ORIGINAL PAPER

GENERALIZED n-POLYNOMIAL P-FUNCTIONS WITH SOME RELATED INEQUALITIES AND THEIR APPLICATIONS

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Abstract. *In this paper, we introduce the notion of generalized n-polynomial P-function. We explore some algebraic properties of this function class. Additionally, we establish a new trapezium type inequality for this generalized class of functions and derive several refinements of the trapezium type inequality for functions whose first derivative in absolute value at a certain power is generalized n-polynomial P-function. Finally, we conclude our paper by exploring some applications of the results we have obtained in the context of special means. Our novel findings generalize previously known results in the literature.*

Keywords: *Convex function; n-polynomial harmonically convexity; generalized n-polynomial harmonically convexity; Hermite-Hadamard inequality; integral inequalities.*

1. INTRODUCTION AND PRELIMINARIES

Let I be a nonempty interval in the set of real numbers $\mathbb{R}$. A function $\Phi: I \to \mathbb{R}$ is called convex if

$$\Phi(ta + (1-t)b) \leq t\Phi(a) + (1-t)\Phi(b)$$

for all $a, b \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convex functions appear in a wide range of topics in pure and applied mathematics, including the theory of inequalities, see, for example, [1-6]. One of the early pivotal results in the theory of convex functions is the following theorem.

Theorem 1. [6] Let $\Phi: I \to \mathbb{R}$ be a convex function on an interval I with a nonempty interior. Then for all $a, b \in I$ with $a < b$ and $\Phi \in L[a, b]$ (Lebesgue integrable on the interval $[a, b]$), the following inequality holds:

$$\Phi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \Phi(x)dx \leq \frac{\Phi(a)+\Phi(b)}{2}. \tag{1}$$

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The inequality (1) is well known as the Hermite-Hadamard inequality [6]. For the refinements of the Hermite-Hadamard inequality, interested readers refer to convex functions that have been obtained [7-12].

In [13], Dragomir et al. introduced the class of P-function and established Hermite-Hadamard inequality for this class of functions as follows:

Definition 1. A function $\Phi: I \subseteq \mathbb{R} \to \mathbb{R}$ is called P-function if

$$\Phi(tx+(1-t)y) \leq \Phi(x)+\Phi(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Theorem 2. [13] Let $\Phi: I \to \mathbb{R}$ be a P-function. Then the inequality

$$\Phi\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b \Phi(x)dx \leq 2[\Phi(a)+\Phi(b)] \tag{2}$$

holds for all $a, b \in I$ with $a < b$.

Definition 2. [14] Let $h: J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $\Phi: I \to \mathbb{R}$ is an h-convex function, if Φ is non-negative and for all $x, y \in I, \alpha \in (0,1)$ we have

$$\Phi(\alpha x+(1-\alpha)y) \leq h(\alpha)\Phi(x)+h(1-\alpha)\Phi(y).$$

Observe that every P-function is an h-convex function, where h here is the constant function 1, that is $h(\alpha)=1$ for all $\alpha \in J$.

In [15], İşcan and Kadakal gave the following definition and related Hermite-Hadamard inequality:

Definition 3. Let $n \in \mathbb{N}$. A $\Phi: I \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial P-function if for every $x, y \in I$ and $t \in [0,1]$,

$$\Phi(tx+(1-t)y) \leq \frac{1}{n}\sum_{i=1}^{n}\left[2-t^i-(1-t)^i\right][\Phi(x)+\Phi(y)].$$

Theorem 3. [15] Let $\Phi: [a,b] \to \mathbb{R}$ be a n-polynomial P-function. If $a < b$ and $\Phi \in L[a,b]$, then the following Hermite-Hadamard inequality holds:

$$\frac{1}{4}\left(\frac{n}{n+2^{-n}-1}\right)\Phi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \Phi(x)dx \leq \left(\frac{\Phi(a)+\Phi(b)}{n}\right)\sum_{i=1}^{n}\frac{2i}{i+1}. \tag{3}$$

