mathbb{C}(i)$, and combining them with the imaginary unit j.

(II) The second conjugation $W^{\dagger} = w_1 - jw_2$ is similar to the previous one, but with the opposite sign for the imaginary part.

(III) The third one $W^{*} = \overline{w}_1 - j\overline{w}_2$ is the composition of the previous two conjugations, $\overline{W}$ and $W^{\dagger}$,

where $\overline{w}_1$ and $\overline{w}_2$ are the usual complex conjugates of $w_1, w_2 \in \mathbb{C}(i)$, respectively. In summary, $\overline{W}$ and $W^{\dagger}$ are individual conjugations obtained by taking the complex conjugate of the components and combining them with the imaginary unit j. The third conjugation, W^{*} is obtained by composing $\overline{W}$ and $W^{\dagger}$, resulting in a new expression involving both real and imaginary parts.

For any bicomplex number W, there exist the following three moduli in [1, 9, 14]:

(I) $|W|_i^2 = W \cdot W^{\dagger} = w_1^2 + w_2^2 \in \mathbb{C}(i)$

(II) $|W|_j^2 = W \cdot \overline{W} = (|w_1|^2 - |w_2|^2) + j\left(2\mathrm{Re}\left(w_1\overline{w_2}\right)\right) \in \mathbb{C}(j),$

(III) $|W|_k^2 = W \cdot W^* = (|w_1|^2 + |w_2|^2) + k\left(-2\mathrm{Im}\left(w_1\overline{w_2}\right)\right) \in \mathbb{D}.$

Furthermore, $\mathbb{BC}$ is a normed space with the norm $\|\cdot\|_{\mathbb{BC}}$, where

$$\|W\|_{\mathbb{BC}} = \sqrt{|w_1|^2 + |w_2|^2}$$

for every $w_1 + jw_2 = W$ and

$$\|\zeta_1\zeta_2\|_{\mathbb{BC}} \leq \sqrt{2}\|\zeta_1\|_{\mathbb{BC}}\|\zeta_2\|_{\mathbb{BC}}$$

for every $\zeta_1,\ \zeta_2 \in \mathbb{BC}$. Finally $\mathbb{BC}$ is a modified Banach algebra [1, 14].

If the hyperbolic numbers e_1 and e_2 defined as

$$e_1 = \frac{1+k}{2} \text{ and } e_2 = \frac{1-k}{2},$$

then it is easy to see that the set $\{e_1, e_2\}$ is a fundamental set in $\mathbb{C}(i)$-vector space $\mathbb{BC}$ and linearly independent. The set $\{e_1, e_2\}$ also satisfies the following properties:

$$\begin{aligned} e_1^2 &= e_1, \\ e_2^2 &= e_2, \\ e_1^* &= e_1, \\ e_2^* &= e_2, \\ e_1 + e_2 &= 1, \\ e_1 \cdot e_2 &= 0 \end{aligned}$$

with $\|e_1\|_{\mathbb{BC}} = \|e_2\|_{\mathbb{BC}} = \frac{\sqrt{2}}{2}$. By using this set, any $W = w_1 + jw_2 \in \mathbb{BC}$ can be written as a linear combination of e_1 and e_2 uniquely. That is, $W = w_1 + jw_2$ can be written as

$$W = w_1 + jw_2 = e_1z_1 + e_2z_2 \tag{1}$$

where $z_1 = w_1 - iw_2$ and $z_2 = w_1 + iw_2$, [1]. Here z_1 and z_2 are elements of $\mathbb{C}(i)$ and the formula in (1) is called the *idempotent representation* of the bicomplex number W.

Besides the Euclidean-type norm $\|\cdot\|_{\mathbb{BC}}$, another norm named with ($\mathbb{D}$-valued) hyperbolic-valued norm $|W|_k$ of any bicomplex number $W = e_1z_1 + e_2z_2$ is defined as

$$|W|_k = e_1|z_1| + e_2|z_2|.$$

For any hyperbolic number $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, an idempotent representation can also be written as $\mathbb{D} \subset \mathbb{BC}$. Thus $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$ can be written as

$$\alpha = e_1\alpha_1 + e_2\alpha_2$$

where $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_1 - \beta_2$ are real numbers. If $\alpha_1 > 0$ and $\alpha_2 > 0$ for any $\alpha = \beta_1 + k\beta_2 \in \mathbb{D}$, then we say that α is a positive hyperbolic number. Thus, the set of non-negative hyperbolic numbers $\mathbb{D}^+ \cup \{0\}$ is defined by

$$\mathbb{D}^+ \cup \{0\} = \{\alpha = \beta_1 + k\beta_2 : \beta_1^2 - \beta_2^2 \geq 0, \beta_1 \geq 0\}$$

$$= \{\alpha = e_1\alpha_1 + e_2\alpha_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}.$$

Now, let α and γ be any two elements of $\mathbb{D}$. In [1,4,5,9] and [14], a relation "$\preccurlyeq$" is defined on $\mathbb{D}$ by

$$\alpha \preccurlyeq \gamma \Leftrightarrow \gamma - \alpha \in \mathbb{D}^+ \cup \{0\}.$$

It is showed in [1] that this relation "$\preccurlyeq$" has reflexive, anti-symmetric and transitive properties. Therefore "$\preccurlyeq$" defines a partial order on $\mathbb{D}$. If idempotent representations of the hyperbolic numbers α and γ are written as $\alpha = e_1\alpha_1 + e_2\alpha_2$ and $\gamma = e_1\gamma_1 + e_2\gamma_2$, then $\alpha \preccurlyeq \gamma$ implies that $\alpha_1 \leq \gamma_1$ and $\alpha_2 \leq \gamma_2$. By $\alpha \prec \gamma$, we mean $\alpha_1 < \gamma_1$ and $\alpha_2 < \gamma_2$. For more details on hyperbolic numbers $\mathbb{D}$ and partial order "$\preccurlyeq$", one can refer [1, Section 1.5] and [9, 14].

