ORIGINAL PAPER

HG-CONVEX FUNCTIONS

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Abstract. *In this paper, the concept of Hg-convex function is given for the first time in the literature. Some inequalities of Hadamard's type for Hg-convex functions are given. Some algebraic properties of Hg-convex functions and special cases are discussed. In addition, we establish some new integral inequalities for Hg-convex functions by using an integral identity.*

Keywords: *Convex function; Hg-convex; Hermite-Hadamard inequality.*

1. INTRODUCTION

Convexity theory plays a central and fundamental role in the fields of mathematical finance, economics, engineering, management sciences, and optimization theory. In recent years, the concept of convexity has been extended and generalized in several directions using novel and innovative ideas; see, for example, [1-9] and the references therein.

Definition 1. A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well-known in the literature. Denote by $C(I)$ the set of the convex functions on the interval I.

Definition 2. $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

In [6], Kadakal and İşcan gave the concept of the Ag-convexity as follow:

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Definition 3. Let $I \subset \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$, $g: J \to \mathbb{R}$, $J \supset f(I)$. f is said to be Ag-convex if the inequality

$$f(tx + (1-t)y) \leq tg(f(x)) + (1-t)g(f(y))$$

is valid for all $x, y \in I$ and $t \in 0,1]$. Denote by $AgC(I)$ the set of the Ag-convex functions on the interval I.

Definition 4. [4] A function $f: I \subseteq \mathbb{R}\backslash\{0\}$ is said to be harmonically convex (or HA-convex) on interval I if

$$f\left(\frac{xy}{ty + (1-t)x}\right) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$, where $xy/(ty + (1-t)x)$ and $tf(x) + (1-t)f(y)$ are, respectively, the weighted harmonic mean of two positive numbers x and y and the weighted arithmetic mean of f(x) and f(y). We will denote by $HAC(I)$ the set of the GA-convex functions on the interval I.

In [4], İşcan proved the following lemma and established new inequalities of Hermite-Hadamard type for harmonically convex functions:

Lemma 1. Let $f: I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then

$$\frac{f(a) + f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)dx}{x^2} = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb + (1-t)a)^2} f'\left(\frac{ab}{tb + (1-t)a}\right) dt.$$

The main purpose of this paper is to give a new class of convex functions called as Hg-convex function (or (HA, g)-convex) and establish both the Hermite-Hadamard type integral inequalities and we establish some new integral inequalities for Hg-convex functions by using an integral identity. The results obtained in special cases are reduced to the results obtained in the literature.

2. MAIN RESULTS FOR Hg-CONVEX FUNCTIONS

Definition 5. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f: I \to \mathbb{R}$, $J \supset f(I)$, $g: J \to \mathbb{R}$ be differentiable functions. If,

$$f\left(\frac{xy}{ty + (1-t)x}\right) \leq t[g(f(x))] + (1-t)[g(f(y))] \tag{2}$$

holds for all $x, y \in I$ and $t \in 0,1]$, then the function f is called Hg-convex (or (Harmonic, g)-convex). If the inequality (2) holds in reverse, then the function f is called Hg-concave. The set of Hg-convex functions on the interval I is denoted by $HgC(I)$.

From the above inequality, it is possible to write

$$(f \circ \phi^{-1})(t\phi(x) + (1-t)\phi(y)) \leq t[g(f(x))] + (1-t)[g(f(y))]$$

where $\phi(x) = \frac{1}{x}$ for $x \neq 0$.

Based on this, the following proposition can be formulated. Additionally, if $g(x) = x$, the definition of Hg-convex functions reduces to the definition of harmonic convex functions.

Proposition 1. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f: I \to \mathbb{R}$, $g: J \to \mathbb{R}$, $J \supset f(I)$. If f is Hg-convex on I if and only if $f \circ \phi^{-1}$ is A_g-convex on $Q(I) = \{\phi(x) = \frac{1}{x} : x \in I\}$.

Proposition 2. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f: I \to \mathbb{R}$, $g: J \to \mathbb{R}$, $J \supset f(I)$. If f is Hg-convex, then $y \leq g(y)$ for every $y \in f(I)$.

