ORIGINAL PAPER

MAIN RESULTS ON FUZZY PROBLEMS

HÜLYA GÜLTEKİN ÇİTİL[1]

Manuscript received: 07.03.2024. Accepted paper: 03.10.2024;
Published online: 30.12.2024.

Abstract. *In this paper, fuzzy problems with positive and negative coefficients are investigated by fuzzy Laplace transform method under the generalized Hukuhara differentiability. Main results are found on the problems. Examples are solved to illustrate the problems. Conclusions are given.*

Keywords: *Fuzzy problem; fuzzy Laplace transform method; generalized Hukuhara differentiability.*

1. INTRODUCTION

The topic of fuzzy differential equation has been rapidly growing in recent years. The fuzzy differential equations are extensively used in applied mathematics, physics and engineering. Many researchers have been worked in theoretical and numerical solution of fuzzy differential equations [1-11].

Fuzzy Laplace transform method is practically important method for fuzzy problems. The problems are solved directly by fuzzy Laplace transform method. Thus, many researchers used fuzzy Laplace transform in many papers to solve fuzzy differential equations [12-18].

The aim of this paper is to research solutions of fuzzy problems with positive and negative coefficients by fuzzy Laplace transform method under the generalized Hukuhara differentiability.

Definition 1.1. [19] A fuzzy number is a function $u: \mathbb{R} \to [0,1]$ satisfying the following properties:
u is normal,
u is convex fuzzy set,
u is upper semi-continuous on $\mathbb{R}$
$cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ be the space of fuzzy numbers.

Definition 1.2. [19] Let $u \in \mathbb{R}_F$. The α-level set of u is

$$[u]^\alpha = \left[\underline{u}_\alpha, \overline{u}_\alpha\right] = \{x \in \mathbb{R} | u(x) \geq \alpha\}, 0 < \alpha \leq 1.$$

If $\alpha = 0$, the support of u is

[1] Department of Mathematics, Faculty of Sciences and Arts, Giresun University, Giresun, Turkey.
E-mail: hulya.citil@giresun.edu.tr.

https://doi.org/10.46939/J.Sci.Arts-24.4-a04

$$[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}.$$

Definition 1.3. [18] A fuzzy number u in parametric form is a pair $[\underline{u}_\alpha, \overline{u}_\alpha]$ of functions $\underline{u}_\alpha, \overline{u}_\alpha$, $0 \le \alpha \le 1$, which satisfty the following requirements:
1. $\underline{u}_\alpha$ is bounded non-decreasing left-continuous in $(0,1]$, right-continuous at $\alpha = 0$.
2. $\overline{u}_\alpha$ is bounded non-increasing left-continuous in $(0,1]$, right-continuous at $\alpha = 0$.
3. $\underline{u}_\alpha \le \overline{u}_\alpha$, $0 \le \alpha \le 1$.

Definition 1.4. [19] If A is a symmetric triangular number with support $[\underline{a}, \overline{a}]$, the α-level set of A is

$$[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right].$$

Definition 1.5. [11] Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that $u = v + w$ then w is called the H-difference of u and v and it is denoted $u \ominus v$.

Definition 1.6. [11] Let $f: [a, b] \longrightarrow \mathbb{R}_F$ and $t_0 \in [a, b]$. If there exists $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits hold

$$\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0),$$

f is (1)-differentiable at t_0. If there exists $f'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, $\exists f(t_0) \ominus f(t_0 + h), f(t_0 - h) \ominus f(t_0)$ and the limits hold

$$\lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 + h)}{-h} = \lim_{h \to 0^+} \frac{f(t_0 - h) \ominus f(t_0)}{-h} = f'(t_0),$$

f is (2)-differentiable.