Definition 1. Let A be a subset of $\mathbb{D}$. A is called a $\mathbb{D}$-bounded above set if there is a hyperbolic number δ such that $\delta \succcurlyeq \alpha$ for all $\alpha \in A$. If $A \subset \mathbb{D}$ is $\mathbb{D}$-bounded from above, then the $\mathbb{D}$-supremum of A is defined as the smallest member of the set of all upper bounds of A. In other words, a hyperbolic number λ is an upper bound of the set A, can be described by the following two properties:
(i) $\alpha \preccurlyeq \lambda$, for each $\alpha \in A$
(ii) For any $\varepsilon \succ 0$, there exists $\theta \in A$ such that $\theta \succ \lambda - \varepsilon$ [4].

Remark 2. Let A be a $\mathbb{D}$-bounded above subset of $\mathbb{D}$, $A_1 = \{\alpha_1 : e_1\alpha_1 + e_2\alpha_2 \in A\}$ and $A_2 = \{\alpha_2 : e_1\alpha_1 + e_2\alpha_2 \in A\}$. Then, $\sup_{\mathbb{D}} A$ is given by

$$\sup_{\mathbb{D}} A = e_1 \sup A_1 + e_2 \sup A_2.$$

Similarly, for any $\mathbb{D}$-bounded below set A, $\mathbb{D}$-infimum is defined as

$$\inf_{\mathbb{D}} A = e_1 \inf A_1 + e_2 \inf A_2$$

where A_1 and A_2 are defined as above [1, Remark 1.5.2].

Definition 3. Let $(X,+)$ be an abelian group and $(X,+,\cdot)$ be a $\mathbb{BC}$-module. If there is a topology τ_X in X, such that the operations $+: X\times X\to X$ and $\cdot:\mathbb{BC}\times X\to X$ are continuous, then $(X,+,\cdot)$ is called a topological $\mathbb{BC}$-module.

Remark 4. Any $\mathbb{BC}$-module space or $\mathbb{D}$-module space Y can be decomposed as

$$Y = e_1Y_1 + e_2Y_2 \tag{2}$$

where $Y_1 = e_1Y$ and $Y_2 = e_2Y$ are $\mathbb{R}$-vector or $\mathbb{C}(i)-$vector spaces. The spelling in (2) is called as the idempotent decomposition of the space Y [1, 9].

Definition 5. Let $\mathcal{M}$ be a σ-algebra on a set Ω and $\mu = \mu_1e_1 + \mu_2e_2$ be a bicomplex-valued function defined on Ω. Then μ is called a bicomplex measure on $\mathcal{M}$ if μ_1 and μ_2 are both complex measures on $\mathcal{M}$. Nevertheless if μ_1 and μ_2 are positive measures on $\mathcal{M}$ namely, range of both μ_1, μ_2 are $[0,\infty]$, then μ is called a $\mathbb{D}$-measure on $\mathcal{M}$. Also, μ is called a $\mathbb{D}^+$-measure on $\mathcal{M}$, if μ_1, μ_2 are real measures on $\mathcal{M}$ i.e. $\mu_1(\cdot)$, $\mu_2(\cdot)\in[0,\infty)$, [15].

Assume that $\Omega = (\Omega,\mathcal{M},\mu)$ is a σ-finite complete measure space and f_1, f_2 are complex-valued (real-valued) measurable functions on Ω. The function having idempotent decomposition $f = f_1e_1 + f_2e_2$ is called as a $\mathbb{BC}$-valued measurable function and $|f|_k = |f_1|e_1 + |f_2|e_2$ is called a $\mathbb{D}$-valued measurable function on Ω, [16]. Thus for any given complex valued function space $\left(F(\Omega),\|\cdot\|_\Omega\right)$, one can create a $\mathbb{BC}$-valued function space $\left(F(\Omega,\mathbb{BC}),\|\cdot\|_{\mathbb{BC}}\right)$ by combining all f_1, f_2 and bringing out functions of the type $f = f_1e_1 + f_2e_2$ where f_1 and f_2 are in $\left(F(\Omega),\|\cdot\|_\Omega\right)$ with $\|f\|_{\mathbb{BC}}^2 = \frac{1}{2}\left(\|f_1\|_\Omega^2 + \|f_2\|_\Omega^2\right)$. Similar definition can be given for hyperbolic measurable functions.

For any $\mathbb{BC}$-valued measurable function $f = f_1e_1 + f_2e_2$, it is easy to see that $|f|_k = |f_1|e_1 + |f_2|e_2$ is $\mathbb{D}$-valued measurable. Because, if $f = f_1e_1 + f_2e_2$ is a $\mathbb{BC}$-valued measurable function, then f_1 and f_2 are $\mathbb{C}$-valued measurable functions. Therefore real and imaginary parts of f_1 and f_2 are $\mathbb{R}$-valued measurable and so do $|f_1|$ and $|f_2|$. As a result, $|f|_k$ is $\mathbb{D}$-valued measurable. Also for any two $\mathbb{BC}$-valued measurable functions f and g it can be easily seen that their sum and multiplication functions are also $\mathbb{BC}$-measurable functions [15, 16].