Proof: Let $y \in f(I)$ be arbitrary. Then, there exists $x \in I$ such that $y = f(x)$. Since is Hg-convex on I, for every $t \in 0,1]$, f being Hg-convex implies

$$f(x) = f\left(\frac{xx}{tx + (1-t)x}\right) \leq t[g(f(x))] + (1-t)[g(f(x))] = g(f(x))$$

for $x \in I$ and every $t \in 0,1]$. This shows that $y \leq g(y)$ for every $y \in f(I)$.

Remark 1. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f: I \to \mathbb{R}$, $g: J \to \mathbb{R}$ and $J \supset f(I)$.
i) If the function g satisfies the condition $y \leq g(y), y \in f(I)$ and the function f is harmonically convex, then f is Hg-convex function. Indeed, for every $t \in [0,1]$ and every $a, b \in I$

$$\begin{aligned} f\left(\frac{ab}{tb + (1-t)a}\right) &\leq tf(a) + (1-t)f(b) \\ &\leq tg(f(a)) + (1-t)g(f(b)) \\ &\leq t(g \circ f)(a) + (1-t)(g \circ f)(b). \end{aligned}$$

ii) If the function g satisfies the condition $g(y) \leq y, y \in f(I)$ and the function f is Hg-convex, then f is harmonically convex function. Indeed, for every $t \in [0,1]$ and every $a, b \in I$

$$f\left(\frac{ab}{tb + (1-t)a}\right) \leq tg(f(a)) + (1-t)g(f(b)) \leq tf(a) + (1-t)f(b).$$

iii.) It is obvious that $HgC(I) = HAC(I) \Leftrightarrow g(x) = x$.

Theorem 1. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval and $c \in [0, \infty)$. If $f \in HgC(I)$ and g is linear, then cf is Hg-convex function.

Proof: For $c \in [0, \infty)$,

$$\begin{aligned} (cf)(x^t y^{1-t}) &\leq c[tg(f(x)) + (1-t)g(f(y))] \\ &= tg(cf(x)) + (1-t)g(cf(y)) \\ &= t(g \circ (cf))(x) + (1-t)(g \circ (cf))(y) \end{aligned}$$

This completes the proof of the theorem.

Theorem 2. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval. If the functions $f, h \in HgC(I)$ and g is linear, then $f + h \in HgC(I)$.

Proof: For $x, y \in I$ and $t \in 0,1]$,

$$(f+h)\left(\frac{xy}{ty + (1-t)x}\right) = f\left(\frac{xy}{ty + (1-t)x}\right) + h\left(\frac{xy}{ty + (1-t)x}\right)$$

$$\begin{aligned}&\leq [tg(f(x)) + (1-t)g(f(y))] + [tg(h(x)) + (1-t)g(h(y))]\\ &= t[g(f(x)) + g(h(x))] + (1-t)[g(f(y)) + g(h(y))]\\ &= tg(f(x) + h(x)) + (1-t)g\big(f(y) + h(y)\big)\\ &= t(g\circ(f+h))(x) + (1-t)(g\circ(f+h))(y).\end{aligned}$$

This completes the proof of the theorem.

Theorem 3. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval. If the function $f \in HgC(I)$ and monotone increasing, and h is HH-convex, then $f\circ h \in HgC(I)$.

Proof: For $x, y \in I$ and $t \in 0,1]$,

$$\begin{aligned}(foh)\left(\frac{xy}{ty+(1-t)x}\right) &= f\left(h\left(\frac{xy}{ty+(1-t)x}\right)\right)\\ &\leq f\left(\frac{h(x)h(y)}{th(y)+(1-t)h(x)}\right)\\ &\leq tg(f(h(x))) + (1-t)g(f(h(y)))\\ &\leq t(g\circ(f\circ h))(x) + (1-t)(g\circ(f\circ h))(y).\end{aligned}$$

This completes the proof of the theorem.

Theorem 4. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f, h: I \to \mathbb{R}$ are both nonnegative, monotone (increasing or decreasing) and $g: J \to \mathbb{R}$, $J \supset f(I)$, is monotone (increasing or decreasing) and satisfies the condition $g(u)g(v) \leq g(uv)$. If $f, h \in HgC(I)$, then $fh \in HgC(I)$.