Definition 1.7. [18] The fuzzy Laplace transform of fuzzy function f is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt = \left[\lim_{\rho \to \infty} \int_0^\rho e^{-st} \underline{f}(t)dt \, , \lim_{\rho \to \infty} \int_0^\rho e^{-st} \overline{f}(t)dt\right],$$

$$F(s, \alpha) = L([f(t)]^\alpha) = \left[L\left(\underline{f}_\alpha(t)\right), L\left(\overline{f}_\alpha(t)\right)\right],$$

$$L\left(\underline{f}_\alpha(t)\right) = \int_0^\infty e^{-st} \underline{f}_\alpha(t)dt = \lim_{\rho \to \infty} \int_0^\rho e^{-st} \underline{f}_\alpha(t)dt,$$

$$L\left(\overline{f}_\alpha(t)\right) = \int_0^\infty e^{-st} \overline{f}_\alpha(t)dt = \lim_{\rho \to \infty} \int_0^\rho e^{-st} \overline{f}_\alpha(t)dt.$$

Theorem 1.1. [12] Let $f'(t)$ be an integrable fuzzy function and $f(t)$ is primitive of $f'(t)$ on $(0, \infty]$.
1. If f is (1)-differentiable, $L(f'(t)) = sL(f(t)) \ominus f(0)$.
2. If f is (2)-differentiable, $L(f'(t)) = (-f(0)) \ominus \left(-sL(f(t))\right)$.

2. MAIN RESULTS

We consider solutions of fuzzy problems

$$y'(x) = [\lambda]^{\alpha} y(x), x > 0 \tag{1}$$

$$\mathrm{y}(0) = [\beta]^{\alpha}, \tag{2}$$

and

$$y'(x) = -[\lambda]^{\alpha} y(x), x > 0 \tag{3}$$

$$\mathrm{y}(0) = [\beta]^{\alpha}, \tag{4}$$

by the fuzzy Laplace transform method and the generalized Hukuhara differentiability, where $[\lambda]^{\alpha} = \left[\underline{\lambda}_{\alpha}, \overline{\lambda}_{\alpha}\right], (\underline{\lambda}_{\alpha} > 0, \overline{\lambda}_{\alpha} > 0)$, $[\beta]^{\alpha} = \left[\underline{\beta}_{\alpha}, \overline{\beta}_{\alpha}\right]$ are symmetric triangular fuzzy numbers and $y(x)$ is positive fuzzy function. In this work, (i)-solution means that y is (i)-differentiable, i=1,2.

I) The problem (1)-(2):
1) Let y be (1)-differentiable. Then, from the differential equation (1), we obtain the equation

$$sL\big(y(x)\big) \ominus y(0) = [\lambda]^{\alpha} L\big(y(x)\big)$$

by the fuzzy Laplace transform method. From this, we have the equations

$$sL\left(\underline{y}_{\alpha}(x)\right) - \underline{y}_{\alpha}(0) = \underline{\lambda}_{\alpha} L\left(\underline{y}_{\alpha}(x)\right),$$

$$sL\left(\overline{y}_{\alpha}(x)\right) - \overline{y}_{\alpha}(0) = \overline{\lambda}_{\alpha} L\left(\overline{y}_{\alpha}(x)\right).$$

Using the initial condition (2),

$$L\left(\underline{y}_{\alpha}(x)\right) = \frac{\underline{\beta}_{\alpha}}{s - \underline{\lambda}_{\alpha}},$$

$$L\left(\overline{y}_{\alpha}(x)\right) = \frac{\overline{\beta}_{\alpha}}{s - \overline{\lambda}_{\alpha}}$$

are obtained. From this, (1)-solution is obtained as

$$\underline{y}_{\alpha}(x) = \underline{\beta}_{\alpha} e^{\underline{\lambda}_{\alpha} x},$$

$$\overline{y}_{\alpha}(x) = \overline{\beta}_{\alpha} e^{\overline{\lambda}_{\alpha} x},$$

$$[y(x)]^{\alpha} = \left[\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right].$$

2) Let y be (2)-differentiable. Then, from the fuzzy differential equation (1)

$$-y(0) \ominus \left(-sL\big(y(x)\big)\right) = [\lambda]^{\alpha} L\big(y(x)\big)$$

is obtained. That is, the equations

$$-\overline{y}_{\alpha}(0) + sL\left(\overline{y}_{\alpha}(x)\right) = \underline{\lambda}_{\alpha} L\left(\underline{y}_{\alpha}(x)\right),$$

$$-\underline{y}_{\alpha}(0) + sL\left(\underline{y}_{\alpha}(x)\right) = \overline{\lambda}_{\alpha} L\left(\overline{y}_{\alpha}(x)\right)$$

are obtained. Using the initial value (2) and taking the necessary operations, we have

$$L\left(\underline{y}_{\alpha}(x)\right) = \frac{\overline{\lambda}_{\alpha}\overline{\beta}_{\alpha}}{s^2 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}} + \frac{s\underline{\beta}_{\alpha}}{s^2 - \underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}}$$