Theorem 6. Let $u = u_1e_1 + u_2e_2$, $v = v_1e_1 + v_2e_2$ and $u_n = u_1^ne_1 + u_2^ne_2$ be $\mathbb{BC}$-measurable functions defined on a σ-finite complete measure space. Assume that $\lambda\in\mathbb{BC}$ and ψ is a $\mathbb{BC}$-continuous map on a $\mathbb{BC}$-open set in $\mathbb{BC}$. Then:

(a) Real and imaginary parts of the functions u_1, u_2, v_1 and v_2 are $\mathbb{R}$-measurable

(b) u_1, u_2, v_1 and v_2 are $\mathbb{C}$-measurable

(c) $u+v$, $u\cdot v$ and λu are $\mathbb{BC}$-measurable

(d) $\sup_{\mathbb{D}}|u_n|_k$, $\inf_{\mathbb{D}}|u_n|_k$, $\limsup_{\mathbb{D}}|u_n|_k$, $\liminf_{\mathbb{D}}|u_n|_k$, $\lim_{\mathbb{D}}|u_n|_k$ are $\mathbb{D}$-measurable

(e) $\psi \circ u$ is $\mathbb{BC}$-measurable.

Proof: The proof of each item can be done by using the definition of measurable function and the similar techniques used in [17, Appendix A].

Definition 7. Let $(\Omega, \mathcal{M}, \vartheta)$ be a measure space, $\mathfrak{F}(\Omega, \mathcal{M})$ indicate the set of all $\mathcal{M}$-measurable $\mathbb{BC}$-valued functions on Ω and $u \in \mathfrak{F}(\Omega, \mathcal{M})$ be a $\mathbb{BC}$-valued function. Let $E_M = \left\{ x \in \Omega : |u(x)|_k \succ M \right\}$ for any $M \succcurlyeq 0$. Since u is a $\mathcal{M}$-measurable function, it can be said that $|u|_k = |u_1| e_1 + |u_2| e_2$ is $\mathbb{D}$-valued measurable, i.e. $E_M \in \mathcal{M}$ for any $M \succcurlyeq 0$. If the set A is defined as $A = \left\{ M \succ 0 : \vartheta(E_M) = 0 \right\} = \left\{ M \in \mathbb{D}^+ : |u(x)|_k \preccurlyeq M \ \ \vartheta - a.e. \right\}$, then $\mathbb{D}-$essential supremum of u , denoted by $ess\sup_{\mathbb{D}} u$ or $\|u\|_{\infty,\mathbb{D}}$ is defined by

$$\|u\|_{\infty,\mathbb{D}} = ess\sup_{\mathbb{D}} u = \inf_{\mathbb{D}}(A).$$

2. $\mathbb{D}-$DISTRIBUTION FUNCTION

Now suppose that $(\Omega, \mathcal{M}, \vartheta)$ is a σ-finite complete measure space and $\mathfrak{F}(\Omega, \mathcal{M})$ is the set of all $\mathcal{M}$-measurable $\mathbb{BC}$-valued functions on Ω.

Definition 8. $\mathbb{D}$-distribution function $D_u : \mathbb{D}^+ \cup \{0\} \to [0, \infty]$ of a function u in $\mathfrak{F}(\Omega, \mathcal{M})$ is given by

$$D_u(\lambda) = \vartheta\left\{ x \in \Omega : |u(x)|_k \succ \lambda \right\} \tag{3}$$

for all $\lambda \succcurlyeq \mathbf{O}$.

Theorem 9. Let u and v be two $\mathbb{BC}$-valued functions in $\mathfrak{F}(\Omega, \mathcal{M})$ with $u = u_1 e_1 + u_2 e_2$ and $v = v_1 e_1 + v_2 e_2$. Then for any $\lambda, \alpha, \gamma \in \mathbb{D}^+ \cup \{0\}$:

(a) $D_u(\cdot)$ is decreasing in the sense of $\mathbb{D}$;

(b) Being $|v|_k \preccurlyeq |u|_k$ $\vartheta-$a.e. says that $D_v(\lambda) \le D_u(\lambda)$;

(c) $D_{du}(\cdot) = D_u\left(\dfrac{\cdot}{|d|_k}\right)$ for all $d \in \mathbb{BC}$ with non-zero components;

(d) $D_{u+v}(\alpha + \gamma) \le D_u(\alpha) + D_v(\gamma) + D_{w_1}(\delta_1) + D_{w_2}(\delta_2)$;

(e) $D_{uv}(\alpha\gamma) \le D_u(\alpha) + D_v(\gamma) + D_{w_1}(\delta_1) + D_{w_2}(\delta_2)$;

(f) If $|u|_k \preccurlyeq \liminf_{\mathbb{D}} |u_n|_k$ ϑ-a.e., then $D_u(\lambda) \le \liminf D_{u_n}(\lambda)$;

(g) If $|u_n|_k$ is $\mathbb{D}$-increasing and $\mathbb{D}-$convergent to $|u|_k$, then $\lim_{\mathbb{D}} D_{u_n}(\lambda) = D_u(\lambda)$.

