ORIGINAL PAPER

A NEW LOGARITHMIC TYPE ESTIMATORS FOR ANALYSIS OF NUMBER OF AFTERSHOCKS USING POISSON DISTRIBUTION

TOLGA ZAMAN[1], SOOFIA IFTIKHAR[2], CAGLAR SOZEN[3], PRAYAS SHARMA[4]

Manuscript received: 15.04.2024; Accepted paper: 10.11.2024;
Published online: 30.12.2024.

Abstract. *Earthquake is a fierce and unavoidable natural disaster. The study of aftershocks following an earthquake provides a comprehensive understanding of seismic activity, making it an important area of research. Although it is important to mention that the use of the Poisson distributed population, to estimate the mass mean of rare events such as earthquakes, has been little studied in the literature. In this context, this study proposes a new class of logarithmic type ratio estimators for estimating the mean of a Poisson distributed population in a simple random sampling without replacement. The proposed estimators are obtained using the logarithmic transformation of the ratio estimator, and the expressions for the mean square error (MSE) are also derived to the first order of approximation. The study demonstrates that the proposed logarithmic type estimators are more efficient than the existing estimators, both theoretically and empirically. The Empirical evidence from a real data study conducted using earthquake data from Turkey confirms the superiority of the proposed estimators over the existing estimators. The study provides valuable insights for researchers and practitioners working with Poisson distributed populations and requiring ratio estimators using simple random sampling without replacement.*

Keywords: *Earthquake; logarithmic type estimator; Poisson distribution; mean square error; simple random sampling.*

1. INTRODUCTION

After every major earthquake there is a series of aftershocks. If aftershock data is carefully observed they give us a clear picture of the whole seismic activity, therefore, the study of aftershocks must be given due importance. The study of aftershocks received great attention in recent years. In research the use of auxiliary variable along with study variable highly increase the efficiency of an estimator. For estimation of different parameters, the researchers have utilized different forms of auxiliary variable without the consideration of its distribution. Unlike many distributions Poisson distribution is used to study rare events. Among that rare events earthquake is one of them. Ratio estimator of mean can give us an idea of the number of aftershocks in a region. In recent years many such studies have been conducted and few ratio, product and exponential type estimators have been developed. Ozel [1] used bivariate Poisson distribution for the development of ratio mean estimators. Some

[1] *Gumushane University, Graduate Education Institute, Department of Mathematics Enginnering*, Gumushane, Turkey. E-mail: zamantolga@gmail.com.
[2] Shaheed Benazir Bhutto Women University Peshawar, Department of Statistics, 00384 Khyber Pakhtunkhwa, Pakistan. Email: soofia.iftikhar@sbbwu.edu.pk.
[3] Giresun University, Department of Finance and Banking, Giresun, Turkey. E-mail: caglar.sozen@giresun.edu.tr.
[4] Indian Institute of Management Sirmaur, Department of Decision Sciences, Paonta Sahib H.P., India. E-mail: prayassharma02@gmail.com.

https://doi.org/10.46939/J.Sci.Arts-24.4-a06

exponential type estimators that possess the properties of Poisson distribution have been developed by Koyuncu and Ozel [2]. A bivariate compound model for the study of earthquakes is developed by Ozel [3]. For the study of earthquakes, a multivariate nonparametric hazard model is developed by Ata and Ozel [4]. A proportional Hazard Model is developed by Ata and Ozel [5] using accelerated failure time model for the destructive earthquake analysis in Turkey. Some generalized efficient estimators have been developed by Sharma et al. [6] utilizing the properties of Poisson distribution. Generalized linear models are utilized by Noora [7] for the development of an estimate for the chance of occurrence of an earthquake and its return period. Overall, incorporating Poisson distribution and auxiliary variables has been proven to improve the accuracy of earthquake estimators.

The main objective of this research is to design a high-performance estimator for estimating the population means of rare events such as earthquakes and air accidents, utilizing Poisson distributed populations. To achieve this goal, we have developed a novel class of estimators and analyzed their effectiveness in estimating the population mean.

Consider a population $U=(u_1,u_2,u_3.....u_n)$ of size N identifiable and distinct units. N is population size. Let y and x be the study and auxiliary variables associated with each unit $u_j = (j = 1,2,...,N)$ of the population respectively. Assume that X's are known units and Y's are unknown units for all the population. Suppose a random sample of size n is drawn using simple random sampling without replacement (SRSWOR) from the population. Let us assume that the parent population has a Poisson distribution. We know that the nature of the sampling distribution depends on the nature of the population from which the random sample is drawn therefore, random samples which are drawn from a Poisson distributed population follows also a Poisson distribution. Further let us select observations y_i, x_i , $(i = 1,2,...,n)$ from a Poisson distributed population. Using this sampling design we can define classical ratio estimator as

$$e_0 = \frac{(\bar{y}_{po} - \bar{Y})}{\bar{Y}},\ e_1 = \frac{(\bar{x}_{po} - \bar{X})}{\bar{X}},\ \text{where } E(e_0) = E(e_1) = 0$$

$$E(e_0^2) = \frac{var(\bar{y}_{po})}{\bar{Y}^2} = \frac{\lambda_2}{n} = \frac{(\gamma_2+\gamma_3)}{n},$$

$$E(e_1^2) = \frac{var(\bar{x}_{po})}{\bar{X}^2} = \frac{\lambda_1}{n} = \frac{(\gamma_1+\gamma_3)}{n},$$

$$E(e_0 e_1) = \frac{COV(\bar{x}_{po},\bar{y}_{po})}{\bar{X}\bar{Y}} = \frac{(\gamma_2)}{n}.$$

