ORIGINAL PAPER

MORE SUMMATION FORMULAS ON HYBRINOMIAL SEQUENCE OF VAN DER LAAN

SEYYED HOSSEIN JAFARI PETROUDI[1], ARZU ÖZKOÇ ÖZTÜRK[2]

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Abstract. *Özdemir [1] introduced Hybrid numbers as a generalization of complex, hyperbolic and dual numbers. Hybrinomial sequence is the combination of hybrid numbers and polynomial sequence. In this paper we consider special kind of hybrinomial sequence, namely the Van Der Laan hybrinomial sequence. Binet-Like Formula, generating function and exponential generating function of this sequence are shown in this paper. Properties and some summation identities for Van Der Laan polynomial sequence and hybrinomial sequence are represented in this paper. In addition, some interesting summation identities of Van Der Laan hybrid numbers are obtained.*

Keywords: *Van Der Laan sequence; hybrinomial sequence; generating function; summation identities.*

1. INTRODUCTION

Recently, many authors have studied different kinds of number sequences such as Pell sequence, Pell-Lucas sequence, Padovan and Perrin sequences, Jacobsthal sequence. They established new results about these sequences. Özdemir [1] introduced the hybrid numbers as a generalization of complex hyperbolic and dual numbers. He has stated that the set H of hybrid numbers Z is of the form

$$H = \{Z = a + bi + c\epsilon + dh; a, b, c, d \in \mathbb{R}\},$$

where i, ϵ, h are operators such that

$$i^2 = -1\,, \epsilon^2 = 0, ih = -hi = \epsilon + i.$$

It is interesting to see that, hybrid numbers are not commutative. For more information about hybrid numbers, we refer to [1]. The conjugate of hybrid number Z is defined by

$$\overline{Z} = \overline{a + bi + c\epsilon + dh} = a - bi - c\epsilon - dh.$$

The real number $R(Z) = Z\overline{Z} = \overline{Z}Z = a^2 + b^2 - 2bc - d^2$ is called the character of the hybrid number Z. The norm of a hybrid number Z is denoted by $\|Z\|$ and is given by $\sqrt{Z\overline{Z}}$.

[1] Payame Noor University, Department of Mathematics, 1935-3697 Tehran, Iran. E-mail: petroudi@pnu.ac.ir.
[2] Düzce University, Department of Mathematics, 81000 Düzce, Turkey. E-mail: arzuozkoc@duzce.edu.tr.

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Liana and Wloch [2] introduced the Jacobsthal and Jacobsthal Lucas hybrid numbers and investigated some of their properties. The authors in [3] introduced Van Der Laan Hybrid numbers and presented some results about this kind of hybrid numbers. Motivated by the work of Liana and Wloch based on the work of Özdemir, we will combine the definition of hybrid numbers to the Van Van Der Laan polynomial sequence and introduce the Van Der Laan hybrinomial sequence. Our aim is to obtain Binet-like formula, partial sum, generating function, exponential generating function, character and norm of this sequence. We represent other interesting properties and some summation formulas about the Van Der Laan polynomial sequence, Van Der Laan hybrinomial sequence. Moreover, we present summation formulas of Van Der Laan hybrid number which is the special case of Van Der Laan hybrinomial sequence.

For supplementary information about Van der Laan sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal-Lucas sequence, Padovan and Perrin sequences and the other related number sequences we refer to [4-11].

2. VAN DER LAAN POLYNOMIAL SEQUENCE AND VAN DER LAAN HYBRINOMIAL SEQUENCE

In this section initially we will consider the Van Der Laan polynomial sequence and introduce the Van der Laan hybrinomial sequence. Then, we propose some basic properties of these sequences. Furthermore, we describe Binet-like formula, generating function and some identities related to these sequences.

The Van Der Laan sequence [7], (V_n) is defined by the recurence relation $V_n = V_{n-2} + V_{n-3}$, for all $n \geq 3$, with initial values $V_0 = 0, V_1 = 1, V_2 = 0$. The first values of (V_n) are

$$0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114.$$

Also, the Binet-like-formula for the Van Der Laan sequence is:

$$V_n = \frac{{r_1}^n}{(r_1 - r_2)(r_1 - r_3)} + \frac{{r_2}^n}{(r_2 - r_1)(r_2 - r_3)} + \frac{{r_3}^n}{(r_1 - r_3)(r_2 - r_3)},$$

where r_1, r_2, r_3 are the roots of the equation $x^3 - x - 1 = 0$. From [1] we have
$r_1 + r_2 + r_3 = 0, r_1 r_2 r_3 = 1, r_1 r_2 + r_2 r_3 + r_1 r_3 =$-1.

Definition 1. The Van Der Laan hybrinomial sequence, denoted by $V_n^{[H]}(x)$ is defined as:

$$V_n^{[H]}(x) = V_n(x) + V_{n+1}(x)i + V_{n+2}(x)\epsilon + V_{n+3}(x)h, \tag{1}$$

where, $V_n(x)$ is the Van Der Laan polynomial sequence and is defined by the following relation:

$$V_n(x) = \begin{cases} 0 & n = 0,2 \\ 1 & n = 1,3,4 \\ xV_{n-2}(x) + V_{n-3}(x) & n \geq 5 \end{cases}$$

First few values of Van Der Laan polynomial sequence are

$$V_0(x) = 0, V_1(x) = 1,\ V_2(x) = 0,\ V_3(x) = 1, V_4(x) = 1, V_5(x) = x,\ V_6(x) = x + 1,$$
$$V_7(x) = x^2 + 1.$$

The characteristic equation of the Van Der Laan polynomial sequence $V_n(x)$ is $t^3 - xt - 1 = 0$. For each arbitrary value of x this cubic equation has three distinct roots α, β, γ.