Proof: If $x \leq y$ (the case $y \leq x$ runs in the same fashion) then

$$[g(f(x)) - g(f(y))][g(h(y)) - g(h(x))] \leq 0$$

which implies

$$g\big(f(x)\big)g\big(h(y)\big) + g\big(f(y)\big)g\big(h(x)\big) \leq g\big(f(x)\big)g\big(h(x)\big) + g\big(f(y)\big)g\big(h(y)\big). \tag{2.1}$$

On the other hand for $x, y \in I$ and $t \in 0,1]$, we have

$$\begin{aligned}(fh)(x^t y^{1-t}) &= f(x^t y^{1-t})h(x^t y^{1-t})\\ &\leq \big[tg\big(f(x)\big) + (1-t)g\big(f(y)\big)\big]\big[tg\big(h(x)\big) + (1-t)g\big(h(y)\big)\big]\\ &= t^2 g(f(x))g(h(x)) + t(1-t)g(f(x))g(h(y)) + t(1-t)g(f(y))g(h(x)) + (1-t)^2 g(f(y))g(h(y)).\end{aligned}$$

Using now (2.1), we obtain,

$$\begin{aligned}(fh)(x^t y^{1-t}) &\leq t^2 g(f(x))g(h(x)) + (1-t)^2 g(f(y))g(h(y))\\ &\quad + t(1-t)\big[g\big(f(x)\big)g\big(h(x)\big) + g\big(f(y)\big)g\big(h(y)\big)\big]\\ &\leq t[t + (1-t)]g(f(x))g(h(x)) + (1-t)[t + (1-t)]g(f(y))g(h(y))\\ &= tg\big(f(x)\big)g\big(h(x)\big) + (1-t)g\big(f(y)\big)g\big(h(y)\big)\\ &\leq tg(fh)(x) + (1-t)g(fh)(y).\end{aligned}$$

This completes the proof of the theorem.

3. HERMITE-HADAMARD INEQUALITY FOR Hg-CONVEX FUNCTIONS

Theorem 5. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval. If $f: I \to \mathbb{R}$ is a Hg-convex function, $g: J \to \mathbb{R}$, $J \supset f(I)$, $a, b \in I$ with $a < b$ and $g \circ f \in L[a,b]$. The following inequality holds.

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{2ab}{b-a}\int_a^b \frac{(g\circ f)(u)}{u^2}du.$$

Proof: By the definition of Hg-convexity of the function f on the interval $[a,b]$, we write

$$f\left(\frac{2ab}{a+b}\right) \leq f\left(\frac{2H_tH_{1-t}}{H_t+H_{1-t}}\right) \leq \frac{1}{2}(g\circ f)(H_t) + \frac{1}{2}(g\circ f)(H_{1-t}).$$

Now, if we integrate the last inequality on $t \in 0,1]$, we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{2ab}{b-a}\int_a^b \frac{(g\circ f)(u)}{u^2}du.$$

This completes the proof of the theorem.

Remark 2. If we take $g(x) = x$ in the Theorem 5, then we have the left side of the Hermite-Hadamard integral inequality for the harmonik convex functions.

Theorem 6. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval. If $f: I \to \mathbb{R}$ is a Hg-convex function, $g: J \to \mathbb{R}$, $J \supset f(I)$, $a, b \in I$ with $a < b$ and $f \in L[a,b]$, then the following inequality holds:

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{(g\circ f)(a) + (g\circ f)(b)}{2}. \quad (3)$$

Proof: If we use Hg-convexity of the function f and changing $x = \frac{ab}{tb+(1-t)a}$, we get

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt = \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \int_0^1 [tg(f(a)) + (1-t)g(f(b))]dt$$
$$= \frac{(g\circ f)(a) + (g\circ f)(b)}{2}$$

for $t \in [0,1]$.

This completes the proof of the theorem.

Remark 3. If we take $g(x) = x$ in the Theorem 6, then we obtain the left side of the Hermite-hadamard integral inequality for the harmonic-convex functions.