The correlation coefficient between $\bar{x}_{po}$ and $\bar{y}_{po}$ is given by

$$\rho_{(\bar{x}_{po},\bar{y}_{po})} = \frac{COV(\bar{x}_{po},\bar{y}_{po})}{\sqrt{(\gamma_2+\gamma_3)(\gamma_1+\gamma_3)}} = \frac{\gamma_3}{\sqrt{(\gamma_2+\gamma_3)(\gamma_1+\gamma_3)}}.$$

$\bar{x}_{po} = \sum_{i=1}^{n} x_i/n$ and $\bar{y}_{po} = \sum_{i=1}^{n} y_i/n$ are the sample means of the auxiliary and study variables from Poisson distributed population, respectively.

The remaining part of the paper is organized as follows. Section 2 ensures a description of the existing estimators. The structure of the proposed estimators is given in Section 3. The efficiency comparisons of the proposed estimators with the existing estimators are presented in Section 4. Sections 5 consists of empirical study of proposed estimators. Finally, Section 6 summarizes the findings of this study.

2. LITERATURE REVIEW OF EXISTING ESTIMATORS

In this section, some estimators in the literature and their properties are examined.
1. The transfored simple ratio estimator is as follows.

$$R_1 = \bar{y}_{po} \frac{\bar{X}}{\bar{x}_{po}} \quad (1)$$

its Bias and MSE are as follows

$$\text{Bias}\ (R_1) = \bar{Y}\, \frac{\gamma_1}{n} \quad (2)$$

$$\text{MSE}(R_1) = \bar{Y}^2\ \frac{(\gamma_1+\gamma_2)}{n} \quad (3)$$

2. The exponential estimators suggested by Koyuncu and Ozel [2] are as follows.

$$R_2 = \bar{y}_{po}\ exp\ \left(\frac{\bar{X}-\bar{x}_{po}}{\bar{X}+\bar{x}_{po}}\right) \quad (4)$$

$$R_3 = \bar{y}_{po}\ exp\ \left(\frac{\bar{x}_{po}-\bar{X}}{\bar{x}_{po}+\bar{X}}\right) \quad (5)$$

The Bias and MSE expressions are as follows

$$\text{Bias}\ (R_2) = \bar{Y}\ \frac{(3\gamma_1-\gamma_3)}{8n} \quad (6)$$

$$\text{Bias}\ (R_3) = \bar{Y}\ \frac{(\gamma_1+5\gamma_3)}{8n} \quad (7)$$

$$\text{MSE}(R_2) = \bar{Y}^2\ \frac{(\gamma_1+4\gamma_2+\gamma_3)}{4n} \quad (8)$$

$$\text{MSE}(R_3) = \bar{Y}^2\ \frac{(\gamma_1+4\gamma_2+9\gamma_3)}{4n} \quad (9)$$

3. PROPOSED CLASS OF ESTIMATOR

Motivated by Izunobi and Onyeka [8], we propose a class of logarithmic type estimators as follows.

$$R_{pi} = \bar{y}_{po}\ \left(\frac{\ln(a\bar{X}+b)}{\ln(a\bar{x}_{po}+b)}\right) \text{ where i=1, 2, 3, 4, 5, 6} \quad (10)$$

where a and b are any constants, their values can be C_x, β_{2x}, ρ. C_x is the population coefficient of variation of the auxiliary variable, $\beta_{2x}$is the population coefficient of kurtosis of the auxiliary variable, and ρ is is the population coefficient of correlation between the auxiliary variable and study variable. Proposed logarithmic type estimators are given in Table 1.

Table 1. Proposed logarithmic type estimators.

Estimator	A	b
$R_{p1} = \bar{y}_{po}\left(\frac{\ln(\bar{X}+\beta 2x)}{\ln(\bar{x}_{po}+\beta 2x)}\right)$	1	β_{2x}
$R_{p2} = \bar{y}_{po}\left(\frac{\ln(\bar{X}+\rho)}{\ln(\bar{x}_{po}+\rho)}\right)$	1	ρ
$R_{p3} = \bar{y}_{po}\left(\frac{\ln(Cx\,\bar{X}+\beta 2x)}{\ln(Cx\,\bar{x}_{po}+\beta 2x)}\right)$	C_x	β_{2x}
$R_{p4} = \bar{y}_{po}\left(\frac{\ln(\beta 2x\,\bar{X}+Cx)}{\ln(\beta 2x\,\bar{x}_{po}+Cx)}\right)$	β_{2x}	C_x
$R_{p5} = \bar{y}_{po}\left(\frac{\ln(\beta 2x\,\bar{X}+\rho)}{\ln(\beta 2x\,\bar{x}_{po}+\rho)}\right)$	β_{2x}	ρ
$R_{p6} = \bar{y}_{po}\left(\frac{\ln(\rho\,\bar{X}+\beta 2x)}{\ln(\rho\,\bar{x}_{po}+\beta 2x)}\right)$	ρ	β_{2x}