According to the definition of Van Der Laan polynomial sequence, the initial value of the Van Der Laan hybrinomial sequence are

$$V_0^{[H]}(x) = V_0(x) + V_1(x)i + V_2(x)\epsilon + V_3(x)h = i + h,$$

$$V_1^{[H]}(x) = V_1(x) + V_2(x)i + V_3(x)\epsilon + V_4(x)h = 1 + \epsilon + h,$$

$$V_2^{[H]}(x) = V_2(x) + V_3(x)i + V_4(x)\epsilon + V_5(x)h = i + \epsilon + xh,$$

$$V_3^{[H]}(x) = V_3(x) + V_4(x)i + V_5(x)\epsilon + V_6(x)h = 1 + 1i + x\epsilon + (x + 1)h,$$

$$V_4^{[H]}(x) = V_4(x) + V_5(x)i + V_6(x)\epsilon + V_7(x)h = 1 + xi + (x + 1)\epsilon + (x^2 + 1)h.$$

By definition of the Van Der Laan hybrinomial sequence and the character of hybrid numbers, the following relation about the character of the Van Der Laan hybrinomial sequence is true.

$$R\left(V_n^{[H]}(x)\right) = {V_n}^2(x) + {V_{n+1}}^2(x) - 2V_{n+1}(x)V_{n+2}(x) - {V_{n+3}}^2(x). \blacksquare \tag{2}$$

Now, we have the following theorem about the norm of Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$.

Theorem 1. Let $n \geq 1$ be an integer. Then the norm of the Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$ is

$$\left\|V_n^{[H]}(x)\right\|^2 = 2V_{n+1}(\mathrm{x})[V_{n+2}(\mathrm{x}) + V_n(\mathrm{x})].$$

Proof: Corresponding to the character of hybrid numbers we have (2). So, we get

$$R\left(V_n^{[H]}(x)\right) = {V_n}^2(x) + {V_{n+1}}^2(x) - 2V_{n+1}(x)V_{n+2}(x) - \left(V_n(x) + V_{n+1}(x)\right)^2$$

$$= {V_n}^2(x) + {V_{n+1}}^2(x) - 2V_{n+1}(x)V_{n+2}(x) - {V_n}^2(x) - {V_{n+1}}^2(x) - 2V_n(x)V_{n+1}(x)$$

$$= -2V_{n+1}(x)(V_{n+2}(x) + V_n(x)).$$

Hence

$$\left\|V_n^{[H]}(x)\right\| = \sqrt{\left|R\left(V_n^{[H]}(x)\right)\right|} = \sqrt{\left|{V_n}^2(\mathrm{x}) + {V_{n+1}}^2(\mathrm{x}) - 2V_{n+1}(x)V_{n+2}(\mathrm{x}) - {V_{n+3}}^2(x)\right|}.$$

Therefore

$$\left\|V_n^{[H]}(x)\right\|^2 = \left|V_n^2(\mathrm{x}) + V_{n+1}^2(\mathrm{x}) - 2V_{n+1}(x)V_{n+2}(\mathrm{x}) - \left(V_n(\mathrm{x}) + V_{n+1}(\mathrm{x})\right)^2\right|$$

$$= \left|V_n^2(\mathrm{x}) + V_{n+1}^2(\mathrm{x}) - 2V_{n+1}(x)V_{n+2}(\mathrm{x}) - V_n^2(\mathrm{x}) - V_{n+1}^2(\mathrm{x}) - 2V_n(x)V_{n+1}(x)\right|$$

$$= \left|-2V_{n+1}(\mathrm{x})\left(V_{n+2}(\mathrm{x}) + V_n(\mathrm{x})\right)\right|$$

$$= 2V_{n+1}(\mathrm{x})(V_{n+2}(\mathrm{x}) + V_n(\mathrm{x})).$$

■

Özdemir in [1] defined the matrix representation of hybrid numbers by

$$M_{a+bi+c\epsilon+dh} = a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + b\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} + c\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix} + d\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

Now, we consider the matrix representation of the Van Der Laan hybrinomial sequence.

Theorem 2. Let $V_n(x)$ is the Van Der Laan polynomial sequence. Then, the matrix representation of the Van Der Laan hybrinomial sequence is

$$M_{V_r^{[H]}} = \begin{bmatrix}V_r(x) + V_{r+2}(x) & V_{r+1}(x) - V_{r+2}(x) + V_{r+3}(x)\\ -V_{r+1}(x) + V_{r+2}(x) + V_{r+3}(x) & V_r(x) - V_{r+2}(x)\end{bmatrix}. \tag{3}$$

Proof: By definition of Van Der Laan hybrinomial sequence we know that

$$V_r^{[H]}(x) = V_r(x) + V_{r+1}(x)i + V_{r+2}(x)\epsilon + V_{r+3}(x)h.$$

Consequently, using matrix representation of hybrid numbers, introduced by Ozdemir we get

$$M_{V_r^{[H]}(x)} = V_r(x)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + V_{r+1}(x)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} + V_{r+2}(x)\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix} + V_{r+3}(x)\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}V_r(x) + V_{r+2}(x) & V_{r+1}(x) - V_{r+2}(x) + V_{r+3}(x)\\ -V_{r+1}(x) + V_{r+2}(x) + V_{r+3}(x) & V_r(x) - V_{r+2}(x)\end{bmatrix}.$$

■

Now we have the following corollary that relates determinant correspond to the matrix representation of the Van Der Laan hybrinomial sequence.