Theorem 7. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f: I \to \mathbb{R}$ be Hg-convex functions and $g: J \to \mathbb{R}$, $J \supset f(I)$, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. If the function f is symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality holds:

$$f(x) \leq \frac{(g\circ f)(a) + (g\circ f)(b)}{2}. \quad (4)$$

Proof: Let $x \in [a,b]$. Then, there is at least $t \in [0,1]$ such that

$$x = \frac{ab}{tb + (1-t)a}$$

and

$$\frac{abx}{(a+b)x - ab} = \frac{ab}{ta + (1-t)b}.$$

Since the function f is symmetric with respect to $\frac{2ab}{a+b}$ and is Hg-convex function, we write

$$\begin{aligned} 2f(x) &= f(x) + f\left(\frac{abx}{(a+b)x - ab}\right) \\ &= f\left(\frac{ab}{tb + (1-t)a}\right) + f\left(\frac{ab}{ta + (1-t)b}\right) \\ &\leq t[g(f(a))] + (1-t)[g(f(b))] + t[g(f(b))] + (1-t)[g(f(a))] \\ &= g(f(a)) + g(f(b)). \end{aligned}$$

This completes the proof of the theorem.

Theorem 8. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f, h: I \to \mathbb{R}$ be Hg-convex functions and $g: J \to \mathbb{R}$, $J \supset f(I) \cup h(I)$, $a, b \in I$ with $a < b$ and $f, h \in L[a,b]$. If the function f is symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality holds:

$$\begin{aligned} &\left[\frac{(g\circ f)(a) + (g\circ f)(b)}{2}\right]\left[\frac{(g\circ h)(a) + (g\circ h)(b)}{2}\right] + \frac{1}{b-a}\int_a^b f(x)h(x)dx \\ &\geq \frac{(g\circ f)(a) + (g\circ f)(b)}{2}\frac{1}{b-a}\int_a^b h(x)dx + \frac{(g\circ h)(a) + (g\circ h)(b)}{2}\frac{1}{b-a}\int_a^b f(x)dx. \end{aligned}$$

Proof: Since $f, h: I \to \mathbb{R}$ are Hg-convex functions, we write the following inequalities by using the inequality (4)

$$\frac{(g\circ f)(a) + (g\circ f)(b)}{2} - f(x) \geq 0$$

and

$$\frac{(g\circ h)(a) + (g\circ h)(b)}{2} - h(x) \geq 0.$$

Thus, we get

$$\begin{aligned} &\left[\frac{(g\circ f)(a) + (g\circ f)(b)}{2} - f(x)\right]\left[\frac{(g\circ h)(a) + (g\circ h)(b)}{2} - h(x)\right] \geq 0 \\ &\left[\frac{(g\circ f)(a) + (g\circ f)(b)}{2}\right]\left[\frac{(g\circ h)(a) + (g\circ h)(b)}{2}\right] + f(x)h(x) \\ &\geq \frac{(g\circ f)(a) + (g\circ f)(b)}{2}h(x) + \frac{(g\circ h)(a) + (g\circ h)(b)}{2}f(x) \end{aligned}$$

for all $x \in [a,b]$. If we integrate the last inequality on $t \in 0,1]$, we get

$$\begin{aligned} &\left[\frac{(g\circ f)(a) + (g\circ f)(b)}{2}\right]\left[\frac{(g\circ h)(a) + (g\circ h)(b)}{2}\right] + \frac{1}{b-a}\int_a^b f(x)h(x)dx \\ &\geq \frac{(g\circ f)(a) + (g\circ f)(b)}{2}\frac{1}{b-a}\int_a^b h(x)dx + \frac{(g\circ h)(a) + (g\circ h)(b)}{2}\frac{1}{b-a}\int_a^b f(x)dx. \end{aligned}$$

This completes the proof of the theorem.

