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EXISTENCE OF SOLUTIONS FOR NONHOMOGENEOUS DIRICHLET PROBLEMS IN ORLICZ-SOBOLEV SPACES

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Abstract. *In this paper, by using variational methods and critical point theory in an appropriate Orlicz-Sobolev space, we establish the existence of infinitely many nontrivial solutions to a nonhomogeneous problem. Precisely, we use the* $\mathbb{Z}_2$*-symmetric version for the well-known Mountain Pass theorem, to prove the existence of such solutions.*

Keywords: *Nonhomogeneous problem;Orlicz-Sobolev space; critical points; variational methods.*

1. INTRODUCTION

In this paper, we study the following nonhomogeneous problem

$$(P_\lambda)\begin{cases}-\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x)) + \alpha(|u(x)|)u(x) = \lambda f(x,u(x)) - g(x)|u(x)|^{q(x)-2}u(x), x \in \Omega,\\ u(x) = 0, x \in \partial\Omega,\end{cases}$$

where $\lambda > 0, \Omega$ is a bounded domain in $\mathbb{R}^N(N \geq 2), g \in L^\infty(\Omega)$ and $f: \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function which satisfies some additional conditions. The function $\alpha: (0, \infty) \longrightarrow \mathbb{R}$ is such that the mapping $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$, defined by

$$\varphi(t) = \begin{cases}\alpha(|t|)t, & \text{for } t \neq 0,\\ 0, & \text{for } t = 0,\end{cases} \tag{1.1}$$

is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

Problem (P_λ) appears in many branches of mathematics such as partial differential equations, calculus of variations, quasi-conformal mappings, non-Newtonian fluids, image processing, differential geometry, and probability theory we refer the interested readers to [1-2], and the references therein. In the case when $\alpha(t) = t^{p-2}(p > 1)$, the operator used in problem (P_λ) becomes the well-known p-Laplacian operator. This means that some complicated analysis has to be carefully carried out in this paper. The most adequate functional framework to deal with this problem is an appropriate Orlicz-Sobolev space, which is a Sobolev space constructed from an Orlicz L_Φ space instead of L^p space. So, the Orlicz-Lebesgue spaces are a generalization of the classical Lebesgue spaces. Also, we define the

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Orlicz-Sobolev space $W^m L_\Phi(\Omega)$, which are a generalization of the classical Sobolev spaces $W^{m,p}(\Omega)$. Many properties of Lebesgue and Sobolev spaces have been extended to Orlicz-Sobolev space see for example [3-7].

Very recently, many researchers have paid their attention to studying the existence of solutions for several problems through Orlicz-Sobolev space see [8-15] and references therein. At this point, we briefly recall literature concerning related elliptic problems in Orlicz-Sobolev space. In [16], Clément et al. studied the following Dirichlet problem

$$\begin{cases} -\operatorname{div}\big(\alpha(|\nabla u(x)|)\nabla u(x)\big) = f\big(x,u(x)\big), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and the function $s \longmapsto s\alpha(s)$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. By using Mountain Pass geometry in an Orlicz-Sobolev space, the authors investigated the existence of nontrivial solutions to the problem (1.2). Later, Clément et al. studied the problem (1.2). By using Orlicz-Sobolev spaces theory combined with a variant of the Mountain Pass lemma of Ambrosetti-Rabinowitz, the authors proved the existence of a nontrivial solution for such a problem. Mihailescu and Radulescu [17], employed Orlicz-Sobolev space theory and variational methods to investigate the existence and multiplicity of solutions for the following problem

$$\begin{cases} -\operatorname{div}\big(\alpha(|\nabla u(x)|)\nabla u(x)\big) = f\big(u(x)\big), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases}$$

Motivated by the above-mentioned works, in this paper, by combining Orlicz-Sobolev spaces theory with adequate variational methods and $\mathbb{Z}_2$-symmetric version (for even functionals) of the Mountain Pass Lemma (see Theorem 9.12 in [18]), we establish the existence of infinitely many solutions for the problem (P_λ) as λ is positive and small enough.
The rest of this paper is organized as follows: In Section 2, we present some preliminaries on function spaces and variational settings. In Section 3, we present and prove the main result of this paper.

2. PRELIMINARIES

We recall in what follows some definitions and basic properties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Sobolev spaces $W^{1,p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. In that context we refer to the book of Musielak [19] and the paper of Fan et al. [20].

Set

$$C_+(\bar{\Omega}) = \{h; h \in C(\bar{\Omega}), h(x) > 1 \text{ for all } x \in \bar{\Omega}\}.$$

For any $p \in C_+(\bar{\Omega})$ such that

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty, \tag{2.1}$$

we define the variable exponent Lebesgue space $L^{p(x)}(\Omega) = \{u$: is a Borel real-valued function on Ω: $\int_\Omega |u(x)|^{p(x)} dx < \infty\big\}$.

