ORIGINAL PAPER

SEIDEL LAPLACIAN ENERGY of ZERO-DIVISOR GRAPH $\Gamma[\mathbb{Z}_n]$

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Abstract. *The Seidel Laplacian energy of graphs has recently been defined. In the present work, we compute the Seidel Laplacian energy of the zero-divisor graph* $\Gamma[\mathbb{Z}_n]$ *for* $n = p^2, n = pq, n = 2q$, *and* $n = p^3$, *where* p, q *are distinct prime numbers.*
Keywords: *Seidel Laplacian energy; zero-divisor graph; graph energy.*

1. INTRODUCTION AND PRELIMINARIES

Let G be a simple graph with n vertices and e edges. The vertices v_i and v_j are called adjacent and denoted by $v_i \sim v_j$ if they are joined by an edge. The degree of a vertex v_i is denoted by d_i. A complete graph has n vertices denoted by K_n, and its complement $\overline{K}_n$ has n vertices and no edges. A complete bipartite graph $K_{n,m}$ is a kind of bipartite graph that the set of vertices consists of two disjoint subsets of cardinality n and m, where two vertices in the same set are not adjacent, and every pair of vertices in the two sets are adjacent.

The zero-divisor graph of a commutative ring $\mathscr{R}$ was first defined in [1]. The standard definition of the zero-divisor graph $\Gamma[\mathscr{R}]$ of $\mathscr{R}$ was given in [2]. Let $Z(\mathscr{R})$ be the set of zero-divisors of $\mathscr{R}$. The vertex set of $\Gamma[\mathscr{R}]$ is $Z^*(\mathscr{R}) = Z(\mathscr{R}) \backslash \{0\}$ and $x \sim y$ iff $xy = 0$ for $x \neq y$ and $x, y \in Z^*(\mathscr{R})$.

The spectrum of a matrix is a multiset consisting of eigenvalues θ_i of multiplicities m_i $(1 \leq i \leq n)$ and are denoted by $\{\theta_1^{(m_1)}, \dots, \theta_n^{(m_n)}\}$. Consider the diagonal matrix $D(G) = diag(d_1, d_2, \dots, d_n)$. The Laplacian matrix $L(G)$ is known as $L(G) = D(G) - A(G)$, let ϑ_i $(1 \leq i \leq n)$ be its eigenvalues, briefly L-eigenvalues. Throughout the study, the expression Seidel Laplacian will denoted by SL. The SL matrix of G is defined in [3] as $SL = D_S - S$, where $D_S = diag(n - 1 - 2d_1, n - 1 - 2d_2, \dots, n - 1 - 2d_n)$ and S is the Seidel matrix of G. Moreover, the SL-energy of G is presented [3] as

$$E_{SL}(G) = \sum_{i=1}^{n} \left| \sigma_i - \frac{n(n-1) - 4e}{n} \right|,$$

where σ_i are the SL-eigenvalues of G. Let $N = \frac{n(n-1)-4e}{n}$. Thus, we have

$$E_{SL}(G) = \sum_{i=1}^{n} |\sigma_i - N|.$$

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Some bounds are presented for $E_{SL}(G)$ (see [3]), further studies on $E_{SL}(G)$ can be followed from [4-6].

Let $\mathbb{Z}_n$ be the commutative ring of all residue classes of integers modulo n. The matrix representations, energies, and topological indices of commutative rings, especially the zero divisor graph of $\mathbb{Z}_n$, are some of the most studied subjects in recent years. The energy and Wiener index of $\Gamma[\mathbb{Z}_n]$ are computed for $n = pq$ and $n = p^2$, where p, q are primes [7]. The adjacency matrix of $\Gamma[\mathbb{Z}_n]$ is considered for $n = p^2q$ and $n = p^3$, where p, q are primes and the Wiener index of $\Gamma[\mathbb{Z}_n]$ is calculated [8], for more work refer to [9-10]. Further, in [11], the degree distance of $\Gamma[\mathbb{Z}_n]$ is computed. The L-eigenvalues of $\Gamma[\mathbb{Z}_n]$ is studied in detail (see [12-13]). These studies establish a connection between $\Gamma[\mathbb{Z}_n]$ and spectral graph theory and are the source of our motivation.

In the present work, we calculate the SL-energy of $\Gamma[\mathbb{Z}_n]$ for $n = p^2$, $n = pq$, $n = 2q$, $n = p^3$. Now, we state the essential lemmas.

Lemma 1.1. ([7])
i. If $n = p^2$ ($p > 2$ is a prime), then $Z^*(\mathbb{Z}_{p^2}) = \{p, 2p, \dots, (p-1)p^2\}$. For any $x, y \in Z^*(\mathbb{Z}_{p^2})$, $xy = 0$, and we have $\Gamma[\mathbb{Z}_{p^2}] \cong K_{p-1}$.
ii. If $n = pq$ such that p, q are distinct primes, then $Z^*(\mathbb{Z}_{pq}) = B \cup C$, where $B = \{pt: t = 1,2, \dots, q-1\}$ and $C = \{qt: t = 1,2, \dots, p-1\}$. For any $x, y \in Z^*(\mathbb{Z}_{pq})$, $xy = 0$ iff $x \in B, y \in C$ or $x \in C, y \in B$, and $\Gamma[\mathbb{Z}_{pq}] \cong K_{p-1,q-1}$.