Theorem 9. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f: I \to \mathbb{R}$ be Hg-convex functions and $g: J \to \mathbb{R}$, $J \supset f(I)$, $a, b \in I$ with $a < b$. If the function f is symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality holds for all $x \in [a, b]$:

$$f\left(\frac{2ab}{a+b}\right) \leq (g \circ f)(x).$$

Proof: Let $x \in [a, b]$. Then, there is at least $t \in [0,1]$ such that

$$x = \frac{ab}{tb + (1-t)a} = H_t.$$

Since the function f is symmetric with respect to $\frac{2ab}{a+b}$, we get

$$f\left(\frac{2ab}{a+b}\right) \leq f\left(\frac{2H_tH_{1-t}}{H_t + H_{1-t}}\right) \leq \frac{1}{2}(g \circ f)(H_t) + \frac{1}{2}(g \circ f)(H_{1-t}).$$

From here, we write

$$f(x) = f(H_t) = f(H_{1-t}).$$

This completes the proof of the theorem. With the help of the above theorem, the following theorem is obtained for two Hg-convex functions that are harmonically symmetric with respect to $\frac{2ab}{a+b}$.

Theorem 10. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f, h: I \to \mathbb{R}$ be Hg-convex functions and $g: J \to \mathbb{R}$, $J \supset f(I) \cup h(I)$, $a, b \in I$ with $a < b$ and $f, h \in L[a, b]$. If the functions f, h are symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality holds:

$$\frac{1}{b-a}\int_a^b (g \circ f)(x)(g \circ h)(x)dx + f\left(\frac{2ab}{a+b}\right)h\left(\frac{2ab}{a+b}\right)$$
$$\geq h\left(\frac{2ab}{a+b}\right)\frac{1}{b-a}\int_a^b (g \circ f)(x)dx + f\left(\frac{2ab}{a+b}\right)\frac{1}{b-a}\int_a^b (g \circ h)(x)dx.$$

Proof: Since $f, h: I \to \mathbb{R}$ are Hg-convex functions, we write the following inequalities by using the inequality (3.2)

$$(g \circ f)(x) - f\left(\frac{2ab}{a+b}\right) \geq 0$$

and

$$(g \circ h)(x) - h\left(\frac{2ab}{a+b}\right) \geq 0$$

for all $x \in [a, b]$. So, we get

$$\left[(g \circ f)(x) - f\left(\frac{2ab}{a+b}\right)\right]\left[(g \circ h)(x) - h\left(\frac{2ab}{a+b}\right)\right] \geq 0$$
$$(g \circ f)(x)(g \circ h)(x) + f\left(\frac{2ab}{a+b}\right)h\left(\frac{2ab}{a+b}\right) \geq (g \circ f)(x)h\left(\frac{2ab}{a+b}\right) + (g \circ h)(x)f\left(\frac{2ab}{a+b}\right).$$

If we take integral the last inequality on $t \in 0,1]$, we have

$$\frac{1}{b-a}\int_a^b (g\circ f)(x)(g\circ h)(x)dx + f\left(\frac{2ab}{a+b}\right)h\left(\frac{2ab}{a+b}\right)$$
$$\geq h\left(\frac{2ab}{a+b}\right)\frac{1}{b-a}\int_a^b (g\circ f)(x)dx + f\left(\frac{2ab}{a+b}\right)\frac{1}{b-a}\int_a^b (g\circ h)(x)dx.$$

This completes the proof of the theorem.

Theorem 11. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f: I \to \mathbb{R}$ be Hg-convex functions and $g: J \to \mathbb{R}$, $J \supset f(I)$, $a, b \in I$ with $a < b$. Then, the following inequality holds:

$$\frac{ab}{b-a}\int_a^b f(x)f\left(\frac{abx}{(a+b)x-ab}\right)\frac{1}{x^2}dx$$
$$\leq \frac{2}{3}\left[g\big(f(a)\big)g\big(f(b)\big)\right] + \frac{1}{6}\left[g^2\big(f(a)\big) + g^2\big(f(b)\big)\right] \tag{5}$$

for all $x \in [a, b]$.

Proof: If we change the variable $x = \frac{ab}{tb+(1-t)a}$ in the integral on the left side of inequality (5) and use the Hg-convexity of the function f, we get

$$\frac{ab}{b-a}\int_a^b f(x)f\left(\frac{abx}{(a+b)x-ab}\right)\frac{1}{x^2}dx$$
$$= \int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)f\left(\frac{ab}{ta+(1-t)b}\right)dt$$
$$\leq \int_0^1 \{t[gf(a)] + (1-t)[gf(b)]\}\{t[gf(b)] + (1-t)[gf(a)]\}dt$$
$$= \frac{2}{3}[g(f(a))g(f(b))] + \frac{1}{6}[g^2(f(a)) + g^2(f(b))].$$

This completes the proof of the theorem.