Then, the lower solution is obtained as

$$\underline{y}_{\alpha}(x) = \frac{\overline{\lambda}_{\alpha}\overline{\beta}_{\alpha}}{\sqrt{\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}}} sinh\left(\sqrt{\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}}x\right) + \underline{\beta}_{\alpha} cosh\left(\sqrt{\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}}x\right). \quad (5)$$

Similarly, the upper solution is obtained as

$$\overline{y}_{\alpha}(x) = \frac{\underline{\lambda}_{\alpha}\underline{\beta}_{\alpha}}{\sqrt{\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}}} sinh\left(\sqrt{\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}}x\right) + \overline{\beta}_{\alpha} cosh\left(\sqrt{\underline{\lambda}_{\alpha}\overline{\lambda}_{\alpha}}x\right). \quad (6)$$

That is, (2)-solution is $[y(x)]^{\alpha} = \left[\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right]$, where $\underline{y}_{\alpha}(x)$ is (5) and $\overline{y}_{\alpha}(x)$ is (6).

Example 1. Consider the problem

$$y'(x) = [1]^{\alpha} y(x), x > 0 \quad (7)$$

$$\mathrm{y}(0) = [2]^{\alpha}, \quad (8)$$

by the fuzzy Laplace transform method, where $[1]^{\alpha} = [\alpha, 2-\alpha], [2]^{\alpha} = [1+\alpha, 3-\alpha]$.
(1)-solution of the problem is

$$\underline{y}_{\alpha}(x) = (1+\alpha)e^{\alpha x}, \quad (9)$$

$$\overline{y}_{\alpha}(x) = (3-\alpha)e^{(2-\alpha)x}, \quad (10)$$

$$[y(x)]^{\alpha} = \left[\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right], \quad (11)$$

and (2)-solution of the problem is

$$\underline{y}_\alpha(x) = \frac{(2-\alpha)(3-\alpha)}{\sqrt{\alpha(2-\alpha)}} sinh\left(\sqrt{\alpha(2-\alpha)}x\right) + (1+\alpha)cosh\left(\sqrt{\alpha(2-\alpha)}x\right), \quad (12)$$

$$\overline{y}_\alpha(x) = \frac{\alpha(1+\alpha)}{\sqrt{\alpha(2-\alpha)}} sinh\left(\sqrt{\alpha(2-\alpha)}x\right) + (3-\alpha)cosh\left(\sqrt{\alpha(2-\alpha)}x\right), \quad (13)$$

$$[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right]. \quad (14)$$

From Definition 1.3, according to Fig. 1, (1)-solution is a valid fuzzy function, according to Fig. 2, (2)-solution is a valid fuzzy function for $x\epsilon[0.34308503003499247]$.