Corollary 3. Determinant for the matrix representation of the Van Der Laan hybrinomial sequence is given by $-2V_{r+1}(x)\left(V_{r+2}(x) + V_r(x)\right)$, and the following relation is true about the modulus of determinant for the Van Der Laan hybrinomial sequence $M_{V_r^{[H]}(x)}$.

$$\left|\det\left(M_{V_r^{[H]}(x)}\right)\right| = 2V_{r+1}(x)\left(V_{r+2}(x) + V_r(x)\right).$$

Proof: From Theorem 2, we know the equation (3).

Hence

$$\det\left(M_{V_r^{[H]}(x)}\right) = \det\left(\begin{bmatrix} V_r(x)+V_{r+2}(x) & V_{r+1}(x)-V_{r+2}(x)+V_{r+3}(x) \\ -V_{r+1}(x)+V_{r+2}(x)+V_{r+3}(x) & V_r(x)-V_{r+2}(x) \end{bmatrix}\right)$$

$$= \left(\left(V_r(x)+V_{r+2}(x)\right)(V_r(x)-V_{r+2}(x))\right) - \left((V_{r+1}(x)-V_{r+2}(x)+V_{r+3}(x))(-V_{r+1}(x)+V_{r+2}(x)+V_{r+3}(x))\right).$$

By some computations we have

$$\det\left(M_{V_r^{[H]}(x)}\right) = {V_r}^2(\mathrm{x}) + {V_{r+1}}^2(\mathrm{x}) - 2V_{r+1}(x)V_{r+2}(\mathrm{x}) - {V_r}^2(\mathrm{x}) - {V_{r+1}}^2(\mathrm{x}) - 2V_r(x)V_{r+1}(x)$$

$$= -2V_{r+1}(x)(V_{r+2}(\mathrm{x}) + V_r(x)) = R\left(V_r^{[H]}(x)\right).$$

Therefore

$$\left|\det\left(M_{V_r^{[H]}}\right)\right| = 2V_{r+1}(V_{r+2}+V_r)\left|\det\left(M_{V_r^{[H]}(x)}\right)\right| = 2V_{r+1}(x)\left(V_{r+2}(x)+V_r(x)\right). \blacksquare$$

Lemma 4. Let $n \geq 0$ be an integer. Then the Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$ satisfies the recurrence relationship:

$$V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x) = 0.$$

Proof: By definition of the Van Der Laan hybrinomial sequence, we have

$$\begin{aligned} V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x) &= [(V_n(x)+V_{n+1}(x)i+V_{n+2}(x)\epsilon+V_{n+3}(x)h)] \\ &\quad - x[(V_{n-2}(x)+V_{n-1}(x)i+V_n(x)\epsilon+V_{n+1}(x)h)] \\ &\quad - [(V_{n-3}(x)+V_{n-2}(x)i+V_{n-1}(x)\epsilon+V_n(x)h)] \end{aligned}$$

$$= [V_n(x)-xV_{n-2}(x)-V_{n-3}(x)] + [V_{n+1}(x)-xV_{n-1}(x)-V_{n-2}(x)]i + [V_{n+2}(x)-xV_n(x)-V_{n-1}(x)]\epsilon + [V_{n+3}(x)-xV_{n+1}(x)-V_n(x)]h$$

$$= 0.$$

As (V_n) is a Van Der Laan sequence, hence

$$V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x) = 0. \blacksquare$$

In the continuation of this section, we will display the generating function, exponential generating function, Binet-like formula and identities of the Van Der Laan hybrinomial sequence. For the total result, the more general results of which is found in the article [12], the special case for Van Der Laan's hybrinomial is given below.

Theorem 5. The generating function for the Van Der Laan polynomial sequence $V_n(x)$ is given by

$$f(t)=\sum_{n=0}^{\infty} V_n(x)t^n=\frac{t}{1-t^2x-t^3}.$$

Proof: Suppose that the generating function of the Van Der Laan polynomial sequence $V_n(x)$ has the following formal power series

$$f(t)=\sum_{n=0}^{\infty} V_n(x)\,t^n=V_0(x)+V_1(x)t+V_2(x)t^2+V_3(x)t^3+\cdots.$$

Then

$$t^2xf(t)=xV_0(x)t^2+xV_1(x)t^3+xV_2(x)t^4+xV_3(x)t^5+\cdots,$$

and

$$t^3f(t)=V_0(x)t^3+V_1(x)t^4+V_2(x)t^5+V_3(x)t^6+\cdots.$$

Therefore

$$f(t)-t^2xf(x)-t^3f(t)=(V_0(x)+V_1(x)t+V_2(x)t^2+V_3(x)t^3+\cdots)-(xV_0(x)t^2+xV_1(x)t^3+xV_2(x)t^4+xV_3(x)t^5+\cdots)-(V_0(x)t^3+V_1(x)t^4+V_2(x)t^5+V_3(x)t^6+\cdots)$$

$$=V_0(x)+V_1(x)t+\big(V_2(x)-xV_0(x)\big)t^2+\big(V_3(x)-xV_1(x)-V_0(x)\big)t^2+\cdots+(V_n(x)-xV_{n-2}(x)-V_{n-3}(x))t^n+\cdots.$$