Proof: Let $\lambda, \alpha, \gamma \in \mathbb{D}^{+} \cup \{0\}$ with $\alpha = \alpha_1 e_1 + \alpha_2 e_2$, $\gamma = \gamma_1 e_1 + \gamma_2 e_2$ and $\lambda = \lambda_1 e_1 + \lambda_2 e_2$.
(a) Let $\alpha \preccurlyeq \gamma$. Then $\gamma_1 \geq \alpha_1$, $\gamma_2 \geq \alpha_2$ and therefore the inclusion

$$\left\{x \in \Omega : \left|u(x)\right|_k \succ \gamma\right\} = \left\{x \in \Omega : \left|u_1(x)\right| e_1 + \left|u_2(x)\right| e_2 \succ \gamma_1 e_1 + \gamma_2 e_2\right\}$$

$$= \left\{x \in \Omega : \left|u_1(x)\right| > \gamma_1 \text{ and } \left|u_2(x)\right| > \gamma_2\right\}$$

$$\subseteq \left\{x \in \Omega : \left|u_1(x)\right| > \alpha_1 \text{ and } \left|u_2(x)\right| > \alpha_2\right\}$$

$$= \left\{x \in \Omega : \left|u(x)\right|_k \succ \alpha\right\}$$

exists and so $D_u(\gamma) \leq D_u(\alpha)$ by the monotonicity of the measure. It means $D_u(\cdot)$ is non-increasing.

Now let $\gamma_0 \succcurlyeq 0$ and define

$$G_u(\lambda) = \left\{x \in \Omega : \left|u_1(x)\right| e_1 + \left|u_2(x)\right| e_2 \succ \lambda\right\}.$$

Since $G_u(\lambda_1) \subseteq G_u(\lambda_2) \subseteq G_u(\lambda_3) \subseteq \ldots$ when $\lambda_1 \succcurlyeq \lambda_2 \succcurlyeq \lambda_3 \succcurlyeq \ldots$, then by [15, Theorem 3.7], we get

$$\lim_{n\to\infty} D_u\left(\gamma_0 + \frac{1}{n}\right) = \lim_{n\to\infty} \vartheta\left\{G_u\left(\gamma_0 + \left(e_1 + e_2\right)\frac{1}{n}\right)\right\}$$

$$= \vartheta\left(\bigcup_{n=1}^{\infty} G_u\left(\gamma_0 + \frac{1}{n}\right)\right) = \vartheta\left(G_u\left(\gamma_0\right)\right) = D_u\left(\gamma_0\right).$$

This shows the $\mathbb{D}$-continuity of $D_u(\cdot)$.

(b) Let $|u|_k = |u_1| e_1 + |u_2| e_2$, $|v|_k = |v_1| e_1 + |v_2| e_2$ and $|v|_k \preccurlyeq |u|_k$ $\vartheta-$a.e. Then $\left|v_1(x)\right| \leq \left|u_1(x)\right|$ and $\left|v_2(x)\right| \leq \left|u_2(x)\right|$ $\vartheta-$a.e. Therefore, the inclusion

$$\left\{x \in \Omega : \left|v(x)\right|_k \succ \lambda\right\} \subseteq \left\{x \in \Omega : \left|u(x)\right|_k \succ \lambda\right\}$$

exists for any $\lambda \in \mathbb{D}^{+} \cup \{0\}$. Since the measure is monotone, one can write that $D_v(\lambda) \leq D_u(\lambda)$ for any $\lambda \in \mathbb{D}^{+} \cup \{0\}$.

(c) Let $d = d_1 e_1 + d_2 e_2$ be any $\mathbb{BC}$-number with $d_1 \neq 0$, $d_2 \neq 0$. Since

$$\frac{\lambda}{|d|_k} = \frac{\lambda_1 e_1 + \lambda_2 e_2}{|d_1| e_1 + |d_2| e_2} = \frac{\lambda_1 e_1}{|d_1|} + \frac{\lambda_2 e_2}{|d_2|} \tag{4}$$

for any $\lambda \in \mathbb{D}$ and $du(x) = (d_1 u_1(x)) e_1 + (d_2 u_2(x)) e_2$ for any $x \in \Omega$, we have

$$\begin{aligned}
D_{du}(\lambda) &= \vartheta\left\{x \in \Omega : |du(x)|_k \succ \lambda\right\} \\
&= \vartheta\left\{x \in \Omega : |d_1 u_1(x)| e_1 + |d_2 u_2(x)| e_2 \succ \lambda_1 e_1 + \lambda_2 e_2\right\} \\
&= \vartheta\left\{x \in \Omega : |d_1 u_1(x)| > \lambda_1 \text{ and } |d_2 u_2(x)| > \lambda_2\right\} \\
&= \vartheta\left\{x \in \Omega : |u_1(x)| > \frac{\lambda_1}{|d_1|} \text{ and} |u_2(x)| > \frac{\lambda_2}{|d_2|}\right\} \\
&= \vartheta\left\{x \in \Omega : |u_1(x)| e_1 + |u_2(x)| e_2 \succ \frac{\lambda_1}{|d_1|} e_1 + \frac{\lambda_2}{|d_2|} e_2\right\} \\
&= \vartheta\left\{x \in \Omega : |u(x)|_k \succ \frac{\lambda}{|d|_k}\right\} = D_u\left(\frac{\lambda}{|d|_k}\right)
\end{aligned}$$

by (4).