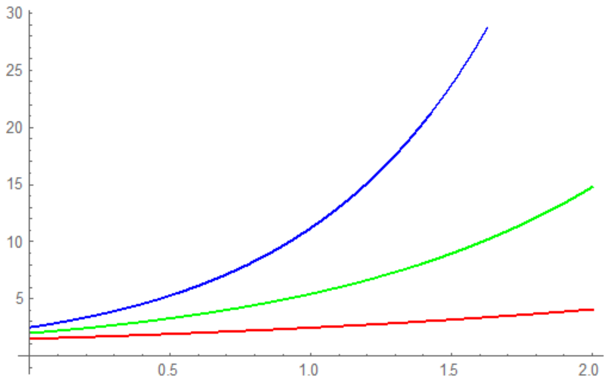

Figure 1. Graphic of (9)-(11) for $\alpha = 0.5$

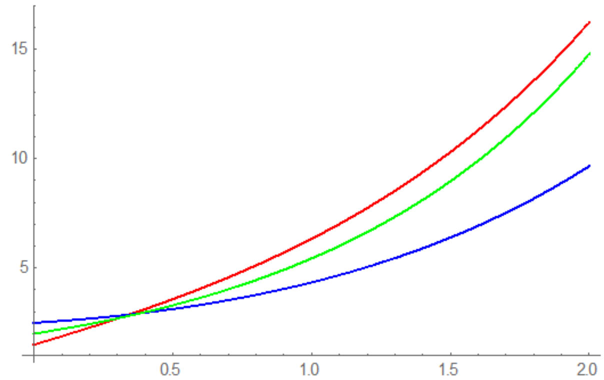

Figure 2. Graphic of (12)-(14) for $\alpha = 0.5$

Legend: red $\rightarrow \underline{y}_\alpha(x)$; ***blue*** $\rightarrow \overline{y}_\alpha(x)$; ***green*** $\rightarrow y(x)$.

II) The problem (3)-(4):

1) Let y be (1)-differentiable. Then, the fuzzy differential equation (3) yields the equation

$$sL\big(y(x)\big) \ominus y(0) = -[\lambda]^\alpha L\big(y(x)\big).$$

From this, the equations

$$sL\left(\underline{y}_\alpha(x)\right) - \underline{y}_\alpha(0) = -\overline{\lambda}_\alpha L\left(\overline{y}_\alpha(x)\right),$$

$$sL\left(\overline{y}_\alpha(x)\right) - s\overline{y}_\alpha(0) = -\underline{\lambda}_\alpha L\left(\underline{y}_\alpha(x)\right)$$

are obtained. Thus, using the initial value (4) and taking the necessary operations, the equation

$$L\left(\underline{y}_\alpha(x)\right) = -\frac{\overline{\lambda}_\alpha \overline{\beta}_\alpha}{s^2 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha} + \frac{s\underline{\beta}_\alpha}{s^2 - \underline{\lambda}_\alpha \overline{\lambda}_\alpha}$$

is obtained. From this, the lower solution is

$$\underline{y}_\alpha(x) = -\frac{\overline{\lambda}_\alpha \overline{\beta}_\alpha}{\sqrt{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}} sinh\left(\sqrt{\underline{\lambda}_\alpha \overline{\lambda}_\alpha} x\right) + \underline{\beta}_\alpha cosh\left(\sqrt{\underline{\lambda}_\alpha \overline{\lambda}_\alpha} x\right). \quad (15)$$

Similarly, the upper solution is

$$\overline{y}_\alpha(x) = -\frac{\underline{\lambda}_\alpha \underline{\beta}_\alpha}{\sqrt{\underline{\lambda}_\alpha \overline{\lambda}_\alpha}} sinh\left(\sqrt{\underline{\lambda}_\alpha \overline{\lambda}_\alpha} x\right) + \overline{\beta}_\alpha cosh\left(\sqrt{\underline{\lambda}_\alpha \overline{\lambda}_\alpha} x\right). \quad (16)$$

(2)-solution is $[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right]$, where $\underline{y}_\alpha(x)$ is (15) and $\overline{y}_\alpha(x)$ is (16).

2) Let y be (2)-differentiable. From the fuzzy differential equation (3), we have the equation

$$-y(0) \ominus \left(-sL\big(y(x)\big)\right) = -[\lambda]^\alpha L\big(y(x)\big).$$

That is, we have the equations

$$-\overline{y}_\alpha(0) + sL\left(\overline{y}_\alpha(x)\right) = -\overline{\lambda}_\alpha L\left(\overline{y}_\alpha(x)\right),$$

$$-\underline{y}_\alpha(0) + sL\left(\underline{y}_\alpha(x)\right) = -\underline{\lambda}_\alpha L\left(\underline{y}_\alpha(x)\right)$$

are obtained. From this, using the initial value (4), (2)-solution is obtained as

$$\underline{y}_\alpha(x) = \underline{\beta}_\alpha e^{-\underline{\lambda}_\alpha x},$$

$$\overline{y}_\alpha(x) = \overline{\beta}_\alpha e^{-\overline{\lambda}_\alpha x}$$

$$[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \overline{y}_\alpha(x)\right].$$

Example 2. Consider the problem

$$y'(x) = -[1]^\alpha y(x), x > 0 \quad (17)$$

$$\mathrm{y}(0) = [2]^\alpha, \quad (18)$$

by the fuzzy Laplace transform method, where $[1]^\alpha = [\alpha, 2-\alpha], [2]^\alpha = [1+\alpha, 3-\alpha]$.