We recall the following so-called Luxemburg norm on this space defined by the formula

$$|u|_{p(x)} := \inf\left\{\mu > 0; \int_{\Omega} \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble to classical Lebesgue spaces in many respects: It is well known [21] that, in view of (2.1), $L^{p(x)}(\Omega)$ equipped with the above norm, is separable, reflexive, uniformly convex Banach space ([22], Theorem 2.1). The inclusion between Lebesgue spaces is also naturally generalized [22], Theorem 2.8: if $0 < |\Omega| < \infty$ and r_1, r_2 are variables exponents so that $r_1(x) \leq r_2(x)$ almost every where in Ω then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$|\int_{\Omega} \mathrm{uvdx}| \leq (\frac{1}{\mathrm{p}^-} + \frac{1}{\mathrm{p}'^-})|\mathrm{u}|_{p(x)}\ |\mathrm{v}|_{\mathrm{p(x)}'} \tag{2.2}$$

holds.

An important role in manipulating the generalized Lebesgue-Sobolev space is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)}: L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx,$$

and it satisfies the following proposition:

Proposition 2.1. (See [23]) For all $u, v \in L^{p(x)}(\Omega)$, we have
(1) $|u|_{p(x)} < 1$(resp. $= 1, > 1$) $\Leftrightarrow \rho_{p(x)}(u) < 1$ (resp.$= 1, > 1$),
(2) $\min\left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right) \leq \rho_{p(x)}(u) \leq \max\left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right)$,
(3) $\rho_{p(x)}(u - v) \to 0 \Leftrightarrow |u - v|_{p(x)} \to 0$.

We also mention the following important proposition:

Proposition 2.2. (See [24]) Let p and q be two measurable functions such that $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$ and $p \in L^{\infty}(\Omega)$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then
$\min\left(|u|_{p(x)q(x)}^{p^+}, |u|_{p(x)q(x)}^{p^-}\right) \leq \left||u|^{p(x)}\right|_{q(x)} \leq \max\left(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+}\right)$.

The Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

and equipped with the norm

$$\| u \|_{1,p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Since the operator in the divergence form is non-homogeneous, then we introduce an Orlicz-Sobolev space setting for problems of this type. We first recall some basic facts about Orlicz-Sobolev space.

Define

$$\Phi(t) := \int_0^t \varphi(s)ds, \quad \Phi^*(t) := \int_0^t \varphi^{-1}(s)ds, \quad \text{for all } t \in \mathbb{R}.$$

We observe that Φ is a Young function, that is, $\Phi(0) = 0, \Phi$ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0 \text{ and } \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty,$$

then Φ is called an N-function. The function Φ^* is called the complementary function of Φ and it satisfies

$$\Phi^*(t) = \sup\{st - \Phi(s); s \geq 0\}, \quad \text{for all } t \geq 0.$$

We observe that Φ^* is also an N-function and the following Young's inequality holds true:

$$st \leq \Phi(s) + \Phi^*(t), \quad \text{for all } s, t \geq 0.$$

Assume that Φ satisfies the following structural hypotheses

$$1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \leq p^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty, \tag{2.3}$$

$$N < p_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \leq \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{2.4}$$

The Orlicz space $L_\Phi(\Omega)$ defined by the N-function Φ is the space of measurable functions $u: \Omega \to \mathbb{R}$ such that

$$\| u \|_{L_\Phi} := \sup\left\{\int_\Omega u(x)v(x)dx; \int_\Omega \Phi^*(|v(x)|)dx < 1\right\} < \infty.$$

Then $\left(L_\Phi(\Omega), \|\cdot\|_{L_\Phi}\right)$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\| u \|_\Phi = \inf\left\{\mu > 0: \int_\Omega \Phi\left(\frac{u(x)}{\mu}\right) dx \leq 1\right\}.$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz-Sobolev space for the problem (P_λ), defined by

$$W^1L_\Phi(\Omega) := \left\{u \in L_\Phi(\Omega); \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), i = 1, \dots, N\right\}.$$

This is a Banach space with respect to the norm

$$\| u \|_{1,\Phi} := \| u \|_\Phi + \| |\nabla u| \|_\Phi,$$

for more details about this space, see [25].

As mentioned in [26-27], assumption (Φ_0) is equivalent to the fact that both of Φ and Φ^* satisfy the Δ_2-condition (at infinity), see ([28], p.232). In particular, (Φ, Ω) and (Φ^*, Ω) are Δ-regular.