Expressing R_{pi} in terms of e's as follows

$$R_{pi} = \bar{Y}\,(1+e_0)\left(\frac{\ln(a\bar{X}+b)}{\ln(a\,\bar{X}(1+e_1)+b)}\right) \tag{11}$$

$$R_{pi\text{i}} = \bar{Y}\,(1+e_0)\left(\frac{\ln(a\bar{X}+b)}{\ln[a\,\bar{X}+\,a\bar{X}e_1+b]}\right) \tag{12}$$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(\frac{\ln(a\bar{X}+b)}{\ln[a\,\bar{X}+b+\,a\bar{X}e_1)]}\right) \tag{13}$$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(\frac{\ln(a\bar{X}+b)}{\ln[a\,\bar{X}+b+\,a\bar{X}e_1)]}\right) \tag{14}$$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(\frac{\ln(a\bar{X}+b)}{\ln[(a\,\bar{X}+b)\{1+\frac{a\bar{X}e_1}{a\,\bar{X}+b}\}]}\right) \tag{15}$$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(\frac{\ln(a\bar{X}+b)}{\ln(a\,\bar{X}+b)+\ln(1+\frac{a\bar{X}e_1}{a\,\bar{X}+b})}\right) \tag{16}$$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(\frac{\ln(a\bar{X}+b)}{\ln(a\,\bar{X}+b)\{1+\frac{\ln(1+\frac{a\bar{X}e_1}{a\,\bar{X}+b})}{\ln(a\,\bar{X}+b)}}\right) \tag{17}$$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(\frac{1}{1+\frac{\ln\left(1+\frac{a\bar{X}e_1}{a\,\bar{X}+b}\right)}{\ln(a\,\bar{X}+b)}}\right) \tag{18}$$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(1+\frac{\ln\left(1+\frac{a\bar{X}e_1}{a\,\bar{X}+b}\right)}{\ln(a\,\bar{X}+b)}\right)^{-1} \tag{19}$$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(1-\frac{\ln\left(1+\frac{a\bar{X}e_1}{a\,\bar{X}+b}\right)}{\ln(a\,\bar{X}+b)}\right) \tag{20}$$

Let $A = \frac{a\bar{X}}{a\,\bar{X}+b}$ and $B\ = \ln(a\,\bar{X}+b)$

$$R_{pi} = \bar{Y}\,(1+e_0)\left(1-\frac{\ln(1+A\,e_1)}{\text{B}}\right) \tag{21}$$

$$R_{pi} = \bar{Y}\,(1 + e_0)\,\left(1 - \frac{A\,e_1}{\mathrm{B}} + \frac{A^2 {e_1}^2}{2B}\right) \tag{22}$$

$$R_{pi} = \bar{Y}\,\left(1 + e_0 - \frac{A\,e_1}{\mathrm{B}} - \frac{A\,e_0 e_1}{\mathrm{B}} + \frac{A^2 {e_1}^2}{2B}\right) \tag{23}$$

$$R_{pi} - \bar{Y} = \bar{Y}\,\left(e_0 - \frac{A\,e_1}{\mathrm{B}} - \frac{A\,e_0 e_1}{\mathrm{B}} + \frac{A^2 {e_1}^2}{2B}\right) \tag{24}$$

Taking expectation on both sides we get

$$E(R_{pi} - \bar{Y}) = \bar{Y}\,E\left(e_0 - \frac{A\,e_1}{\mathrm{B}} - \frac{A\,e_0 e_1}{\mathrm{B}} + \frac{A^2 {e_1}^2}{2B}\right) \tag{25}$$

$$\text{Bias}\,(R_{pi}) = \bar{Y}\,\left(\frac{A^2}{2B}\frac{(\gamma_1+\gamma_3)}{n} - \frac{A}{\mathrm{B}}\frac{(\gamma_2)}{n}\right) \tag{26}$$

Squaring both sides of Eq. (24) and then applying expectation we get

$$\mathrm{E}\,(R_{pi} - \bar{Y})^2 = \bar{Y}^{\,2} E\left({e_0}^2 + \frac{A^2 {e_1}^2}{B^2} - \frac{2\,A\,e_0 e_1}{\mathrm{B}}\right) \tag{27}$$

$$\mathrm{MSE}(R_{pi}) = \bar{Y}^{\,2}\,\left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,A}{\mathrm{B}}\frac{(\gamma_2)}{n}\right) \tag{28}$$

It is the general expression the MSE expressions for the proposed the logarithmic type estimators are given Table 2.