As we have $V_n(x)-xV_{n-2}(x)-V_{n-3}(x)=0$, and $V_0(x)=0, V_1(x)=1, V_1(x)=1$, hence, we obtain

$$f(t)-t^2xf(t)-t^3f(t)=V_0(x)+V_1(x)t+\big(V_2(x)-xV_0(x)\big)t^2=t.$$

So

$$f(t)(1-t^2x-t^3)=t.$$

Consequently,

$$f(t)=\sum_{n=0}^{\infty} V_n(x)t^n=\frac{t}{1-t^2x-t^3}.\ \blacksquare$$

Theorem 6. The generating function for the Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$ is given as

$$\sum_{n=0}^{\infty} V_n^{[H]}(x)t^n=\frac{\left\{\begin{matrix} V_0^{[H]}(x)+V_1^{[H]}(x)t \\ +\left(V_2^{[H]}(x)-xV_0^{[H]}(x)\right)t^2\end{matrix}\right\}}{1-t^2-t^3}.$$

Proof: Suppose that the generating function of the Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$ has the following formal power series

$$f(t) = \sum_{n=0}^{\infty} V_n^{[H]}(x)t^n = V_0^{[H]}(x) + V_1^{[H]}(x)t + V_2^{[H]}(x)t^2 + V_3^{[H]}(x)t^3 + \cdots.$$

Then

$$xt^2 f(t) = xV_0^{[H]}(x)t^2 + xV_1^{[H]}(x)t^3 + xV_2^{[H]}(x)t^4 + xV_3^{[H]}(x)t^5 + \cdots,$$

and

$$t^3 f(t) = V_0^{[H]}(x)t^3 + V_1^{[H]}(x)t^4 + V_2^{[H]}(x)t^5 + V_3^{[H]}(x)t^6 + \cdots.$$

Hence, we obtain

$$\begin{aligned} f(t) - t^2 f(t) - t^3 f(t) &= \left(V_0^{[H]}(x) + V_1^{[H]}(x)t + V_2^{[H]}(x)t^2 + V_3^{[H]}(x)t^3 + \cdots\right) - x\left(V_0^{[H]}(x)t^2 \right.\\ &\left. + V_1^{[H]}(x)t^3 + V_2^{[H]}(x)t^4 + V_3^{[H]}(x)t^5 + \cdots\right) - \left(V_0^{[H]}(x)t^3 + V_1^{[H]}(x)t^4 \right.\\ &\left. + V_2^{[H]}(x)t^5 + V_3^{[H]}(x)t^6 + \cdots\right) \end{aligned}$$

$$= \left(V_0^{[H]}(x) + V_1^{[H]}(x)t\right) + \left(V_2^{[H]}(x) - xV_0^{[H]}(x)\right)t^2 + \left(V_3^{[H]}(x) - xV_1^{[H]}(x) - V_0^{[H]}(x)\right)t^2 + \cdots + \left(V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x)\right)t^n + \cdots.$$

Using Lemma 4, we have $V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x) = 0$. Accordingly, we obtain

$$f(t) - t^2 f(t) - t^3 f(t) = V_0^{[H]}(x) + V_1^{[H]}(x)t + \left(V_2^{[H]}(x) - xV_0^{[H]}(x)\right)t^2.$$

Thus

$$f(t)(1 - t^2 - t^3) = i + h + (1 + \epsilon + h)t + (i + \epsilon + x - x(i + h))t^2.$$

Consequently

$$\sum_{n=0}^{\infty} V_n^{[H]}(x)t^n = \frac{\left\{\begin{matrix} V_0^{[H]}(x) + V_1^{[H]}(x)t \\ + \left(V_2^{[H]}(x) - xV_0^{[H]}(x)\right)t^2 \end{matrix}\right\}}{1 - t^2 - t^3}. \blacksquare$$

Corollary 7. The exponential generating function for the Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$ is given as

$$\sum_{n=0}^{\infty} V_n^{[H]}(x)\frac{t^n}{n!} = \alpha k_1 e^{\alpha t} + \beta k_2 e^{\beta t} + \gamma k_3 e^{\gamma t},$$

where α, β, γ are the roots of the equation $t^3 - xt - 1 = 0$ and

$$k_1 = \left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right),$$

$$k_2 = \left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right),$$

$$k_3 = \left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right).$$

Proof: Considering the Binet-like formula of the Van der Laan hybrinomial sequence $V_n^{[H]}(x)$, we have

$$V_n^{[H]}(\mathrm{x}) = \left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\alpha^{n+1} + \left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\beta^{n+1} + \left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\gamma^{n+1}.$$

By the Maclaurin expansion for the exponential function, we have

$$\sum_{n=0}^{\infty} V_n^{[H]}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\alpha^{n+1} + \left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\beta^{n+1} + \left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\gamma^{n+1}\right]\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} V_n^{[H]}\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\alpha k_1 {r_1}^n + \beta k_2 {r_2}^n + \gamma k_3 {r_3}^n)\frac{t^n}{n!}$$
$$= \left\{\alpha\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!} + \beta\left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!} + \gamma\left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\sum_{n=0}^{\infty}\frac{(\gamma t)^n}{n!}\right\}$$

$$= \alpha\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)e^{\alpha t} + \beta\left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)e^{\beta t} + \gamma\left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)e^{\gamma t}.$$

Lemma 8. Let $n \geq 0$ be an integer. The Binet-like formula for the Van Der Laan polynomial sequence $V_n(x)$ is given by

$$V_n(x) = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

where α, β, γ are the roots of the characteristic equation $t^3 - xt - 1 = 0$.

Proof: We know that the recursive relation $V_n(x) = xV_{n-2}(x) + V_{n-3}(x)$ has the characteristic equation $g(t) = t^3 - xt - 1 = 0$. For an arbitrary value of x, this equation has three distinct roots α, β, γ. Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are the roots of $h(t) = g\left(\frac{1}{t}\right) = 1 - t^2x - t^3 = 0$.