Lemma 1.2. ([14]) Let $n = p^3$. Then, $\Gamma[\mathbb{Z}_{p^3}] \cong K_{p-1} + \overline{K}_{p^2-p}$, which is the complete split graph with $p^2 - 1$ vertices.

Lemma 1.3. ([6]) If $\vartheta_1, \vartheta_2, \dots, \vartheta_n$ be the L-eigenvalues of G, then the SL-eigenvalues of G are $n - 2\vartheta_i$ for $1 \le i \le n-1$ and 0, for $i = n$ (see [4]).

Lemma 1.4. ([6]) Let $K_{m,n}$ $(1 < n < m)$ be a complete bipartite graph. Then,

$$E_{SL}(K_{m,n}) = \begin{cases} \dfrac{4(m^2n - n^2m + n^2) + 2(m^2 - m - n - mn)}{m+n}, & N > 0 \\ \dfrac{4(m^2n - n^2m) + 2(n^2 + mn)}{m+n}, & N < 0. \end{cases}$$

Lemma 1.5. ([6]) Let S_n $(n \ge 4)$ be a star. Then, $E_{SL}(S_n) = 6n + \frac{16}{n} - 20$.

2. SL-ENERGY of $\Gamma[\mathbb{Z}_n]$

The structure of the zero-divisor graph $\Gamma[\mathbb{Z}_n]$ is stated in the previous section for specific values of n. In this section, we will compute the SL-energy of $\Gamma[\mathbb{Z}_n]$ for $n = p^2$, $n = pq$, $n = 2q$ and $n = p^3$, where p, q are distinct prime numbers. First, we can give the SL-energy of $\Gamma[\mathbb{Z}_{2q}]$.

Corollary 2.1. Let $q > 3$ be a prime. The SL-energy of $\Gamma[\mathbb{Z}_{2q}]$ is $E_{SL}(\Gamma[\mathbb{Z}_{2q}]) = 6q + \frac{16}{q} - 20$.

Proof: The zero-divisor graph for a prime $q > 3$ is the star $K_{1,q-1}$. Setting $n = q$ in Lemma 1.5 leads to the conclusion.

Theorem 2.1. The SL-energy of $\Gamma[\mathbb{Z}_{p^2}]$ is $E_{SL}(\Gamma[\mathbb{Z}_{p^2}]) = 2(p-2)$.

Proof: From Lemma 1.1, $\Gamma[\mathbb{Z}_{p^2}] \cong K_{p-1}$. Clearly $N = \frac{(p-1)(p-2)-2(p-1)(p-2)}{p-1} = -(p-2)$. The Laplacian spectrum of K_{p-1} is $\{0^{(1)}, (p-1)^{(p-2)}\}$. Using this fact in Lemma 1.3 determines the SL spectrum of K_{p-1} as $\{0^{(1)}, -(p-1)^{(p-2)}\}$. Then,

$$E_{SL}(\Gamma[\mathbb{Z}_{p^2}]) = |0-N| + (p-2)|-(p-1)-N|$$

$$= |p-2| + (p-2)|-1|$$

$$= 2(p-2).$$

Now, we compute the SL-energy of the zero-divisor graph $\Gamma[\mathbb{Z}_{p^3}]$.

Theorem 2.2. Let $p > 2$ be a prime. Then, the SL-energy of $\Gamma[\mathbb{Z}_{p^3}]$ is

$$E_{SL}(\Gamma[\mathbb{Z}_{p^3}]) = \frac{4p^4 - 8p^3 - 2p^2 + 5p + 1}{p+1}.$$

Proof: By Lemma 1.2, we have $\Gamma[\mathbb{Z}_{p^3}] \cong K_{p-1} + \overline{K}_{p^2-p}$, which is the complete split graph with $p^2 - 1$ vertices, $e = \frac{(p-1)(p-2)}{2} + p(p-1)^2$ edges. We have $\frac{2e}{n} = \frac{2p^2-p-2}{p+1}$. Then,

$$N = n - 1 - \frac{4e}{n} = p^2 - 2 - \frac{2(2p^2 - p - 2)}{p+1} = \frac{p^3 - 3p^2 + 2}{p+1}.$$