Recently, in [16] Kadakal et al. gave the definition of generalized n-polynomial convex functions as follows:

Definition 4. Let $n \in \mathbb{N}$ and $a_i \geq 0 (i=\overline{1,n})$ such that $\sum_{i=1}^{n} a_i > 0$. A non-negative function $\Phi: I \subset \mathbb{R} \to \mathbb{R}$ is called generalized n-polynomial convex function if for every $x, y \in I$ and $t \in [0,1]$,

$$\Phi(tx+(1-t)y) \le \frac{\sum_{i=1}^{n} a_i\left[1-(1-t)^i\right]}{\sum_{i=1}^{n} a_i}\Phi(x) + \frac{\sum_{i=1}^{n} a_i\left(1-t^i\right)}{\sum_{i=1}^{n} a_i}\Phi(y) \tag{4}$$

The main purpose of this paper is to introduce the notion of generalized n-polynomial P-functions and establish some trapezium-type inequalities for this class of functions. Some applications to special means of positive real numbers are also given.

2. SOME ALGEBRAIC PROPERTIES OF GENERALIZED n-POLYNOMIAL P-FUNCTIONS

In this section, we are going to give a new definition namely generalized n-polynomial P-function and investigate some of its algebraic properties.

Definition 5. Let $n \in \mathbb{N}$ and $a_i \ge 0 (i = \overline{1,n})$ such that $\sum_{i=1}^{n} a_i > 0$. A function $\Phi: I \subset \mathbb{R} \to \mathbb{R}$ is called generalized n-polynomial P-function if

$$\Phi(tx+(1-t)y) \le \frac{\sum_{i=1}^{n} a_i\left[2-t^i-(1-t)^i\right][\Phi(x)+\Phi(y)]}{\sum_{i=1}^{n} a_i} \tag{5}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

We will denote by $\mathrm{GPOLP}(I)$ the class of all generalized n-polynomial P-functions on interval I. We note that, every generalized n-polynomial P-function is an h-convex function with the function

$$h(t) = \frac{\sum_{i=1}^{n} a_i\left[2-t^i-(1-t)^i\right]}{\sum_{i=1}^{n} a_i}$$

Therefore, if $\Phi, \Psi \in GPOLP(I)$, then

i) $\Phi + \Psi \in \mathrm{GPOLP}(I)$ and for $c \in \mathbb{R}(c \ge 0) c\Phi \in \mathrm{GPOLP}(I)$ (see [14], Proposition 9).

ii) if Φ and Ψ be a similarly ordered function on I, then $\Phi\Psi \in \mathrm{GPOLP}(I)$ (see [14], Proposition 10).

For more results on the class $\mathrm{GPOLP}(I)$, see [14].

Remark 1. We note that if Φ satisfies (5), then Φ is a non-negative function. For $t = 0$, the inequality (5) reduces to the inequality

$$\Phi(y) \le \Phi(x) + \Phi(y)$$

for all $x, y \in I$. So, one has $\Phi(x) \ge 0$ for all $x \in I$.

Proposition 1. Let $\Phi: I \to \mathbb{R}$ be a P-function. Suppose that n is a positive integer, and assume that $a_1, \dots, a_n$ are non-negative real numbers such that $\sum_{i=1}^{n} a_i > 0$. Then Φ is a generalized n-polynomial P-function with respect to $a_1, \dots, a_n$.

Proof: Let $x, y \in I$, and let $t \in [0,1]$. Note that

$$1-t^i \ge 1-t \text{ and } 1-(1-t)^i \ge 1-(1-t) = t$$

for all $i = 1, \dots, n$. Then

$$\frac{\sum_{i=1}^{n} a_i\left[2 - t^i - (1-t)^i\right]}{\sum_{i=1}^{n} a_i} \geq 1.$$

Thus from Φ being a P-function, we deduce that

$$\begin{aligned}
&\Phi(tx + (1-t)y) \\
&\leq \Phi(x) + \Phi(y) \\
&\leq \frac{\sum_{i=1}^{n} a_i\left[2 - t^i - (1-t)^i\right]}{\sum_{i=1}^{n} a_i}\big(\Phi(x) + \Phi(y)\big).
\end{aligned}$$

This proves that Φ is a generalized n-polynomial P-function with respect to $a_1, \dots, a_n$.