These spaces generalize the usual spaces $L^p(\Omega)$and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t \mapsto \frac{|t|^p}{p}$ is assumed by a more general convex function Φ.

We also define the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1 L_\Phi(\Omega)$. By Lemma 5.7 in [29], $W_0^1 L_\Phi(\Omega)$ can be equivalent to the following norm

$$\| u \|_{1,\Phi} := \| |\nabla u| \|_\Phi.$$

The space $W_0^1 L_\Phi(\Omega)$ is also a reflexive Banach space. In many applications of Orlicz-Sobolev space to boundary value problems for nonlinear partial differential equations, the compactness of the embeddings plays a central role. Compact embedding theorems for Sobolev or Orlicz-Sobolev space are also intimately connected with the problem of discreteness of spectra of Schrödinger operators (see Benci and Fortunato [30]).

We recall the following useful two lemmas regarding the norms on Orlicz-Sobolev space.

Lemma 2.1. On $W_0^1 L_\Phi(\Omega)$ the norms

$$\| u \|_{1,\Phi} = \| |\nabla u \|_\Phi + \| u \|_\Phi,$$
$$\| u \|_{2,\Phi} = \max\{\| |\nabla u| \|_\Phi, \| u \|_\Phi\},$$
$$\| u \| = \inf\left\{\mu > 0: \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right)\right] dx \leq 1\right\},$$

are equivalent. More precisely, for every $u \in W_0^1 L_\Phi(\Omega)$ we have

$$\| u \| \leq 2 \| u \|_{2,\Phi} \leq 2 \| u \|_{1,\Phi} \leq 4 \| u \|.$$

Lemma 2.2.([31], Lemma 2.3). Put

$$D(u) = \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)] dx.$$

For $u \in W_0^1 L_\Phi(\Omega)$, we have

(1) $\| u \| < 1 \Rightarrow \| u \|^{p^0} \leq D(u) \leq \| u \|^{p_0}$,

(2) $\| u \| > 1 \Rightarrow \| u \|^{p_0} \leq D(u) \leq \| u \|^{p^0}$.

Definition 2.1. Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the (PS) condition if any sequence $\{u_n\} \subset X$, such that

$$\varphi(u_n) \text{ is bounded, and } \varphi'(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty,$$

contains a convergent subsequence.

In order to prove that the problem (P_λ) has infinitely many solutions, we will use the following $\mathbb{Z}_2$-symmetric version of the Mountain Pass theorem (see Theorem 9.12 in [32]).

Theorem 2.1. Let X be an infinite dimensional real Banach space. If $\phi \in C^1(X, \mathbb{R})$, is such that the following conditions are satisfied:

- ϕ is an even functional such that $\phi(0) = 0$.
- ϕ satisfies the (PS)-condition.
- There exist positive constants ρ_0 and α_0, such that if $\| u \| = \rho_0$, then, $\phi(u) \geq \alpha_0$.
- For each finite dimensional subspace $X_1 \subset X$, the set

$$\{u \in X_1, \phi(u) \geq 0\},$$

is bounded in X. Then ϕ has an unbounded sequence of critical values.

Throughout the rest of the paper, the letters $C_i, i = 1,2, ...,$ denote positive constants which may change from line to line.

3. MAIN RESULT AND ITS PROOF

In this section, we will present and prove the multiplicity of solutions to the problem (P_λ). Precisely, by using the $\mathbb{Z}_2$-symmetric version of the Mountain Pass Theorem, we will prove the existence of an unbounded sequence of critical values of the functional energy which are solutions to the corresponding problem.

In order to prove the existence of such solutions, we assume the following hypothesis:
(H_1) The functions r and q are in $C_+(\bar{\Omega})$, and satisfy equation (2.1).
(H_2) There exist $C_0 > 0$ and $K \in L^{\frac{\theta(x)}{\theta(x)-1}}(\Omega)$, such that, for all $(x, u) \in \bar{\Omega} \times \mathbb{R}$, we have

$$C_0 |t|^{r(x)-2} t \leq f(x, t) \leq K(x) |t|^{r(x)-2} t,$$

where $\theta \in C(\bar{\Omega})$, is such that for all $x \in \Omega$, we have

$$1 < r(x) < p(x) < N < \theta'(x), \tag{3.1}$$

and

$$1 < r(x)\theta'(x) < p_0^*\,, \tag{3.2}$$

where $\theta'(x) = \frac{\theta(x)}{\theta(x)-1}$.

In this paper, our main result is the following:

Theorem 3.1. Under hypothesis $(H_1) - (H_2)$. If $1 < p^0 < q^+ < r^-$, then, for each $\lambda > 0$, the problem (P_λ) has infinitely many solutions.