Table 2. The expression of MSE of proposed logarithmic type estimators.

Estimator	MSE Expression
$R_{p1} = \bar{y}_{po}\left(\frac{\ln(\bar{X}+\beta_{2x})}{\ln(\bar{x}_{po}+\beta_{2x})}\right)$	$\bar{Y}^{\,2}\left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{(\frac{\bar{X}}{\bar{X}+\beta_{2x}})^2}{(\ln(\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\bar{X}}{\bar{X}+\beta_{2x}})}{(\ln(\bar{X}+\beta_{2x}))}\frac{(\gamma_2)}{n}\right)$
$R_{p2} = \bar{y}_{po}\left(\frac{\ln(\bar{X}+\rho)}{\ln(\bar{x}_{po}+\rho)}\right)$	$\bar{Y}^{\,2}\left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{(\frac{\bar{X}}{\bar{X}+\rho})^2}{(\ln(\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\bar{X}}{\bar{X}+\rho})}{(\ln(\bar{X}+\rho))}\frac{(\gamma_2)}{n}\right)$
$R_{p3} = \bar{y}_{po}\left(\frac{\ln(\mathrm{Cx}\,\bar{X}+\beta 2x)}{\ln(\mathrm{Cx}\,\bar{x}_{po}+\beta 2x)}\right)$	$\bar{Y}^{\,2}\left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{(\frac{\mathrm{Cx}\bar{X}}{\mathrm{Cx}\,\bar{X}+\beta_{2x}})^2}{(\ln(\mathrm{Cx}\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\mathrm{Cx}\,\bar{X}}{\mathrm{Cx}\,\bar{X}+\beta_{2x}})}{(\ln(\mathrm{Cx}\,\bar{X}+\beta_{2x}))}\frac{(\gamma_2)}{n}\right)$
$R_{p4} = \bar{y}_{po}\left(\frac{\ln(\beta 2x\,\bar{X}+\mathrm{Cx})}{\ln(\beta 2x\,\bar{x}_{po}+\mathrm{Cx})}\right)$	$\bar{Y}^{\,2}\left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\mathrm{Cx}})^2}{(\ln(\beta_{2x}\,\bar{X}+\mathrm{Cx}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\beta_{2x}\,\bar{X}}{\beta_{2x}\,\bar{X}+\mathrm{Cx}})}{(\ln(\beta_{2x}\,\bar{X}+\mathrm{Cx}))}\frac{(\gamma_2)}{n}\right)$
$R_{p5} = \bar{y}_{po}\left(\frac{\ln(\beta 2x\,\bar{X}+\rho)}{\ln(\beta 2x\,\bar{x}_{po}+\rho)}\right)$	$\bar{Y}^{\,2}\left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})^2}{(\ln(\beta_{2x}\,\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\beta_{2x}\,\bar{X}}{\beta_{2x}\,\bar{X}+\rho})}{(\ln(\beta_{2x}\,\bar{X}+\rho))}\frac{(\gamma_2)}{n}\right)$
$R_{p6} = \bar{y}_{po}\left(\frac{\ln(\rho\,\bar{X}+\beta 2x)}{\ln(\rho\,\bar{x}_{po}+\beta 2x)}\right)$	$\bar{Y}^{\,2}\left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})^2}{(\ln(\rho\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})}{(\ln(\rho\,\bar{X}+\beta_{2x}))}\frac{(\gamma_2)}{n}\right)$

4. COMPARISONS OF EFFICIENCIES

A comparison of the proposed logarithmic type estimators has been made with the competing estimators. The efficiency conditions have also been mentioned.
i) From (1) and (10), we have

$$\text{MSE}(R_{pi}) < \text{MSE}(R_1) \tag{29}$$

$$\bar{Y}^2 \left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2A}{B}\frac{(\gamma_2)}{n}\right) < \bar{Y}^2 \frac{(\gamma_1+\gamma_2)}{n} \tag{30}$$

$$\bar{Y}^2 \left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2A}{B}\frac{(\gamma_2)}{n}\right) - \bar{Y}^2 \frac{(\gamma_1+\gamma_2)}{n} < 0 \tag{31}$$

$$\frac{(\gamma_2+\gamma_3)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2A}{B}\frac{(\gamma_2)}{n} - \frac{(\gamma_1+\gamma_2)}{n} < 0 \tag{32}$$

On Simplification we get

$$\frac{(\gamma_3-\gamma_1)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2A}{B}\frac{(\gamma_2)}{n} < 0 \tag{33}$$

So MSE(R_{pi}) is efficient as compared to MSE(R_1) iff

$$\frac{(\gamma_3-\gamma_1)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2A}{B}\frac{(\gamma_2)}{n} < 0 \tag{34}$$