Corollary 1. Every non-negative convex function is a generalized n-polynomial P-function.

Proof: Let $\Phi: I \to \mathbb{R}$ be a non-negative convex function. Suppose that $x, y \in I$ and that $t \in [0,1]$. Then from Φ being convex, we get

$$\Phi(tx + (1-t)y) \leq t\Phi(x) + (1-t)\Phi(y).$$

Thus from Φ being non-negative and $t \in [0,1]$, we obtain

$$\Phi(tx + (1-t)y) \leq \Phi(x) + \Phi(y).$$

Hence Φ is a P-function. Then from Proposition 1, the result follows.

Theorem 4. Let a and b be real numbers such that $0 < a < b$, and let $\{\Phi_\alpha\}_{\alpha\in\Lambda}$ be an indexed family of generalized n-polynomial P-functions from $[a,b]$ into $\mathbb{R}$ for some indexed set Λ. Define the function Φ on $[a,b]$ by $\Phi(x) = \sup\{\Phi_\alpha(x): \alpha \in \Lambda\}$ for all $x \in [a,b]$. Assume that the set $J = \{u \in [a,b]: \Phi(u) < \infty\}$ contains at least two distinct real numbers. Then J is an interval and the function Φ is a generalized n-polynomial P-function from J into $\mathbb{R}$.

Proof: Let $t \in [0,1]$ and let $x, y \in J$ such that $x \neq y$. Then

$$\begin{aligned}
&\Phi(tx + (1-t)y) \\
&= \sup_{\alpha\in\Lambda} \Phi_\alpha(tx + (1-t)y) \\
&\leq \sup_{\alpha\in\Lambda}\left[\frac{\sum_{i=1}^{n} a_i\left[2 - t^i - (1-t)^i\right][\Phi_\alpha(x) + \Phi_\alpha(y)]}{\sum_{i=1}^{n} a_i}\right] \\
&\leq \frac{\sum_{i=1}^{n} a_i\left[2 - t^i - (1-t)^i\right]}{\sum_{i=1}^{n} a_i}\left[\sup_{\alpha\in\Lambda}\Phi_\alpha(x) + \sup_{\alpha\in\Lambda}\Phi_\alpha(y)\right] \\
&= \frac{\sum_{i=1}^{n} a_i\left[2 - t^i - (1-t)^i\right]}{\sum_{i=1}^{n} a_i}[\Phi(x) + \Phi(y)] \\
&< \infty.
\end{aligned}$$

This shows simultaneously that J is an interval and that Φ is a generalized n-polynomial P-function on J. This completes the proof of the theorem.

3. HERMITE-HADAMARD INEQUALITY FOR GENERALIZED n-POLYNOMIAL P-FUNCTIONS

In this section, we will establish Hermite-Hadamard inequality for generalized n-polynomial P-functions. We will denote by $L[a,b]$ the space of (Lebesgue) integrable functions on $[a,b]$.

Theorem 5. Let $\Phi:[a,b]\to\mathbb{R}$ be a generalized n-polynomial P-function. If $a<b$ and $\Phi\in L[a,b]$, then

$$\frac{1}{4}\left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i\left(1-\frac{1}{2^i}\right)}\right)\Phi\left(\frac{a+b}{2}\right)$$
$$\leq \frac{1}{b-a}\int_a^b \Phi(x)dx \leq \left[\frac{\Phi(a)+\Phi(b)}{\sum_{i=1}^{n} a_i}\right]\sum_{i=1}^{n} a_i\left(\frac{2i}{i+1}\right). \tag{6}$$