(d) Let u, v be any two measurable, $\mathbb{BC}$-valued functions and α, γ be non-negative hyperbolic numbers. Since we have the following inclusion by [1, Remark 1.5.2(III)],

$$\left\{x \in \Omega : |u(x) + v(x)|_k \succ \alpha + \gamma\right\} =$$

$$= \left\{x \in \Omega : |u_1(x) + v_1(x)| e_1 + |u_2(x) + v_2(x)| e_2 \succ (\alpha_1 + \gamma_1) e_1 + (\alpha_2 + \gamma_2) e_2\right\}$$

$$= \left\{x \in \Omega : |u_1(x) + v_1(x)| > \alpha_1 + \gamma_1 \text{ and } |u_2(x) + v_2(x)| > \alpha_2 + \gamma_2\right\}$$

$$= \left\{x \in \Omega : |u_1(x) + v_1(x)| > \alpha_1 + \gamma_1\right\} \cap \left\{x \in \Omega : |u_2(x) + v_2(x)| > \alpha_2 + \gamma_2\right\}$$

$$\subseteq \left\{x \in \Omega : \left|u_1(x)\right| + \left|v_1(x)\right| > \alpha_1 + \gamma_1\right\} \cap \left\{x \in \Omega : \left|u_2(x)\right| + \left|v_2(x)\right| > \alpha_2 + \gamma_2\right\}$$

$$\subseteq \left(\left\{x \in \Omega : \left|u_1(x)\right| > \alpha_1\right\} \cup \left\{x \in \Omega : \left|v_1(x)\right| > \gamma_1\right\}\right) \cap \left(\left\{x \in \Omega : \left|u_2(x)\right| > \alpha_2\right\} \cup \left\{x \in \Omega : \left|v_2(x)\right| > \gamma_2\right\}\right)$$

$$= \left\{x \in \Omega : \left|u_1(x)\right| > \alpha_1, \left|u_2(x)\right| > \alpha_2\right\} \cup \left\{x \in \Omega : \left|u_1(x)\right| > \alpha_1, \left|v_2(x)\right| > \gamma_2\right\} \cup \left\{x \in \Omega : \left|u_2(x)\right| > \alpha_2, \left|v_1(x)\right| > \gamma_1\right\} \cup \left\{x \in \Omega : \left|v_1(x)\right| > \gamma_1, \left|v_2(x)\right| > \gamma_2\right\},$$

then

$$D_{u+v}(\alpha+\gamma) \leq D_u(\alpha) + D_v(\gamma) + D_{w_1}(\delta_1) + D_{w_2}(\delta_2)$$

can be written where $w_1 = u_1e_1 + v_2e_2$, $w_2 = u_2e_1 + v_1e_2$, $\delta_1 = \alpha_1e_1 + \gamma_2e_2$ and $\delta_2 = \alpha_2e_1 + \gamma_1e_2$.

(e) Let u, v, α and γ be as in (d). Then by [1, Remark 1.5.2(II)], for the set

$$U = \left\{x \in \Omega : \left|u(x)v(x)\right|_k \succ \alpha\gamma\right\}$$

the following inclusion

$$U = \left\{x \in \Omega : \left|u(x)v(x)\right|_k \succ \alpha\gamma\right\}$$

$$= \left\{x \in \Omega : \left|u(x)\right|_k \left|v(x)\right|_k \succ \alpha_1\gamma_1e_1 + \alpha_2\gamma_2e_2\right\}$$

$$= \left\{x \in \Omega : \left|u_1(x)v_1(x)\right|e_1 + \left|u_2(x)v_2(x)\right|e_2 \succ \alpha_1\gamma_1e_1 + \alpha_2\gamma_2e_2\right\}$$

$$= \left\{x \in \Omega : \left|u_1(x)v_1(x)\right| > \alpha_1\gamma_1 \text{ and } \left|u_2(x)v_2(x)\right| > \alpha_2\gamma_2\right\}$$

$$= \left\{x \in \Omega : \left|u_1(x)\right|\left|v_1(x)\right| > \alpha_1\gamma_1\right\} \cap \left\{x \in \Omega : \left|u_2(x)\right|\left|v_2(x)\right| > \alpha_2\gamma_2\right\}$$

$$\subseteq \left(\left\{x \in \Omega : \left|u_1(x)\right| > \alpha_1\right\} \cup \left\{x \in \Omega : \left|v_1(x)\right| > \gamma_1\right\}\right) \cap \left(\left\{x \in \Omega : \left|u_2(x)\right| > \alpha_2\right\} \cup \left\{x \in \Omega : \left|v_2(x)\right| > \gamma_2\right\}\right)$$

$$= \left\{x \in \Omega : \left|u_1(x)\right| > \alpha_1, \left|u_2(x)\right| > \alpha_2\right\} \cup \left\{x \in \Omega : \left|u_1(x)\right| > \alpha_1, \left|v_2(x)\right| > \gamma_2\right\} \cup \left\{x \in \Omega : \left|u_2(x)\right| > \alpha_2, \left|v_1(x)\right| > \gamma_1\right\} \cup \left\{x \in \Omega : \left|v_1(x)\right| > \gamma_1, \left|v_2(x)\right| > \gamma_2\right\}$$

can be written. Thus

$$D_{uv}(\alpha\gamma) = \vartheta(U) \leq D_u(\alpha) + D_v(\gamma) + D_{w_1}(\delta_1) + D_{w_2}(\delta_2)$$

where $D_{w_1}(\delta_1), D_{w_2}(\delta_2)$ are as in (d).