The Laplacian spectrum of $\Gamma[\mathbb{Z}_{p^3}]$ is $\{0^{(1)}, (p-1)^{(p^2-p-1)}, (p^2-1)^{(p-2)}\}$ (see Corollary 10, [13]). Then, by Lemma 1.3, its SL spectrum is $\{(0)^{(1)}, [(p-1)^2]^{(p^2-p-1)}, (1-p^2)^{(p-2)}\}$. Thus, we get

$$E_{SL}(\Gamma[\mathbb{Z}_{p^3}]) = \left|0 - \frac{p^3-3p^2+2}{p+1}\right| + (p^2-p-1)\left|(p-1)^2 - \frac{p^3-3p^2+2}{p+1}\right|$$

$$+(p-2)\left|1 - p^2 - \frac{p^3-3p^2+2}{p+1}\right|$$

$$= \frac{p^3-3p^2+2}{p+1} + (p^2-p-1)\left|\frac{2p^2-p-1}{p+1}\right|$$

$$+(p-2)\left|\frac{-2p^3+2p^2+p-1}{p+1}\right|.$$

By considering $p^3-3p^2+2>0,\ 2p^2-p-1>0$, and $-2p^3+2p^2+p-1<0$ results as

$$E_{SL}(\Gamma[\mathbb{Z}_{p^3}])=\frac{p^3-3p^2+2+(p^2-p-1)(2p^2-p-1)+(p-2)(2p^3-2p^2-p+1)}{p+1}$$

$$=\frac{p^3-3p^2+2+2p^4-3p^3-2p^2+2p+1+2p^4-6p^3+3p^2+3p-2}{p+1}$$

$$=\frac{4p^4-8p^3-2p^2+5p+1}{p+1}.$$

Finally, we can give the SL-energy of $\Gamma[\mathbb{Z}_{pq}]$ as follows.

Theorem 2.3. Let $q<p$. Then, the SL-energy of $\Gamma[\mathbb{Z}_{pq}]$ is

$$E_{SL}(\Gamma[\mathbb{Z}_{pq}])=\begin{cases}\frac{2\left[(p-1)^2+1-pq+2(q-1)[(p-q)^2+(q-1)(p-q+1)]\right]}{p+q-2}, & N>0\\ \frac{4(p-1)(q-1)(p-q)+2(q-1)^2}{p+q-2}, & N<0.\end{cases}$$

Proof: By Lemma 1.1, $\Gamma[\mathbb{Z}_{pq}]\cong K_{p-1,q-1}$. So, $E_{SL}(\Gamma[\mathbb{Z}_{pq}])=E_{SL}(K_{p-1,q-1})$. Clearly $K_{p-1,q-1}$ has $p+q-2$ vertices and $e=(p-1)(q-1)$ edges. Thus, $N=\frac{(p+q-2)(p+q-3)-4(p-1)(q-1)}{p+q-2}=\frac{(p-q)^2-p-q+2}{p+q-2}$. Setting $m=p-1, n=q-1$ when $N>0$ in Lemma 1.4, we get

$$E_{SL}(\Gamma[\mathbb{Z}_{pq}])=\frac{4[(p-1)^2(q-1)-(q-1)^2(p-1)+(q-1)^2]}{p+q-2}$$

$$+\frac{2[(p-1)^2-(p-1)-(q-1)-(p-1)(q-1)]}{p+q-2}$$

$$=\frac{4\left[(p-1)(q-1)\big(p-1-(q-1)\big)+(q-1)^2\right]}{p+q-2}$$

$$+\frac{2[(p-1)(p-2)-(q-1)(1+(p-1))]}{p+q-2}$$

$$=\frac{4(q-1)[(p-1)(p-q)+(q-1)]+2[p^2-3p+2-(q-1)p]}{p+q-2}$$

$$=\frac{4(q-1)(p^2-pq-p+2q-1)+2(p^2-2p+2-pq)}{p+q-2}$$

$$=\frac{4(q-1)(p^2-2pq+pq+q^2-q^2-p+2q-1)+2[(p-1)^2+1-pq]}{p+q-2}$$

$$= \frac{4(q-1)[(p-q)^2 + q(p-q) + 2q - p - 1] + 2[(p-1)^2 + 1 - pq]}{p+q-2}$$

$$= \frac{4(q-1)((p-q)^2 + q(p-q) - (p-q) + q - 1) + 2[(p-1)^2 + 1 - pq]}{p+q-2}$$

$$= \frac{4(q-1)[(p-q)^2 + (p-q)(q-1) + q - 1] + 2[(p-1)^2 + 1 - pq]}{p+q-2}$$

$$= \frac{4(q-1)[(p-q)^2 + (q-1)(p-q+1)] + 2[(p-1)^2 + 1 - pq]}{p+q-2}.$$

Likewise, for $N < 0$ in Lemma 1.4, we have

$$E_{SL}(\Gamma[\mathbb{Z}_{pq}]) = \frac{4[(p-1)^2(q-1) - (q-1)^2(p-1)] + 2[(q-1)^2 + (p-1)(q-1)]}{p+q-2}$$

$$= \frac{2(p-1)(q-1)[2(p-1) - 2(q-1) + 1] + 2(q-1)^2}{p+q-2}$$

$$= \frac{4(p-1)(q-1)(p-q) + 2(q-1)^2}{p+q-2},$$

which yields the result.

3. CONCLUSION

Studies in recent years show that the matrix representations of the zero-divisor graph $\Gamma[\mathbb{Z}_n]$ have been extensively worked. Several topological indices and types of energies are computed for certain values of n. In this work, the Seidel Laplacian energy of the zero-divisor graph $\Gamma[\mathbb{Z}_n]$ is computed for $n = p^2$, $n = pq$, $n = 2q$, $n = p^3$.

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