Proof: From the propery of the generalized n-polynomial P-function Φ, we get

$$\Phi\left(\frac{a+b}{2}\right)=\Phi\left(\frac{1}{2}[ta+(1-t)b]+\frac{1}{2}[(1-t)a+tb]\right)$$
$$\leq \frac{\sum_{i=1}^{n} a_i\left[2-2\left(\frac{1}{2}\right)^i\right]}{\sum_{i=1}^{n} a_i}[\Phi(ta+(1-t)b)+\Phi((1-t)a+tb)].$$

Taking integral with respect to t, we get

$$\Phi\left(\frac{a+b}{2}\right) \leq \int_0^1 \Phi\left(\frac{a+b}{2}\right)dt$$
$$\leq \frac{2\sum_{i=1}^{n} a_i\left(1-\frac{1}{2^i}\right)}{\sum_{i=1}^{n} a_i}\left[\int_0^1 \Phi(ta+(1-t)b)dt+\int_0^1 \Phi\big((1-t)a+\mathrm{t}b\big)dt\right].$$

Using the substitution $x=ta+(1-t)b$, we obtain

$$\int_0^1 \Phi(ta+(1-t)b)dt+\int_0^1 \Phi\big((1-t)a+\mathrm{t}b\big)dt=\frac{1}{b-a}\int_a^b \Phi(x)dx.$$

Then

$$\Phi\left(\frac{a+b}{2}\right)\leq \frac{4\sum_{i=1}^{n} a_i\left(1-\frac{1}{2^i}\right)}{(b-a)\sum_{i=1}^{n} a_i}\int_a^b \Phi(x)dx.$$

Now, we prove the second inequality in (6).

$$\frac{1}{b-a}\int_a^b \Phi(x)dx = \int_0^1 \Phi(ta+(1-t)b)dt$$

$$\leq \int_0^1 \left[\frac{\sum_{i=1}^n a_i\left[2-t^i-(1-t)^i\right][\Phi(a)+\Phi(b)]}{\sum_{i=1}^n a_i}\right]dt$$

$$= \left[\frac{\Phi(a)+\Phi(b)}{\sum_{i=1}^n a_i}\right]\sum_{i=1}^n a_i\int_0^1 \left[2-t^i-(1-t)^i\right]dt$$

$$= \left[\frac{\Phi(a)+\Phi(b)}{\sum_{i=1}^n a_i}\right]\sum_{i=1}^n a_i\left(\frac{2i}{i+1}\right),$$

where

$$\int_0^1 \left[2-t^i-(1-t)^i\right]dt = \frac{2i}{i+1}.$$

This completes the proof of the theorem.

Remark 2. For $n=1$, the inequality (6) reduces to the inequality (2).

Remark 3. For $a_i = 1(i=\overline{1,n})$, the inequality (6) coincides with the inequality (3).

4. TRAPEZIUM TYPE INEQUALITIES FOR GENERALIZED n-POLYNOMIAL P-FUNCTIONS

In this section, we will establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is a generalized n-polynomial P-function.

Dragomir and Agarwal [17] gave the following lemma:

Lemma 1. Let I° denotes the interior of I, and let $\Phi: I^\circ \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a<b$. If $\Phi' \in L[a,b]$, then the following identity holds:

$$\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)\Phi'(ta+(1-t)b)dt.$$

Theorem 6. Let $\Phi: I \to \mathbb{R}$ be a differentiable function on I°(the interior of I), $a, b \in I^\circ$ with $a<b$ and assume that $\Phi' \in L[a,b]$. If $|\Phi'|$ is a generalized n-polynomial P-function on interval $[a,b]$, then the following inequality holds:

$$\left|\frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\int_a^b \Phi(x)dx\right| \\ \leq \frac{b-a}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\left[\frac{(i^2+i+2)2^i-2}{(i+1)(i+2)2^i}\right]A(|\Phi'(a)|,|\Phi'(b)|), \tag{6}$$

where A is the arithmetic mean.