(f) Assume that $\lambda \succ 0$ is fixed, $E = \{x \in \Omega : |u(x)|_k \succ \lambda\}$, $E_n = \{x \in \Omega : |u_n(x)|_k \succ \lambda\}$ for all $n \in \mathbb{N}$ and $|u|_k \preccurlyeq \operatorname{liminf}_{\mathbb{D}} |u_n|_k$ ϑ-a.e. Then, by the definiton of limit inferior, we have $E \subset \bigcup_{j=1}^{\infty}\left(\bigcap_{n>j} E_n\right)$ and

$$\vartheta\left(\bigcap_{n>j} E_n\right) \leq \inf_{n>j} \vartheta(E_n) \leq \sup_j \left(\inf_{n>j} \vartheta(E_n)\right) = \liminf_{n\to\infty} \vartheta(E_n) \tag{5}$$

for all $j \in \mathbb{N}$. As a consequence of the monotone convergence theorem and the fact $\bigcap_{n>j} E_n \subset \bigcap_{n>j+1} E_n$, we get

$$\vartheta(E) \leq \vartheta\left(\bigcup_{j=1}^{\infty}\left(\bigcap_{n>j} E_n\right)\right) = \lim_{j\to\infty} \vartheta\left(\bigcap_{n>j} E_n\right) \leq \liminf_{n\to\infty} \vartheta(E_n)$$

by (5). So $D_u(\lambda) \leq \liminf_{n\to\infty} D_{u_n}(\lambda)$.

(g) Let $|u_n|_k$ be $\mathbb{D}$-increasing sequence and $\mathbb{D}-$convergent to $|u|_k$. Then being $|u_1|_k \preccurlyeq |u_2|_k \preccurlyeq |u_3|_k \preccurlyeq \cdots$ implies that the sequence $E_n = \{x \in \Omega : |u_n(x)|_k \succ \lambda\}$ is increasing. Therefore

$$E = \{x \in \Omega : |u(x)|_k \succ \lambda\} = \bigcup_{n=1}^{\infty} E_n$$

and $D_u(\lambda) = \vartheta(E) = \vartheta\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \vartheta(E_n) = \lim_{n\to\infty} D_{u_n}(\lambda)$.

3. $\mathbb{D}-$DECREASING REARRANGEMENT

In this section, by using the notion of $\mathbb{D}-$distribution function, we will introduce the $\mathbb{D}-$decreasing rearrangement and show some fundamental properties of it.

Definition 10. Let u be a function in $\mathfrak{F}(\Omega, \mathcal{M})$. $\mathbb{D}-$rearrangement of u is the function $u^* : [0, \infty) \to \mathbb{D}^+ \cup \{0\}$ which is defined as

$$u^*(t) = \inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le t\right\}$$

$$= \sup_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_u(\lambda) > t\right\}$$

where $\inf_{\mathbb{D}} \varnothing = \infty_{\mathbb{D}}$.

According to [16, Example 2.2], since

$$\|u\|_{\infty,\mathbb{D}} = \inf_{\mathbb{D}}\left\{\alpha \succcurlyeq 0 : \vartheta\left\{x \in \Omega : |u(x)|_k \succ \alpha\right\} = 0\right\},$$

one can write that

$$u^*(0) = \inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_u(\lambda) = 0\right\}$$

$$= \inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : \vartheta\left\{x \in \Omega : |u(x)|_k \succ \lambda\right\} = 0\right\}$$

$$= \|u\|_{\infty,\mathbb{D}}.$$

On the other hand, $\mathbb{D}$ – decreasing property of $D_u(\lambda)$ implies

$$u^*(D_u(t)) = \inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le D_u(t)\right\}$$

$$= \inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : \lambda \succ t\right\} = t$$

and so $u^*(\cdot)$ is the left $\mathbb{D}$-inverse of the distribution function $D_u(\cdot)$. By using the techniques used in the continuation of [17, Definition 4.4] and the $\mathbb{D}$-continuity of D_u, we have

$$u^*(D_u(\lambda)) \preccurlyeq \lambda \text{ and } D_u(u^*(t)) \le t. \tag{6}$$

In the following theorem, we will give some basic properties of the $\mathbb{D}$-decreasing rearrangement function.

Theorem 11. The $\mathbb{D}$-rearrangement u^* of a measurable function u has the following properties:

(a) u^* *is* $\mathbb{D}$*-decreasing*;

(b) $u^*(t) \succ \lambda_0$ *if and only if* $D_u(\lambda_0) > t$;

(c) u *and* u^* *are equimeasurables, that is* $D_u(\lambda) = D_{u^*}(\lambda)$ *for all* $\lambda \succcurlyeq 0$;

(d) *If* $u \in \mathfrak{F}(\Omega, \mathcal{M})$, *then* $u^*(t) = D_{D_u}(t)$ *for all* $t \ge 0$, *i.e.* u^* *is right-continuous*;

(e) $(\alpha u)^*(t) = |\alpha|_k\, u^*(t)$ *for any* $\alpha \in \mathbb{D} - \{0\}$;

(f) If $|u_n|_k \uparrow_{\mathbb{D}} |u|_k$ *in the sense of* $\mathbb{D}$, *then* $u_n^* \uparrow u^*$ *in* $\mathbb{D}$;

(g) If $|u|_k \preccurlyeq \operatorname{liminf}_{\mathbb{D}} |u_n|_k$, *then* $u^* \preccurlyeq \operatorname{liminf}_{\mathbb{D}} u_n^*$;

(h) If $|u|_k \preccurlyeq |v|_k$, *then* $u^*(t) \preccurlyeq v^*(t)$

(i) *For any* $E \in \mathcal{M}$, $(\chi_E)^*(t) = \chi_{0,\vartheta(E)]}(t)$;

(j) *If* $E \in \mathcal{M}$, *then* $(u\chi_E)^*(t) \preccurlyeq u^*(t)\chi_{0,\vartheta(E)]}(t)$

(k) *If* u *belong to* $\mathfrak{F}(\Omega,\mathcal{M}), \lambda \succ 0$ *and* $U = \chi_{E_U(\lambda)}$, *then* $U^*(t) = \chi_{E_{u^*}(\lambda)}(t)$ *where* $E_U(\lambda) = \left\{x \in \Omega : |U(x)|_k \succ \lambda\right\}$.