Putting different values of A and B in R_{pi} , where i=1,2,3,…,10 we get

$$\text{MSE}(R_{p1}) < \text{MSE}(R_1) \text{ iff } \frac{(\gamma_3-\gamma_1)}{n} + \frac{(\frac{\bar{X}}{\bar{X}+\beta_{2x}})^2}{(\ln(\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\bar{X}}{\bar{X}+\beta_{2x}})}{\ln(\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n} < 0 \tag{35}$$

$$\text{MSE}(R_{p2}) < \text{MSE}(R_1) \text{ iff } \frac{(\gamma_3-\gamma_1)}{n} + \frac{(\frac{\bar{X}}{\bar{X}+\rho})^2}{(\ln(\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\bar{X}}{\bar{X}+\rho})}{\ln(\bar{X}+\rho)}\frac{(\gamma_2)}{n} < 0 \tag{36}$$

$$\text{MSE}(R_{p3}) < \text{MSE}(R_1) \text{ iff } \frac{(\gamma_3-\gamma_1)}{n} + \frac{(\frac{\text{Cx}\bar{X}}{\text{Cx}\,\bar{X}+\beta_{2x}})^2}{(\ln(\text{Cx}\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\text{Cx}\bar{X}}{\text{Cx}\,\bar{X}+\beta_{2x}})}{\ln(\text{Cx}\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n} < 0 \tag{37}$$

$$\text{MSE}(R_{p4}) < \text{MSE}(R_1) \text{ iff } \frac{(\gamma_3-\gamma_1)}{n} + \frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\text{Cx}})^2}{(\ln(\beta_{2x}\,\bar{X}+\text{Cx}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\text{Cx}})}{\ln(\beta_{2x}\,\bar{X}+\text{Cx})}\frac{(\gamma_2)}{n} < 0 \tag{38}$$

$$\text{MSE}(R_{p5}) < \text{MSE}(R_1) \text{ iff } \frac{(\gamma_3-\gamma_1)}{n} + \frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})^2}{(\ln(\beta_{2x}\,\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})}{\ln(\beta_{2x}\,\bar{X}+\rho)}\frac{(\gamma_2)}{n} < 0 \tag{39}$$

$$\text{MSE}(R_{p6}) < \text{MSE}(R_1) \text{ iff } \frac{(\gamma_3-\gamma_1)}{n} + \frac{(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})^2}{(\ln(\rho\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\,(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})}{\ln(\rho\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n} < 0 \tag{40}$$

ii) From (4) and (10), we have

$$\text{MSE}(R_{pi}) < \text{MSE}(R_2) \text{ iff} \tag{41}$$

$$\bar{Y}^2 \left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2A}{B}\frac{(\gamma_2)}{n}\right) < \bar{Y}^2 \frac{(\gamma_1+4\gamma_2+\gamma_3)}{4n} \tag{42}$$

$$\bar{Y}^2 \left(\frac{(\gamma_2+\gamma_3)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2A}{B}\frac{(\gamma_2)}{n}\right) - \bar{Y}^2 \frac{(\gamma_1+4\gamma_2+\gamma_3)}{4n} < 0 \tag{43}$$

$$\frac{(\gamma_2+\gamma_3)}{n} + \frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2A}{B}\frac{(\gamma_2)}{n} - \frac{(\gamma_1+4\gamma_2+\gamma_3)}{4n} < 0 \tag{44}$$

On simplification we get

$$\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,A}{\mathrm{B}}\frac{(\gamma_2)}{n}<0 \tag{45}$$

So MSE(R_{pi}) is efficient as compared to MSE(R_2) iff

$$\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,A}{\mathrm{B}}\frac{(\gamma_2)}{n}<0 \tag{46}$$

Putting different values of A and B in R_{pi}, where $i=1,2,3,\dots,10$ we get

$$\mathrm{MSE}(R_{p1})<\mathrm{MSE}(R_2)\text{ iff }\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\bar{X}}{\bar{X}+\beta_{2x}})^2}{(\ln(\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\bar{X}}{\bar{X}+\beta_{2x}})}{\ln(\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0 \tag{47}$$

$$\mathrm{MSE}(R_{p2})<\mathrm{MSE}(R_2)\text{ iff }\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\bar{X}}{\bar{X}+\rho})^2}{(\ln(\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\bar{X}}{\bar{X}+\rho})}{\ln(\bar{X}+\rho)}\frac{(\gamma_2)}{n}<0 \tag{48}$$

$$\mathrm{MSE}(R_{p3})<\mathrm{MSE}(R_2)\text{ iff }\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\mathrm{Cx}\bar{X}}{\mathrm{Cx}\,\bar{X}+\beta_{2x}})^2}{(\ln(\mathrm{Cx}\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\mathrm{Cx}\bar{X}}{\mathrm{Cx}\,\bar{X}+\beta_{2x}})}{\ln(\mathrm{Cx}\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0 \tag{49}$$

$$\mathrm{MSE}(R_{p4})<\mathrm{MSE}(R_2)\text{ iff }\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\mathrm{Cx}})^2}{(\ln(\beta_{2x}\,\bar{X}+\mathrm{Cx}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\mathrm{Cx}})}{\ln(\beta_{2x}\,\bar{X}+\mathrm{Cx})}\frac{(\gamma_2)}{n}<0 \tag{50}$$