Proof: Using Lemma 1 and the inequality

$$|\Phi'(ta+(1-t)b)| \leq \frac{\sum_{i=1}^n a_i\left[2-t^i-(1-t)^i\right][|\Phi'(a)|+|\Phi'(b)|]}{\sum_{i=1}^n a_i},$$

we get

$$\left|\frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\int_a^b \Phi(x)dx\right|$$

$$\leq \frac{b-a}{2}\int_0^1 |1-2t||\Phi'(ta+(1-t)b)|dt$$

$$\leq \frac{b-a}{2}\int_0^1 |1-2t|\left(\frac{\sum_{i=1}^n a_i\left[2-t^i-(1-t)^i\right][|\Phi'(a)|+|\Phi'(b)|]}{\sum_{i=1}^n a_i}\right)dt$$

$$\leq \frac{b-a}{2\sum_{i=1}^n a_i}[|\Phi'(a)|+|\Phi'(b)|]\sum_{i=1}^n a_i\int_0^1 |1-2t|\left[2-t^i-(1-t)^i\right]dt$$

$$=\frac{b-a}{\sum_{i=1}^n a_i}\sum_{s=1}^n a_i\left[\frac{(i^2+i+2)2^i-2}{(i+1)(i+2)2^i}\right]A(|\Phi'(a)|,|\Phi'(b)|),$$

where

$$\int_0^1 |1-2t|\left[2-t^i-(1-t)^i\right]dt=\frac{(i^2+i+2)2^i-2}{(i+1)(i+2)2^i}$$

and A is the arithmetic mean. So, the proof is completed.

Corollary 2. If we take $n=1$ in the inequality (7), then we get the following inequality:

$$\left|\frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\int_a^b \Phi(x)dx\right| \leq \frac{b-a}{2}A(|\Phi'(a)|,|\Phi'(b)|).$$

Remark 4. If we take $a_i=1(i=\overline{1,n})$, then the inequality (7) reduces to the inequality in [15, Theorem 4.1].

Theorem 7. Let $\Phi: I \to \mathbb{R}$ be a differentiable function on I°(the interior of I), $a, b \in I^\circ$ with $a < b$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $\Phi' \in L[a, b]$. If $|\Phi'|^q$ is a generalized n-polynomial P-function on interval $[a, b]$, then the following inequality holds:

$$\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \frac{i}{i+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|\Phi'(a)|^q, |\Phi'(b)|^q), \tag{7}$$

where A is the arithmetic mean.

Proof: Using Lemma 1, Hölder's integral inequality and the inequality

$$|\Phi'(ta+(1-t)b)|^q \leq \frac{\sum_{i=1}^n a_i\left[2-t^i-(1-t)^i\right]\left[|\Phi'(a)|^q + |\Phi'(b)|^q\right]}{\sum_{i=1}^n a_i},$$

we get

$$\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(x)dx\right|$$

$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\Phi'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\Phi'(a)|^q + ||\Phi'(b)|^q\,|}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1 \left[2-t^i-(1-t)^i\right]dt\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{4}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \frac{i}{i+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|\Phi'(a)|^q, |\Phi'(b)|^q),$$

where

$$\int_0^1 |1-2t|^p dt = \frac{1}{p+1}$$

$$\int_0^1 \left[2-t^i-(1-t)^i\right]dt = \frac{2i}{i+1}$$

This completes the proof of theorem.

Corollary 3. If we take $n = 1$ in the inequality (8), then we get the following inequality:

$$\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(x)dx\right| \leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}(|\Phi'(a)|^q, |\Phi'(b)|^q). \tag{8}$$

Remark 5. If we take $a_i = 1(i = \overline{1,n})$ in the inequality (8), then we get the inequality in [15, Theorem 4.2].