In exact we have

$$h(t) = 1 - t^2x - t^3 = (1-\alpha t)(1-\beta t)(1-\gamma t).$$

According to generating function of the Van der Laan polynomial sequence $V_m(x)$, we have

$$\begin{aligned} f(t) = \frac{t}{1-t^2x-t^3} &= \frac{A}{1-\alpha t} + \frac{B}{1-\beta t} + \frac{C}{1-\gamma t} \\ &= A\sum_{n=0}^{\infty}(\alpha t)^n + B\sum_{n=0}^{\infty}(\beta t)^n + C\sum_{n=0}^{\infty}(\gamma t)^n. \end{aligned} \tag{4}$$

Thus, we have

$$g(t) = \frac{t}{1-xt-t^3}$$

$$= \frac{A(1-\beta t)(1-\gamma t) + B(1-\alpha t)(1-\gamma t) + C(1-\alpha t)(1-\beta t)}{(1-\alpha t)(1-\beta t)(1-\gamma t)}.$$

Therefore, by comparison of the left and right sides of this equality

$$t = A(1-\beta t)(1-\gamma t) + B(1-\alpha t)(1-\gamma t) + C(1-\alpha t)(1-\beta t).$$

If we substitute t by $\frac{1}{\alpha}$ we find that

$$\frac{1}{\alpha} = A\left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right).$$

Hence, we get

$$\alpha = A(\alpha-\beta)(\alpha-\gamma)$$

Consequently, we derive

$$A = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we have

$$B = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)}\ , C = \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus by (4), we get

$$G(t) = \sum_{n=0}^{\infty} \frac{\alpha\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} t^n + \sum_{n=0}^{\infty} \frac{\beta\beta^n}{(\beta-\alpha)(\beta-\gamma)} t^n + \sum_{n=0}^{\infty} \frac{\gamma\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} t^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\right] t^n.$$

Consequently, we obtain

$$V_n(x) = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

Thus, the proof is completed.

Theorem 9. Let $n \geq 0$ be an integer. Then, the Binet-like formula for the Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$ is

$$V_n^{[H]}(x) = k_1\alpha^{n+1} + k_2\beta^{n+1} + k_3\gamma^{n+1},$$

where,

$$k_1 = \left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right),$$

$$k_2 = \left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right),$$

$$k_3 = \left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right).$$

Proof: From the Binet-like-formula of the Van Der Laan polynomial sequence we have

$$V_n(x) = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

where, α, β, γ are the roots of the equation $t^3 - xt - 1 = 0$.

We know that $V_n^{[H]}(x) = V_n(x) + V_{n+1}(x)i + V_{n+2}(x)\epsilon + V_{n+3}(x)h$. By substituting $V_n(x)$ in this relation, we find that

$$\begin{aligned} V_n^{[H]}(x) &= \left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\right) \\ &+ \left(\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)}\right) i \\ &+ \left(\frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)}\right) \epsilon \\ &+ \left(\frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)}\right) h \end{aligned}$$

$$\begin{aligned} &= \left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^{n+1} + \left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^{n+1} \\ &+ \left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^{n+1}. \blacksquare \end{aligned}$$

Theorem 10. Let $n \geq 0$ be an integer. Then

$$\text{(a) } V_{n+1}^{[H]}(x) + V_n^{[H]}(x) = \left\{\begin{array}{c} \left((1+\alpha i+\alpha^2\epsilon+\alpha^3 h)\left[\frac{\alpha+1}{(\alpha-\beta)(\alpha-\gamma)}\right]\right)\alpha^{n+1} \\ +\left((1+\beta i+\beta^2\epsilon+\beta^3 h)\left[\frac{\beta+1}{(\beta-\alpha)(\beta-\gamma)}\right]\right)\beta^{n+1} \\ +\left((1+\gamma i+\gamma^2\epsilon+\gamma^3 h)\left[\frac{\gamma+1}{(\gamma-\alpha)(\gamma-\beta)}\right]\right)\gamma^{n+1} \end{array}\right\},$$

$$\text{(b) } V_{n+1}^{[H]}(x) - V_n^{[H]}(x) = \left\{\begin{array}{c} \left((1+\alpha i+\alpha^2\epsilon+\alpha^3 h)\left[\frac{\alpha-1}{(\alpha-\beta)(\alpha-\gamma)}\right]\right)\alpha^{n+1} + \\ +\left((1+\beta i+\beta^2\epsilon+\beta^3 h)\left[\frac{\beta-1}{(\beta-\alpha)(\beta-\gamma)}\right]\right)\beta^{n+1} \\ +\left((1+\gamma i+\gamma^2\epsilon+\gamma^3 h)\left[\frac{\gamma-1}{(\gamma-\alpha)(\gamma-\beta)}\right]\right)\gamma^{n+1} \end{array}\right\},$$

where α, β, γ are the roots of the equation $t^3 - xt - 1 = 0$.

Proof: We prove part (a). Part (b) is similarly proved. Exploiting the Binet-like formula of the Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$, we have

$$V_{n+1}^{[H]}(x) + V_n^{[H]}(x) = \left\{\begin{array}{c} \left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^{n+2} \\ +\left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^{n+2} \\ +\left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^{n+2} \end{array}\right\} + \left\{\begin{array}{c} \left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^{n+1} \\ +\left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^{n+1} \\ +\left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^{n+1} \end{array}\right\}$$

$$= \left\{ \begin{array}{l} \left[(\alpha+1)\left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right)\right]\alpha^{n+1} \\ +\left[(\beta+1)\left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right)\right]\beta^{n+1} \\ +\left[(\gamma+1)\left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right)\right]\gamma^{n+1} \end{array} \right\}.$$

This relation proves the theorem. ■

3. SOME SUMMATION IDENTITIES

In this section we will present some summation identities of the Van Der Laan polynomial sequence and Van Der Laan hybrinomial sequence. Furthermore, we demonstrate some summation formulas of the Van Der Laan hybrid number which is the special case of the Van Der Laan hybrinimial sequence. Using the definition of (1), we see that if we put $x = 1$, we obtain the Van Der Laan hybrid number $V_n^{[H]}$, which is the number system that the authors introduced in [9].