(1)-solution of the problem is

$$\underline{y}_\alpha(x) = -\frac{(2-\alpha)(3-\alpha)}{\sqrt{\alpha(2-\alpha)}} sinh\left(\sqrt{\alpha(2-\alpha)}x\right) + (1+\alpha) cosh\left(\sqrt{\alpha(2-\alpha)}x\right), \quad (19)$$

$$\overline{y}_\alpha(x) = -\frac{\alpha(1+\alpha)}{\sqrt{\alpha(2-\alpha)}} sinh\left(\sqrt{\alpha(2-\alpha)}x\right) + (3-\alpha) cosh\left(\sqrt{\alpha(2-\alpha)}x\right), \quad (20)$$

$$[y(x)]^{\alpha} = \left[\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right]. \tag{21}$$

(2)-solution of the problem is

$$\underline{y}_{\alpha}(x) = (1+\alpha)e^{-\alpha x}, \tag{22}$$

$$\overline{y}_{\alpha}(x) = (3-\alpha)e^{-(2-\alpha)x}, \tag{23}$$

$$[y(x)]^{\alpha} = \left[\underline{y}_{\alpha}(x), \overline{y}_{\alpha}(x)\right]. \tag{24}$$

From Definition 1.3 and since $[y(x)]^{\alpha}$ positive fuzzy function, according to Fig. 3, (1)-solution is a valid fuzzy function for $x\epsilon[0.41726096626595527]$, according to Fig.4, (2)-solution is a valid fuzzy function $x\epsilon[0.5108256237659908]$.

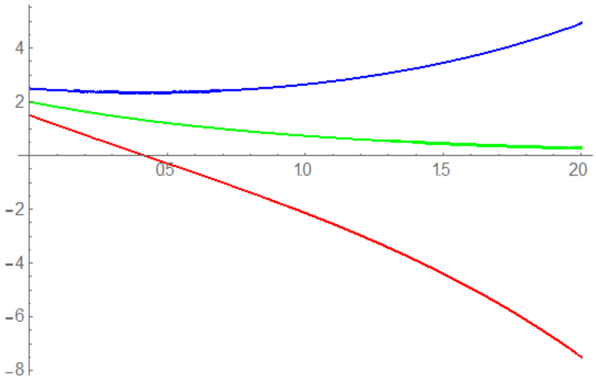

Figure 3. Graphic of (19)-(21) for $\alpha = 0.5$

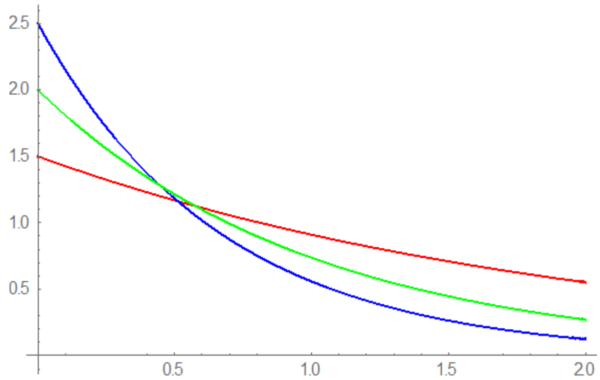

Figure 4. Graphic of (22)-(24) for $\alpha = 0.5$

Legend: red $\rightarrow \underline{y}_{\alpha}(x)$; ***blue*** $\rightarrow \overline{y}_{\alpha}(x)$; ***green*** $\rightarrow y(x)$.

3. CONCLUSION

In this work, we investigate fuzzy problems for fuzzy differential equations with positive and negative coefficients by fuzzy Laplace transform method under the generalized Hukuhara differentiability. We find main results for the solutions and solve examples. We show that solutions are valid fuzzy functions in different intervals for each α −cut.

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