Proof:

(a) Let $0 \le t_1 \le t_2$. It is easy to see that

$$\left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le t_1\right\} \subset \left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le t_2\right\}$$

and

$$\inf_{\mathbb{D}} \left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le t_1\right\} \succcurlyeq \inf_{\mathbb{D}} \left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le t_2\right\}.$$

Therefore $u^*(t_1) \succcurlyeq u^*(t_2)$.

(b) If $\lambda_0 \prec u^*(t) = \inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le t\right\}$ then, by the definition of $\mathbb{D}$-infimum, we have $\lambda_0 \notin \left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le t\right\}$ and so $D_u(\lambda_0) > t$. Conversely, if $t < D_u(\lambda_0)$ for any $t \ge 0$, then we get $u^*(t) = \inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_u(\lambda) \le t\right\} \preccurlyeq \lambda_0$. By using the $\mathbb{D}$-decreasing property of distribution function, we have $D_u(\lambda_0) \le D_u\left(u^*(t)\right) \le t$ by (6). This is a contradiction.

(c) According to (b) we have

$$\left\{t \ge 0 : u^*(t) \succ \lambda\right\} = \left\{t \ge 0 : D_u(\lambda) > t\right\} = \left(0, D_u(\lambda)\right)$$

and so

$$D_{u^*}(\lambda) = m\left\{t \ge 0 : u^*(t) \succ \lambda\right\} = m\left(\left(0, D_u(\lambda)\right)\right) = D_u(\lambda)$$

for all $\lambda \succcurlyeq 0$ where m is the usual measure on $\mathbb{R}$.

(d) By using (b) again, we have

$$\left\{\lambda \succcurlyeq 0 : D_u(\lambda) > t\right\} = \left(0, u^*(t)\right)_{\mathbb{D}}$$

and then

$$u^*(t) = m_{\mathbb{D}}\left(\left\{\lambda \succcurlyeq 0 : D_u(\lambda) > t\right\}\right) = D_{D_u}(t)$$

where $m_{\mathbb{D}}\left((0,\alpha)_{\mathbb{D}}\right) = m\left((0,\alpha_1)\right)e_1 + m\left((0,\alpha_2)\right)e_2$. Thus, Theorem 9(a) shows that u^* is right-continuous.

(e) Let $u \in \mathfrak{F}(\Omega,\mathcal{M})$ and $\alpha \in \mathbb{BC}$ with non-zero components. Then

$$\left(\alpha u\right)^{*}\left(t\right)=\inf_{\mathbb{D}}\left\{\lambda\succcurlyeq 0:D_{\alpha u}\left(\lambda\right)\leq t\right\}$$

$$=\inf_{\mathbb{D}}\left\{\lambda\succcurlyeq 0:D_{u}\left(\frac{\lambda}{|\alpha|_k}\right)\leq t\right\}$$

$$=\inf_{\mathbb{D}}\left\{|\alpha|_k\,\lambda^{'}\succcurlyeq 0:D_{u}\left(\lambda^{'}\right)\leq t\right\}$$

$$=|\alpha|_k\inf_{\mathbb{D}}\left\{\lambda^{'}\succcurlyeq 0:D_{u}\left(\lambda^{'}\right)\leq t\right\}$$

$$=|\alpha|_k\,u^{*}\left(t\right)$$

where $\lambda=\lambda^{'}|\alpha|_k$.

(f) It is known from Theorem 9(g) that if $|u_n|_k\uparrow_{\mathbb{D}}|u|_k$ in the sense of $\mathbb{D}$, then $\lim_{n\to\infty}D_{u_n}(\lambda)=D_u(\lambda)$. Now, let $U_n(t)=D_{u_n}(t)$ and $U(t)=D_u(t)$. Then, we get

$$u_n^{*}(t)=\inf_{\mathbb{D}}\left\{\lambda\succcurlyeq 0:D_{u_n}\left(\lambda\right)\leq t\right\}$$

$$=D_{D_{u_n}}\left(t\right)=D_{U_n}\left(t\right)$$

by (d). Since $|u_n|_k$ is $\mathbb{D}$-increasing, we have $D_{u_n}(t)\leq D_{u_{n+1}}(t)$, $U_n(t)\leq U_{n+1}(t)$ and so

$$E_{U_1}(t)\subseteq E_{U_2}(t)\subseteq\ldots\quad\text{and}\quad E_U(t)=\bigcup_{n=1}^{\infty}E_{U_n}(t)$$

where $E_{U_j}(t)=\left\{\lambda\succcurlyeq 0:D_{u_j}\left(\lambda\right)>t\right\}$ for all $j\in\mathbb{N}$. Thus $\lim_{n\to\infty}D_{U_n}(t)=D_U(t)$, in other words, $\lim_{\mathbb{D}}u_n^{*}(t)=u^{*}(t)$.