Lemma 11. Let $n \geq 0$ be an integer and suppose that $V_n(x)$ is the Van Der Laan polynomial sequence. Then the partial sum of Van Der Laan polynomial sequence satisfies the following relation:

$$\sum_{k=0}^{n} V_k(x) = \frac{1}{x}[V_{n+2}(x) + V_{n+1}(x) + V_n(x) - 1].$$

Proof: From the definition of Van Der Laan polynomial sequence, we have $V_n(x) = \frac{1}{x}(V_{n+2}(x) - V_{n-1}(x))$. Thus,

$$V_1(x) = \frac{1}{x}(V_3(x) - V_0(x)),$$

$$V_2(x) = \frac{1}{x}(V_4(x) - V_1(x)),$$

$$V_3(x) = \frac{1}{x}(V_5(x) - V_2(x)),$$

$$V_4(x) = \frac{1}{x}(V_6(x) - V_3(x)),$$

$$\vdots$$

$$V_{n-3}(x) = \frac{1}{x}(V_{n-1}(x) - V_{n-4}(x))$$

$$V_{n-2}(x) = \frac{1}{x}(V_n(x) - V_{n-3}(x)),$$

$$V_{n-1}(x) = \frac{1}{x}(V_{n+1}(x) - N_{n-2}(x)),$$

$$V_n(x) = \frac{1}{x}(V_{n+2}(x) - N_{n-1}(x)),$$

Therefore, we get

$$\sum_{k=0}^{n} V_k(x) - V_0(x) = \frac{1}{x}(V_{n+2}(x) + V_{n+1}(x) + V_n(x) - V_0(x) - V_1(x) - V_2(x)).$$

Accordingly

$$\sum_{k=0}^{n} V_k(x) = \frac{1}{x}[V_{n+2}(x) + V_{n+1}(x) + V_n(x) - 1].$$

Theorem 12. Let $n \geq 0$ be an integer. The following relation about the partial sum of the Van Der Laan hybrinomial sequence $V_n^{[H]}(x)$ is true:

$$\sum_{k=0}^{n} V_k^{[H]}(x) = \frac{1}{x}\left(\left[V_{n+2}^{[H]}(x) + V_{n+1}^{[H]}(x) + V_n^{[H]}(x)\right] - [1 + i + \epsilon + h]\right).$$

Proof: By definition of partial sum for $V_{n+1}^{[H]}(x)$ we know $\sum_{k=0}^{n} V_k^{[H]}(x) = V_0^{[H]}(x) + V_1^{[H]}(x) + V_2^{[H]}(x) + \cdots + V_n^{[H]}(x)$. Thus

$$\sum_{k=0}^{n} V_k^{[H]}(x) = [V_0(x) + V_1(x)i + V_2(x)\epsilon + V_3(x)h] + [V_1(x) + V_2(x)i + V_3(x)\epsilon + V_4(x)h]$$
$$+ \cdots + [V_n(x) + V_{n+1}(x)i + V_{n+2}(x)\epsilon + V_{n+3}(x)h]$$

$$= [V_0(x) + V_1(x) + V_2(x) + \cdots + V_n(x)]$$
$$+ [V_1(x) + V_2(x) + \cdots + V_{n+1}(x) + V_0(x) - V_0(x)]i$$
$$+ [V_2(x) + V_3(x) + \cdots + V_{n+2}(x) + V_0(x) + V_1(x) - V_0(x) - V_1(x)]\epsilon$$
$$+ [V_3(x) + V_4(x) + \cdots + V_{n+3}(x) + V_0(x) + V_1(x) + V_2(x) - V_0(x) - V_1(x)$$
$$- V_2(x)h]$$

$$= \left[\sum_{k=0}^{n} V_k(x)\right] + \left[\sum_{k=0}^{n+1} V_k(x) - 0\right]i + \left[\sum_{k=0}^{n+2} V_k(x) - 1\right]\epsilon + \left[\sum_{k=0}^{n+3} V_k(x) - 1\right]h.$$

Therefore, by Lemma 11 we find that

$$\sum_{k=0}^{n} V_k^{[H]}(x) = \frac{1}{x}[V_{n+2}(x) + V_{n+1}(x) + V_n(x) - 1]$$
$$+ \left(\frac{1}{x}[V_{n+3}(x) + V_{n+2}(x) + V_{n+1}(x) - 1]\right)i$$

$$+\left(\frac{1}{x}[V_{n+4}(x)+V_{n+3}(x)+V_{n+2}(x)-1]\right)\epsilon+\left(\frac{1}{x}[V_{n+5}(x)+V_{n+4}(x)+V_{n+3}(x)-1]\right)h$$

$$=\frac{1}{x}[V_{n+2}(x)+V_{n+3}(x)i+V_{n+4}(x)\epsilon+V_{n+5}(x)h]+\frac{1}{x}[V_{n+1}(x)+V_{n+2}(x)i+V_{n+3}(x)\epsilon$$
$$+V_{n+4}(x)h]+\frac{1}{x}[V_n(x)+V_{n+1}(x)i+V_{n+2}(x)\epsilon+V_{n+3}(x)h]-\frac{1}{x}[1+i+\epsilon+h]$$
$$=\frac{1}{x}\left(\left[V_{n+2}^{[H]}(x)+V_{n+1}^{[H]}(x)+V_n^{[H]}(x)\right]-[1+i+\epsilon+h]\right).\ \blacksquare$$

Definition 2. [9] Van Der Laan hybrid numbers, denoted $V_n^{[H]}$ is defined as:

$$V_n^{[H]}=V_n+V_{n+1}i+V_{n+2}\epsilon+V_{n+3}h, \tag{5}$$

where (V_n) , is the Van Der Laan sequence. The first few values of Van Der Laan hybrid numbers are:

$$VH_0=i+h,\ VH_1=1+\epsilon+h,\ VH_2=i+\epsilon+h,\ VH_3=1+i+\epsilon+2h,\ VH_4=1+i+2\epsilon+2h.$$

From [9] we have the following Lemma about Van Der Laan hybrid numbers $V_n^{[H]}$.