$$\mathrm{MSE}(R_{p5})<\mathrm{MSE}(R_2)\text{ iff }\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})^2}{(\ln(\beta_{2x}\,\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})}{\ln(\beta_{2x}\,\bar{X}+\rho)}\frac{(\gamma_2)}{n}<0 \tag{51}$$

$$\mathrm{MSE}(R_{p6})<\mathrm{MSE}(R_2)\text{ iff }\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})^2}{(\ln(\rho\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})}{\ln(\rho\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0 \tag{52}$$

iii) From (5) and (10), we have

$$\mathrm{MSE}(R_{pi})<\mathrm{MSE}(R_3)\text{ iff} \tag{53}$$

$$\bar{Y}^2\left(\frac{(\gamma_2+\gamma_3)}{n}+\frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,A}{\mathrm{B}}\frac{(\gamma_2)}{n}\right)<\bar{Y}^2\;\frac{(\gamma_1+4\gamma_2+9\gamma_3)}{4n} \tag{54}$$

$$\bar{Y}^2\left(\frac{(\gamma_2+\gamma_3)}{n}+\frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,A}{\mathrm{B}}\frac{(\gamma_2)}{n}\right)-\bar{Y}^2\;\frac{(\gamma_1+4\gamma_2+9\gamma_3)}{4n}<0 \tag{55}$$

$$\frac{(\gamma_2+\gamma_3)}{n}+\frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,A}{\mathrm{B}}\frac{(\gamma_2)}{n}-\frac{(\gamma_1+4\gamma_2+9\gamma_3)}{4n}<0 \tag{56}$$

On Simplification we get

$$\frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,A}{\mathrm{B}}\frac{(\gamma_2)}{n}-\frac{(5\gamma_3+\gamma_1)}{4n}<0 \tag{57}$$

MSE(R_{pi}) is efficient as compared to MSE(R_2) iff

$$\frac{A^2}{B^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,A}{\mathrm{B}}\frac{(\gamma_2)}{n}-\frac{(5\gamma_3+\gamma_1)}{4n}<0 \tag{58}$$

Putting different values of A and B in R_{pi}, where i=1,2,3,…,10 we get

$$\mathrm{MSE}(R_{p1}) < \mathrm{MSE}(R_3) \text{ iff } \frac{\left(\frac{\bar{X}}{\bar{X}+\beta_{2x}}\right)^2}{(\ln(\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\left(\frac{\bar{X}}{\bar{X}+\beta_{2x}}\right)}{\ln(\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n} - \frac{(5\gamma_3+\gamma_1)}{4n} < 0 \quad (59)$$

$$\mathrm{MSE}(R_{p2}) < \mathrm{MSE}(R_3) \text{ iff } \frac{\left(\frac{\bar{X}}{\bar{X}+\rho}\right)^2}{(\ln(\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\left(\frac{\bar{X}}{\bar{X}+\rho}\right)}{\ln(\bar{X}+\rho)}\frac{(\gamma_2)}{n} - \frac{(5\gamma_3+\gamma_1)}{4n} < 0 \quad (60)$$

$$\mathrm{MSE}(R_{p3}) < \mathrm{MSE}(R_3) \text{ iff } \frac{\left(\frac{Cx\bar{X}}{Cx\,\bar{X}+\beta_{2x}}\right)^2}{(\ln(Cx\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\left(\frac{Cx\bar{X}}{Cx\,\bar{X}+\beta_{2x}}\right)}{\ln(Cx\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n} - \frac{(5\gamma_3+\gamma_1)}{4n} < 0 \quad (61)$$

$$\mathrm{MSE}(R_{p4}) < \mathrm{MSE}(R_3) \text{ iff } \frac{\left(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+Cx}\right)^2}{(\ln(\beta_{2x}\,\bar{X}+Cx))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\left(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+Cx}\right)}{\ln(\beta_{2x}\,\bar{X}+Cx)}\frac{(\gamma_2)}{n} - \frac{(5\gamma_3+\gamma_1)}{4n} < 0 \quad (62)$$

$$\mathrm{MSE}(R_{p5}) < \mathrm{MSE}(R_3) \text{ iff } \frac{\left(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho}\right)^2}{(\ln(\beta_{2x}\,\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\left(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho}\right)}{\ln(\beta_{2x}\,\bar{X}+\rho)}\frac{(\gamma_2)}{n} - \frac{(5\gamma_3+\gamma_1)}{4n} < 0 \quad (63)$$

$$\mathrm{MSE}(R_{p6}) < \mathrm{MSE}(R_3) \text{ iff } \frac{\left(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}}\right)^2}{(\ln(\rho\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n} - \frac{2\left(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}}\right)}{\ln(\rho\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n} - \frac{(5\gamma_3+\gamma_1)}{4n} < 0 \quad (64)$$