Remark 4. If we take the function f as harmonic symmetric with respect to $\frac{2ab}{a+b}$ in the above theorem, we get the following inequality:

$$\frac{ab}{b-a}\int_a^b \frac{f^2(x)}{x^2}dx \leq g^2(f(a)) = g^2(f(b)).$$

Theorem 12. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, $f, h: I \to [0,\infty)$ be Hg-convex functions and $g: J \to \mathbb{R}$, $J \supset f(I) \cup h(I)$, $a, b \in I$ with $a < b$ and $f, h \in L[a, b]$. Then the following inequality holds:

$$\frac{ab}{b-a}\int_a^b f(x)h(x)\frac{1}{x^2}dx \leq \frac{1}{3}M_{g,f,h}(a,b) + \frac{1}{6}N_{g,f,h}(a,b), \tag{6}$$

where

$$M_{g,f,h}(a,b) = g(f(a))g(h(a)) + g(f(b))g(h(b))$$

and

$$N_{g,f,h}(a,b) = g(f(a))g(h(b)) + g(f(b))g(h(a)).$$

Proof: If we change the variable $x = \frac{ab}{tb+(1-t)a}$ in the integral on the left side of inequality (6) and use the Hg-convexity of the f function, we obtain

$$\begin{aligned} &\frac{ab}{b-a}\int_a^b f(x)h(x)\frac{1}{x^2}dx \\ &= \int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)h\left(\frac{ab}{tb+(1-t)a}\right)dt \\ &\leq \int_0^1 \{t[g(f(a))] + (1-t)[g(f(b))]\}\{t[g(h(a))] + (1-t)[g(b(b))]\}dt \\ &= \frac{1}{3}[g(f(a))g(h(a)) + g(f(b))g(h(b))] + \frac{1}{6}[g(f(a))g(h(b)) + g(f(b))g(h(a))]. \end{aligned}$$

This completes the proof of the theorem.

Theorem 13. Let $I \subset \mathbb{R}\backslash\{0\}$ be an interval, f, $h: I \to [0,\infty)$ be Hg-convex functions and $g: [0,\infty) \to \mathbb{R}$, $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $a,b \in I$ with $a < b$ and $f^p, h^q \in L[a,b]$. If the functions f^p, h^q are Hg-convex functions on the interval $[a,b]$, then the following inequality holds:

$$\frac{ab}{b-a}\int_a^b f(x)h(x)\frac{1}{x^2}dx \leq \left(\frac{(g\circ f)^p(a) + (g\circ f)^p(b)}{2}\right)^{\frac{1}{p}}\left(\frac{(g\circ h)^q(a) + (g\circ h)^q(b)}{2}\right)^{\frac{1}{q}}.$$

Proof: If we first apply the Hölder integral inequality to the integral on the left side of (6) inequality, and then use the Hg-convexity of the function f, with the help of (3) inequality, we get

$$\begin{aligned} \frac{ab}{b-a}\int_a^b f(x)h(x)\frac{1}{x^2}dx &= \left(\frac{ab}{b-a}\int_a^b f^p(x)\frac{1}{x^2}dx\right)^{\frac{1}{p}}\left(\frac{ab}{b-a}\int_a^b h^q(x)\frac{1}{x^2}dx\right)^{\frac{1}{q}} \\ &\leq \left(\frac{(g\circ f)^p(a) + (g\circ f)^p(b)}{2}\right)^{\frac{1}{p}}\left(\frac{(g\circ h)^q(a) + (g\circ h)^q(b)}{2}\right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of the theorem.

4. HERMITE-HADAMARD TYPE INEQUALITIES FOR Hg-CONVEX FUNCTIONS

The main purpose of this paper is to prove some new integral inequalities for Hg-convex functions by using the Lemma 1.