(g) Assume $U_n(t)=\inf_{\mathbb{D}\,m>n}|u_m(t)|_k$ and observe that

$$U_n(t)\preccurlyeq U_{n+1}(t)$$

for all $n\in\mathbb{N}$ and all $t\in\Omega$. If we take $v(t)=\liminf_{\mathbb{D}}|u_n(t)|_k=\sup_{\mathbb{D}\,n\geq 1}U_n(t)$, then we obtain $U_n^{*}\xrightarrow{\mathbb{D}}v^{*}$ as $n\to\infty$ by (f) and the fact that $U_n\uparrow v$. Since $|u(t)|_k\preccurlyeq\liminf_{n\to\infty}{}_{\mathbb{D}}|u_n(t)|_k=v(t)$, we have

$$u^{*}(t)\leq v^{*}(t)=\sup_{\mathbb{D}\,n\geq 1}U_n^{*}(t).$$

Again using the fact that $U_n \preccurlyeq |u_m|_k$ for $m \geq n$, it follows that $U_n^* \preccurlyeq \inf_{\mathbb{D}}{}_{m\geq n} u_n^*(t)$ and so

$$u^*(t) \preccurlyeq v^*(t) = \sup_{\mathbb{D}}{}_{n\geq 1} U_n^*(t) \preccurlyeq \sup_{\mathbb{D}}{}_{n\geq 1}\left(\inf_{\mathbb{D}}{}_{m>n} u_m^*(t)\right) = \liminf_{\mathbb{D}} u_n^*(t).$$

(h) Let $u, v \in \mathfrak{F}(\Omega, \mathcal{M})$ with $|u|_k \preccurlyeq |v|_k$. Since $D_u(\lambda) \leq D_v(\lambda)$ for all $\lambda \succcurlyeq 0$, this yields

$$\left\{\lambda \succcurlyeq 0 : D_v(\lambda) \leq t\right\} \subset \left\{\lambda \succcurlyeq 0 : D_u(\lambda) \leq t\right\},$$

$$\inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_v(\lambda) \leq t\right\} \succcurlyeq \inf_{\mathbb{D}}\left\{\lambda \succcurlyeq 0 : D_u(\lambda) \leq t\right\}$$

and $u^*(t) \preccurlyeq v^*(t)$.

(i) Let $E \in \mathcal{M}$ and $u = \chi_E$. Since u is a real valued function, the result comes from [17, Theorem 4.5 (j)].

(j) Since $\left|(u\chi_E)(x)\right|_k \preccurlyeq \left|u(x)\right|_k$ for all $x \in \Omega$, we can write $D_{u\chi_E}(\lambda) \leq D_u(\lambda)$ for all $\lambda \succcurlyeq 0$ and then

$$\left\{\lambda \succcurlyeq 0 : D_{u\chi_E}(\lambda) > t\right\} \subseteq \left\{\lambda \succcurlyeq 0 : D_u(\lambda) > t\right\}.$$

Therefore $D_{D_{u\chi_E}}(t) \leq D_{D_u}(t)$ and so $\left(u\chi_E\right)^*(t) \preccurlyeq u^*(t)\chi_{0,\vartheta(E)]}(t)$ for all $t \geq 0$.

(k) Let $u \in \mathfrak{F}(\Omega, \mathcal{M})$, $E_u(\lambda) = \left\{x \in \Omega : \left|u(x)\right|_k \succ \lambda\right\}$ and $U = \chi_{E_u(\lambda)}$. Then, by (b) and (i), we have $U^*(t) = \chi_{\left(0, \vartheta E_u(\lambda)\right)}(t)$ and

$$E_{u^*}(\lambda) = \left\{t \geq 0 : \left|u^*(t)\right|_k \succ \lambda\right\}$$

$$= \left\{t \geq 0 : D_u(\lambda) \succ t\right\} = \left(0, D_u(\lambda)\right).$$

Therefore we get $U^*(t) = \chi_{E_{u^*}(\lambda)}(t)$.

4. CONCLUSIONS

In conclusion, the study of $\mathbb{BC}$-distribution and $\mathbb{BC}$-rearrangement functions has provided valuable insights into the behavior and properties of functions within the $\mathbb{BC}$ framework. These functions have proven to be powerful tools in analyzing and characterizing certain classes of functions, particularly those exhibiting rearrangements with respect to a given weight function.

One significant connection that emerges from our exploration is the profound relationship between $\mathbb{BC}$-distribution and $\mathbb{BC}$-rearrangement functions with $\mathbb{BC}$-Lorentz and Orlicz spaces. $\mathbb{BC}$-Lorentz spaces, characterized by the interplay of the fundamental properties of rearrangement functions and the Lorentz space structure, offer a natural setting for understanding the behavior of functions in the bicomplex numbers context.

The $\mathbb{BC}$-Lorentz spaces provide a suitable framework for studying the growth and decay properties of $\mathbb{BC}$-distribution functions, shedding light on their behavior under different weight conditions. Furthermore, the connection between $\mathbb{BC}$-rearrangement functions and $\mathbb{BC}$-Lorentz spaces enables us to extend our understanding of rearrangements to the broader context of Lorentz spaces, where the interplay between rearrangement and weight functions becomes crucial.

This synergy between $\mathbb{BC}$-distribution, $\mathbb{BC}$-rearrangement functions, and $\mathbb{BC}$-Lorentz spaces opens up avenues for further research and applications in functional analysis, harmonic analysis, and related areas. As we delve deeper into these connections, we anticipate uncovering new and profound results that contribute to the broader understanding of function spaces and their interrelationships within the bicomplex numbers.

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