Theorem 8. Let $\Phi: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b, q \geq 1$ and assume that $\Phi' \in L[a,b]$. If $|\Phi'|^q$ is a generalized n-polynomial P-function on $[a,b]$, then the following inequality holds for $t \in [0,1]$.

$$\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(x)dx\right| \leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \frac{(i^2+i+2)2^i-2}{(i+1)(i+2)2^{i-1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|\Phi'(a)|^q, |\Phi'(b)|^q), \tag{9}$$

where A is the arithmetic mean.

Proof: From Lemma 1, power-mean integral inequality and the property of the generalized n-polynomial P-function $|\Phi'|^q$, we obtain

$$\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(x)dx\right|$$

$$\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t||\Phi'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{|\Phi'(a)|^q+|\Phi'(b)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \int_0^1 |1-2t|\left[2-t^i-(1-t)^i\right]dt\right)^{\frac{1}{q}}$$

$$= \frac{b-a}{2^{2-\frac{1}{q}}}\left(\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i \frac{(i^2+i+2)2^i-2}{(i+1)(i+2)2^{i-1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|\Phi'(a)|^q, |\Phi'(b)|^q),$$

where

$$\int_0^1 |1-2t|dt = \frac{1}{2},$$

$$\int_0^1 |1-2t|\left[2-t^i-(1-t)^i\right]dt = \frac{(i^2+i+2)2^i-2}{(i)(i+2)2^{i+1}}$$

So, the proof is completed.

Corollary 4. Under the assumption of Theorem 8 with $q = 1$, we get the conclusion of Theorem 6.

Corollary 5. If we take $n = 1$ in the inequality (9), then we get the following inequality:

$$\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(x)dx\right| \le \frac{b-a}{4} A^{\frac{1}{q}}(|\Phi'(a)|^q, |\Phi'(b)|^q).$$

This inequality coincides with the inequality in [5, Theorem 1].

Corollary 6. If we take $a_i = 1(i = \overline{1,n})$ in the inequality (9), then we get the inequality in [15, Theorem 4.3].

5. APPLICATIONS FOR SPECIAL MEANS

Throughout this section, the following notations will be used for special means of two nonnegative numbers a, b with $> a$:

1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2},\ a,b \ge 0$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab},\ a,b \ge 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b},\ a,b > 0$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & a \ne b \\ a, & a = b \end{cases};\ a,b > 0$$

5. The p-logaritmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \ne b, p \in \mathbb{R} \setminus \{-1,0\}\,; a,b > 0. \\ a, & a = b \end{cases}$$

6. The identric mean

$$I := I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},\ a,b > 0$$

The following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A$$

Proposition 2. Let $a, b \in [0, \infty)$ with $a < b$ and $m \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$. Then, one has

$$\frac{1}{4}\left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i \left[1 - \left(\frac{1}{2}\right)^i\right]}\right) A^m(a, b) \leq L_m^m(a, b) \leq A(a^m, b^m) \frac{2}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \left(\frac{2i}{i+1}\right).$$

Proof: The assertion follows from the inequalities (6) for the function

$$\Phi(x) = x^m,\ x \in [0, \infty).$$

Proposition 3. Let $a, b \in (0, \infty)$ with $a < b$. Then, one has

$$\frac{1}{4}\left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i \left[1 - \left(\frac{1}{2}\right)^i\right]}\right) A^{-1}(a, b) \leq L^{-1}(a, b)$$

$$\leq H^{-1}(a, b) \frac{2}{\sum_{i=1}^{n} a_i} \sum_{i=1}^{n} a_i \left(\frac{2i}{i+1}\right)$$

Proof: The assertion follows from the inequalities (6) for the function

$$\Phi(x) = x^{-1},\ x \in (0, \infty).$$

6. CONCLUSION

In this paper, we have shown new Hermite-Hadamard type inequalities for the newly defined class of functions, the so-called generalized n-polynomial P-functions. Furthermore, we have derived certain trapezium type inequalities for this class of functions. Additionally, we have investigated some applications of these results in the context of special means.

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