5. COMPUTATIONAL STUDY

In this study, we conducted a thorough mathematical comparison of the proposed estimators with other existing methods, and derived the conditions under which our estimators outperform the others. To provide a practical demonstration of the superiority of our proposed estimators, we utilized earthquake data from Turkey, which is a highly seismically active region where several pieces of Earth's crust converge and collide. As a result, earthquakes are a frequent and unavoidable natural disaster in Turkey and can cause significant damage and loss of life. By applying our proposed estimators to earthquake data from Turkey [2]. We were able to demonstrate their superior performance compared to other estimators. This empirical analysis provides important insights into the practical applicability and usefulness of our proposed estimators in real-world scenarios. Overall, this study highlights the importance of developing efficient and accurate estimators for rare events such as earthquakes, which have significant implications for public safety and disaster management. To evaluate the performance of our proposed estimators, we focused on earthquakes that occurred in Turkey between the years 1900 and 2011. Specifically, we considered the main shocks with surface wave magnitude MS >0.5, their fore shocks within 5 days with MS >0.3, and after-shocks within one month with MS >0.4. According to our data, a total of 109 main shocks with surface magnitude MS > 0.5 occurred during this period. By analyzing this dataset using our proposed estimators and comparing the results to other existing methods, we were able to demonstrate the superior performance and efficiency of our proposed estimators in estimating the population mean of rare events such as earthquakes. This analysis provides valuable insights into the practical applicability of our proposed estimators in real-world scenarios, particularly in the context of natural disasters like earthquakes. In the population consisting destructive earthquake following are the study and auxiliary variables:

- Study variable (y): number of after shocks
- Auxiliary variable (x): number of fore shocks

To get the distribution of these variables with sample size 20, we fit the Poisson distribution to the earthquake data set. To obtain the correlation between study and auxiliary variables for Poisson distributed data, Turkey is divided into three main neotectonic domains based on the neotectonic zones of Turkey. In this way the parameters γ_1, γ_2 and γ_3 are obtained. Further, using the goodness of fit test it can be easily observed that Poisson distribution fits the number of shocks and we can calculate the following values:

Table 3. Statistics of the number of shocks.

Region 1	$\gamma_1 = 4.181$	$\chi^2 = 0.048$	$p = 0.043$
Region 2	$\gamma_2 = 8.104$	$\chi^2 = 0.014$	$p = 0032$
Region 3	$\gamma_3 = 2.112$	$\chi^2 = 0.013$	$p = 0.025$

We also observed that the correlation between study and auxiliary variables is $\rho = 0.712$. The mean square errors (MSE's) and percentage relative efficiencies (PRE's) of the proposed estimators are computed and presented in the table below, as Table 4. The MSE equations are mentioned in the respective equations and formula for PRE is given as:

$$PRE = \frac{MSE(R_1)}{MSE(R_i)} \qquad i = 1,2,3,4,5,6; \tag{65}$$

Table 4. MSE and PRE values of R_i, $i = 1,2,3,4,5,6$.

Estimators	MSE	PRE
R1	64.1088	100
R2	50.4995	126.9494
R3	72.5417	88.3751
R_{p1}	21.9489	292.0820
R_{p2}	0.8026	7,987.6401
R_{p3}	26.4822	242.0826
R_{p4}	14.5467	440.7103
R_{p5}	14.6449	437.7551
R_{p6}	23.6686	270.8601

It can be noted that the number of foreshocks is related to the number of aftershocks therefore, proposed logarithmic type estimators are preferable in this case. Table 3 gives the MSE values of the ratio estimators, based on Equations (1), (4), (5), and (10). The estimators in Equation (10) have the smallest MSE, which proves that this estimator is more efficient than the estimator in Equations (1), (4), and (5). We present the numerical values of the efficiency conditions which the estimators Equations (1), (4), (5), and (10) for estimating population mean. These values are given in Table 4. From Table 5 it is evident that all the efficiency conditions are also satisfied.

Table 5. Efficiency conditions (Equations 35-64).

Equation number	Condition	Value
35	$\frac{(\gamma_3-\gamma_1)}{n}+\frac{(\frac{\bar{X}}{\bar{X}+\beta_{2x}})^2}{(\ln(\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\bar{X}}{\bar{X}+\beta_{2x}})}{\ln(\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0$	-0.2078<0
36	$\frac{(\gamma_3-\gamma_1)}{n}+\frac{(\frac{\bar{X}}{\bar{X}+\rho})^2}{(\ln(\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\bar{X}}{\bar{X}+\rho})}{\ln(\bar{X}+\rho)}\frac{(\gamma_2)}{n}<0$	-0.4104<0
37	$\frac{(\gamma_3-\gamma_1)}{n}+\frac{(\frac{Cx\bar{X}}{Cx\,\bar{X}+\beta_{2x}})^2}{(\ln(Cx\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{Cx\bar{X}}{Cx\,\bar{X}+\beta_{2x}})}{\ln(Cx\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0$	-0.1644<0
38	$\frac{(\gamma_3-\gamma_1)}{n}+\frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\bar{X}+Cx})^2}{(\ln(\beta_{2x}\bar{X}+Cx))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\bar{X}+Cx})}{\ln(\beta_{2x}\bar{X}+Cx)}\frac{(\gamma_2)}{n}<0$	-0.2787<0
39	$\frac{(\gamma_3-\gamma_1)}{n}+\frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\bar{X}+\rho})^2}{(\ln(\beta_{2x}\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\bar{X}+\rho})}{\ln(\beta_{2x}\bar{X}+\rho)}\frac{(\gamma_2)}{n}<0$	-0.2778<0
40	$\frac{(\gamma_3-\gamma_1)}{n}+\frac{(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})^2}{(\ln(\rho\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})}{\ln(\rho\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0$	-0.1913<0
47	$\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\bar{X}}{\bar{X}+\beta_{2x}})^2}{(\ln(\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\bar{X}}{\bar{X}+\beta_{2x}})}{\ln(\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0$	-0.0774<0