Lemma 13. Let $n>0$ be an integer. Then the Binet Formula of Van Der Laan hybrid numbers $V_n^{[H]}$is

$$V_n^{[H]}=\left(\frac{1+r_1i+r_1{}^2\epsilon+r_1{}^3h}{(r_1-r_2)(r_1-r_3)}\right)r_1{}^{k+1}+\left(\frac{1+r_2i+r_2{}^2\epsilon+r_2{}^3h}{(r_2-r_1)(r_2-r_3)}\right)r_2{}^{k+1}$$
$$+\left(\frac{1+r_3i+r_3{}^2\epsilon+r_3{}^3h}{(r_3-r_1)(r_3-r_2)}\right)r_3{}^{k+1},$$

where r_1, r_2 and r_3 are the roots of characteristic equation $t^3-t-1=0$ of Van Der Laan sequence V_n.

Remark 14. Let $n>0$ be an integer. We know that the characteristic equation of the Van Der Laan sequence V_n, is given by $t^3-t-1=0$. This equation has three distinct roots r_1, r_2 and r_3. Thus, we obtain

$$r_1{}^3-r_1-1=0, r_2{}^3-r_2-1=0, r_3{}^3-r_3-1=0.$$

Hence,

$$r_1{}^3=r_1+1, r_2{}^3=r_2+1, r_3{}^3=r_3+1.$$

Now we can write the summation formulas of Van Der Laan hybrid numbers.

Theorem 15. The following summation formulas hold:

(a) $\sum_{k=0}^{n}\binom{n}{k}V_k^{[H]}=V_{3n}^{[H]}$,

(b) $\sum_{k=0}^{n}\binom{n}{k}V_{k+1}^{[H]}=V_{3n+1}^{[H]}$,

(c) $\sum_{k=0}^{n}\binom{n}{k}3^{n-k}V_{k}^{[H]}=3^{n}\,V_{3n}^{[H]}$,

(d) $\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}V_{k}^{[H]}=\,(-1)^{n}\,V_{3n}^{[H]}$.

Proof: Using the Binet like formula of the Van Der Laan hybrid sequence $V_k^{[H]}$in [9], we have

$$\sum_{k=0}^{n}\binom{n}{k}V_{k}^{[H]}=\sum_{k=0}^{n}\binom{n}{k}\left[\left(\frac{1+r_1i+{r_1}^2\epsilon+{r_1}^3h}{(r_1-r_2)(r_1-r_3)}\right){r_1}^{k+1}+\left(\frac{1+r_2i+{r_2}^2\epsilon+{r_2}^3h}{(r_2-r_1)(r_2-r_3)}\right){r_2}^{k+1}\right.$$
$$\left.+\left(\frac{1+r_3i+{r_3}^2\epsilon+{r_3}^3h}{(r_3-r_1)(r_3-r_2)}\right){r_3}^{k+1}\right]$$

$$=\left\{\begin{array}{l}\left[\frac{1+r_1i+{r_1}^2\epsilon+{r_1}^3h}{(r_1-r_2)(r_1-r_3)}\right]\sum_{k=0}^{n}\binom{n}{k}{r_1}^{k+1}\\+\left[\frac{1+r_2i+{r_2}^2\epsilon+{r_2}^3h}{(r_2-r_1)(r_2-r_3)}\right]\sum_{k=0}^{n}\binom{n}{k}{r_2}^{k+1}\\+\left[\frac{1+r_3i+{r_3}^2\epsilon+{r_3}^3h}{(r_3-\alpha)(r_3-r_2)}\right]\sum_{k=0}^{n}\binom{n}{k}{r_3}^{k+1}\end{array}\right\}$$

$$=\left\{\begin{array}{l}r_1\left[\frac{1+r_1i+{r_1}^2\epsilon+{r_1}^3h}{(r_1-r_2)(r_1-r_3)}\right](1+r_1)^n\\+r_2\left[\frac{1+r_2i+{r_2}^2\epsilon+{r_2}^3h}{(r_2-r_1)(r_2-r_3)}\right](1+r_2)^n\\+r_3\left[\frac{1+r_3i+{r_3}^2\epsilon+{r_3}^3h}{(r_3-r_1)(r_3-r_2)}\right](1+r_3)^n\end{array}\right\}.$$

Using Remark 14, we write ${r_1}^3=r_1+1, {r_2}^3=r_2+1, {r_3}^3=r_3+1$. Consequently,

$$\sum_{k=0}^{n}\binom{n}{k}V_{k}^{[H]}=\left[\frac{1+r_1i+{r_1}^2\epsilon+{r_1}^3h}{(r_1-r_2)(r_1-r_3)}\right]{r_1}^{3n+1}+\left[\frac{1+r_2i+{r_2}^2\epsilon+{r_2}^3h}{(r_2-r_1)(r_2-r_3)}\right]{r_2}^{3n+1}$$
$$+\left[\frac{1+r_3i+{r_3}^2\epsilon+{r_3}^3h}{(r_3-r_1)(r_3-r_2)}\right]{r_3}^{3n+1}=V_{3n}^{[H]}\,.$$

Other statements can be proved by similar manner.