Theorem 14. Let $f: I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$, $q \geq 1$ and $J \supset |f'|^q \in (I^\circ)$, $g: J \to (0,\infty)$. If the functions $f' \in L[a,b]$, $|f'|^q$ are Hg-convex functions on the interval $[a,b]$, then the following inequality holds for all $x \in [a,b]$:

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}[\lambda_2(g\circ|f'|^q)(a)+\lambda_3(g\circ|f'|^q)(b)]^{\frac{1}{q}}$$

where

$$\lambda_1 = \frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right)$$
$$\lambda_2 = \frac{-1}{b(b-a)}+\frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right)$$
$$\lambda_3 = \frac{1}{a(b-a)}-\frac{3a+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right) = \lambda_2-\lambda_1.$$

Proof: If we use the Lemma 1 and well known Power-mean integral inequality, then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{(tb+(1-t)a)^2}\right)\right|dt$$
$$\le \frac{ab(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{(tb+(1-t)a)^2}\right)\right|dt\right)^{\frac{1}{q}}.$$

By using the Hg-convexity of the function $|f'|^q$, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\le \frac{ab(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{|1-2t|[t(g\circ|f'|^q)(a)+(1-t)(g\circ|f'|^q)(b)]}{(tb+(1-t)a)^2}\right|dt\right)^{\frac{1}{q}}$$
$$\le \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}[\lambda_2(g\circ|f'|^q)(a)+\lambda_3(g\circ|f'|^q)(b)]^{\frac{1}{q}}.$$

This completes the proof of the the theorem.

Remark 5. If we take $g(x)=x$ in the Theorem 14, then we obtain the following inequality for the HA-convex functions.

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}[\lambda_2|f'|^q(a)+\lambda_3|f'|^q(b)]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [4].

Corollary 1. If we take $q=1$ in the Theorem 14, we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}[\lambda_2(g\circ|f'|)(a)+\lambda_3(g\circ|f'|)(b)].$$

Theorem 15. Let $f: I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $J\supset|f'|^q\in(I^\circ)$, $g: J\to(0,\infty)$. If the functions $f'\in L[a,b]$, $|f'|^q$ are Hg-convex functions on the interval $[a,b]$, then the following inequality holds for all $x\in[a,b]$:

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mu_1(g\circ|f'|^q)(a)+\mu_1(g\circ|f'|^q)(b)\right)^{\frac{1}{q}}, \tag{7}$$

where

$$\mu_1=\frac{(a^{2-2q}+b^{1-2q}[(b-a)(1-2q)-a])}{2(b-a)^2(1-q)(1-2q)}$$

and

$$\mu_2=\frac{(b^{2-2q}+a^{1-2q}[(b-a)(1-2q)-b])}{2(b-a)^2(1-q)(1-2q)}.$$

Proof: By using the Lemma 1, well known Hölder integral inequality and Hg-convexity of $|f'(x)|^q$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\int_0^1|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{(tb+(1-t)a)^2}\right)\right|^q dt\right)^{\frac{1}{q}} \tag{8}$$

$$\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1\left|\frac{t(g\circ|f'|^q)(a)+(1-t)(g\circ|f'|^q)(b)}{(tb+(1-t)a)^{2q}}\right|dt\right)^{\frac{1}{q}}.$$

By sample calculation give us that

$$\int_0^1\frac{t}{(tb+(1-t)a)^{2q}}dt=\frac{a^{2-2q}+b^{1-q}[(b-a)(1-2q)-a]}{2(b-a)^2(1-q)(1-2q)}, \tag{9}$$

$$\int_0^1\frac{1-2t}{(tb+(1-t)a)^{2q}}dt=\frac{b^{2-2q}+a^{1-q}[(b-a)(1-2q)+b]}{2(b-a)^2(1-q)(1-2q)}. \tag{10}$$

Substituting (9)-(10) in the inequality (8), the required result is obtained. This completes the proof of the theorem.

Remark 6. If we take $g(x)=x$ in the Theorem 15, then we obtain the following inequality for the HA-convex functions.

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}[\mu_1|f'|^q(a)+\mu_1|f'|^q(b)]^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [4].

5. CONCLUSION

In this study, the concept of Hg-convex function is given for the first time in the literature. Some inequalities of Hadamard's type for Hg-convex functions are given. Some properties of Hg-convex functions and special cases are discussed. In addition, Some new integral inequalities for Hg-convex functions are established by using an integral identity.

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