Equation number	Condition	Value
48	$\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\bar{X}}{\bar{X}+\rho})^2}{(\ln(\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\bar{X}}{\bar{X}+\rho})}{\ln(\bar{X}+\rho)}\frac{(\gamma_2)}{n}<0$	-0.2800<0
49	$\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{Cx\bar{X}}{Cx\,\bar{X}+\beta_{2x}})^2}{(\ln(Cx\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{Cx\bar{X}}{Cx\,\bar{X}+\beta_{2x}})}{\ln(Cx\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0$	-0.0340<0
50	$\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+Cx})^2}{(\ln(\beta_{2x}\,\bar{X}+Cx))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+Cx})}{\ln(\beta_{2x}\,\bar{X}+Cx)}\frac{(\gamma_2)}{n}<0$	-0.1484<0
51	$\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})^2}{(\ln(\beta_{2x}\,\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})}{\ln(\beta_{2x}\,\bar{X}+\rho)}\frac{(\gamma_2)}{n}<0$	-0.1474<0
52	$\frac{(3\gamma_3-\gamma_1)}{4n}+\frac{(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})^2}{(\ln(\rho\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})}{\ln(\rho\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}<0$	-0.0609<0
59	$\frac{(\frac{\bar{X}}{\bar{X}+\beta_{2x}})^2}{(\ln(\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\bar{X}}{\bar{X}+\beta_{2x}})}{\ln(\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}-\frac{(5\gamma_3+\gamma_1)}{4n}<0$	-0.1841<0
60	$\frac{(\frac{\bar{X}}{\bar{X}+\rho})^2}{(\ln(\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\bar{X}}{\bar{X}+\rho})}{\ln(\bar{X}+\rho)}\frac{(\gamma_2)}{n}-\frac{(5\gamma_3+\gamma_1)}{4n}<0$	-0.3867<0
61	$\frac{(\frac{Cx\bar{X}}{Cx\,\bar{X}+\beta_{2x}})^2}{(\ln(Cx\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{Cx\bar{X}}{Cx\,\bar{X}+\beta_{2x}})}{\ln(Cx\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}-\frac{(5\gamma_3+\gamma_1)}{4n}<0$	-0.14066<0
62	$\frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+Cx})^2}{(\ln(\beta_{2x}\,\bar{X}+Cx))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+Cx})}{\ln(\beta_{2x}\,\bar{X}+Cx)}\frac{(\gamma_2)}{n}-\frac{(5\gamma_3+\gamma_1)}{4n}<0$	-0.2550<0
63	$\frac{(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})^2}{(\ln(\beta_{2x}\,\bar{X}+\rho))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\beta_{2x}\bar{X}}{\beta_{2x}\,\bar{X}+\rho})}{\ln(\beta_{2x}\,\bar{X}+\rho)}\frac{(\gamma_2)}{n}-\frac{(5\gamma_3+\gamma_1)}{4n}<0$	-0.9461<0
64	$\frac{(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})^2}{(\ln(\rho\,\bar{X}+\beta_{2x}))^2}\frac{(\gamma_1+\gamma_3)}{n}-\frac{2\,(\frac{\rho\,\bar{X}}{\rho\,\bar{X}+\beta_{2x}})}{\ln(\rho\,\bar{X}+\beta_{2x})}\frac{(\gamma_2)}{n}-\frac{(5\gamma_3+\gamma_1)}{4n}<0$	-0.1676<0

6. CONCLUSIONS

This paper provides important insights into improving the accuracy of estimation in Poisson distributed populations, particularly in the context of rare events. The ratio estimators are devised for simple random sampling. Table 3 shows that the estimator in Eq. (10) for estimating the population mean is more efficient. The estimator in Eq. (46) provides lower MSE than MSE of the ratio estimators in Eq. (1), (4), (5) in simple random sampling. By using earthquake data from Turkey, we were able to demonstrate the practical relevance and usefulness of our proposed estimators in real-world scenarios. This highlights the potential for future research to extend the proposed estimators to develop new and improved methods for stratified random sampling.

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