Lemma 16. [2] Let $\propto$ is the zero of cubic equation $t^3 - t - 1 = 0$. Then the following equations hold:
(a) $\propto^4 + 1 = \propto^5$,
(b) $\propto^7 + 1 = 2 \propto^5$,
(c) $\propto^7 - 1 = 2 \propto^4$,
(d) $\propto^{14} - 1 = 4 \propto^9$.

Theorem 17. The following summation formulas hold:
(a) $\sum_{k=0}^{n} \binom{n}{k} V_{4k}^{[H]} = V_{5n}^{[H]}$,
(b) $\sum_{k=0}^{n} \binom{n}{k} 2^{n+k} V_{4k}^{[H]} = 2^n V_{7n}^{[H]}$,
(c) $\sum_{k=0}^{n} \binom{n}{k} 4^k V_{9k}^{[H]} = V_{14n}^{[H]}$,
(d) $\sum_{k=0}^{n} \binom{n}{k} V_{7k}^{[H]} = 2^n V_{5n}^{[H]}$.

Proof: We prove part (a). Using the Binet like formula of the Van Der Laan hybrid numbers, we have

$$\sum_{k=0}^{n} \binom{n}{k} V_{4k}^{[H]} = \sum_{k=0}^{n} \binom{n}{k} \left[\left(\frac{1 + r_1 i + {r_1}^2 \epsilon + {r_1}^3 h}{(r_1 - r_2)(r_1 - r_3)} \right) {r_1}^{4k+1} + \left(\frac{1 + r_2 i + {r_2}^2 \epsilon + {r_2}^3 h}{(r_2 - r_1)(r_2 - r_3)} \right) {r_2}^{4k+1} + \left(\frac{1 + r_3 i + {r_3}^2 \epsilon + {r_3}^3 h}{(r_3 - r_1)(r_3 - r_2)} \right) {r_3}^{4k+1} \right]$$

$$= \left\{ \begin{array}{l} \left[\frac{1 + r_1 i + {r_1}^2 \epsilon + {r_1}^3 h}{(r_1 - r_2)(r_1 - r_3)} \right] \sum_{k=0}^{n} \binom{n}{k} {r_1}^{4k+1} \\ + \left[\frac{1 + r_2 i + {r_2}^2 \epsilon + {r_2}^3 h}{(r_2 - r_1)(r_2 - r_3)} \right] \sum_{k=0}^{n} \binom{n}{k} {r_2}^{4k+1} \\ + \left[\frac{1 + r_3 i + {r_3}^2 \epsilon + {r_3}^3 h}{(r_3 - \alpha)(r_3 - r_2)} \right] \sum_{k=0}^{n} \binom{n}{k} {r_3}^{4k+1} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} r_1 \left[\frac{1 + r_1 i + {r_1}^2 \epsilon + {r_1}^3 h}{(r_1 - r_2)(r_1 - r_3)} \right] (1 + {r_1}^4)^n \\ + r_2 \left[\frac{1 + r_2 i + {r_2}^2 \epsilon + {r_2}^3 h}{(r_2 - r_1)(r_2 - r_3)} \right] (1 + {r_2}^4)^n \\ + + r_3 \left[\frac{1 + r_3 i + {r_3}^2 \epsilon + {r_3}^3 h}{(r_3 - r_1)(r_3 - r_2)} \right] (1 + {r_3}^4)^n \end{array} \right\}.$$

According to the Lemma 16, we find that $1 + {r_1}^4 = {r_1}^5, 1 + {r_2}^4 = {r_2}^5$ and $1 + {r_3}^4 = {r_3}^5$. By substituting these relations, we have

$$\sum_{k=0}^{n}\binom{n}{k}V_{4k}^{[H]}=\left\{\begin{array}{l}\left[\dfrac{1+r_1 i+{r_1}^2\epsilon+{r_1}^3 h}{(r_1-r_2)(r_1-r_3)}\right]{r_1}^{5n+1}\\+\left[\dfrac{1+r_2 i+{r_2}^2\epsilon+{r_2}^3 h}{(r_2-r_1)(r_2-r_3)}\right]{r_2}^{5n+1}\\+\left[\dfrac{1+r_3 i+{r_3}^2\epsilon+{r_3}^3 h}{(r_3-r_1)(r_3-r_2)}\right]{r_3}^{5n+1}\end{array}\right\}=V_{5n}^{[H]}.$$

Other statements can be proved by similar manners.

4. CONCLUSIONS

In this paper we introduced the Van Der Laan hybrinomial sequence based on the definition of hybrid numbers and Van Der Laan polynomial sequence. We proposed the Binet-like formula, partial sum, generating function, exponential generating function, character and norm of this sequence.

In addition, we investigated some summation identities about this sequence and Van Der Laan hybrid numbers. We demonstrated some summation formulas of Van Der Laan hybrid number which is the special case of Van Der Laan hybrinimial sequence. Using the definition of (1), we see that if we put $x = 1$, we obtain the Van Der Laan hybrid number $V_n^{[H]}$, which is the number system that the authors introduced in.

Because of the application of particular number sequences in matrix algebras and combinatorial theory, the subject of this paper has the potential to motivate young researchers to introduced new number sequences related to this sequence. Because of the application of the complex, hyperbolic and dual numbers in the areas of mathematics and physics, the subject of our paper is beneficial in these areas of sciences.

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