Before proving Theorem 3.1, we need to prove several lemmas. First, let as introduce the variational setting of the problem (P_λ).

Definition 3.1. We say that a function $u: \Omega \to \mathbb{R}$, is a weak solution for the problem (P_λ) if for every $\psi \in E: = W_0^1 L_\Phi(\Omega)$, we have

$$\int_\Omega \alpha(|\nabla u(x)|) \nabla u(x) \cdot \nabla \psi(x) dx + \int_\Omega \alpha(|u(x)|) u(x) \psi(x) dx$$
$$-\lambda \int_\Omega f(x, u(x)) \psi(x) dx + \int_\Omega g(x) |u(x)|^{q(x)-2} u(x) \psi(x) dx = 0.$$

Associated to problem (P_λ), we define the energy functional $J_\lambda : E \to \mathbb{R}$, by

$$J_\lambda(u) := I(u) - \lambda J_1(u) + J_2(u),$$

where

$$I(u) := \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx,$$
$$J_1(u) = \int_\Omega F\big(x, u(x)\big)dx,$$
$$J_2(u) = \int_\Omega \frac{g(x)}{q(x)}|u|^{q(x)}dx,$$

and

$$F(x,\xi) = \int_0^\xi f(x,t)dt.$$

By hypothesis (H_2) we see that for all x in Ω, we have

$$F(x,u(x)) \le C\frac{|K_1(x)|}{r(x)}|u|^{r(x)}, \tag{3.3}$$

where $K_1(x) = \max(C_0, K(x))$. We note that since Ω is bounded, then, $K_1 \in L^{\frac{\theta(x)}{\theta(x)-1}}(\Omega)$.

By standard arguments, we obtain $J_\lambda \in C^1(E,\mathbb{R})$. Moreover, for all $u, \psi \in E$, we have

$$\begin{aligned}\langle dJ_\lambda(u),\psi\rangle &= \lim_{t\to 0^+}\frac{J_\lambda(u+t\psi) - J_\lambda(u)}{t},\\ &= \int_\Omega \alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla\psi(x)dx + \int_\Omega \alpha(|u(x)|)u(x)\psi(x)dx\\ &-\lambda\int_\Omega f(x,u(x))\psi(x)dx + \int_\Omega g(x)|u(x)|^{q(x)-2}u(x)\psi(x)dx.\end{aligned}$$

Thus, according to Definition 3.1, weak solutions of problem (P_λ) coincide with critical points of the functional J_λ.

Lemma 3.1. Assume that hypothesis $(H_1)-(H_2)$ are satisfied. If $1 < p^0 < r^-$ then, for all $\lambda > 0$, there exist $\varrho > 0$ and $0 < \delta < 1$, such that for all $u \in E$, we have

$$\| u \| = \delta \Longrightarrow J_\lambda(u) \ge \varrho > 0.$$

Proof. Let $u \in E$, with $\| u \| < 1$. Using Proposition 2.2, the Hölder inequality and (3.3), we obtain

$$J_\lambda(u) \ge \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx - \lambda\int_\Omega F\big(x,u(x)\big)dx,$$

$$\ge \| u \|^{p^0} - \frac{\lambda}{r^-}\int_\Omega K(x)|u|^{r(x)}dx,$$

$$\geq \| u \|^{p^0} - \frac{\lambda C_1}{r^-} |K|_{\theta'(x)} \left| |u|^{r(x)} \right|_{\theta(x)},$$

$$\geq \| u \|^{p^0} - \frac{\lambda C_1}{r^-} |K|_{\theta'(x)} \max\left(|u|^{r^+}_{r(x)\theta(x)}, |u|^{r^-}_{r(x)\theta(x)} \right),$$

$$\geq \| u \|^{p^0} - \frac{\lambda C_1}{r^-} |K|_{\theta'(x)} \max\left(C_2^{r^+} \| u \|^{r^+}, C_2^{r^-} \| u \|^{r^-} \right),$$

$$\geq \| u \|^{p^0} - C_1 \frac{\lambda}{r^-} |K|_{\theta'(x)} \max\left(\| u \|^{r^+}, \| u \|^{r^-} \right),$$

$$= \| u \|^{p^0} - C_1 \frac{\lambda}{r^-} |K|_{\theta'(x)} \| u \|^{r^-}.$$

Put

$$h_\lambda(t) = t^{p^0} - C_1 \frac{\lambda}{r^-} |K|_{\theta'(x)} t^{r^-}, t > 0.$$

Since $p^0 < r^-$, then, it is easy to see that $h_\lambda(t) > 0$ for all $t \in (0, t_1)$, where

$$t_1 = \min\left\{ \left(\frac{r^-}{\lambda C_1 |K|_{\theta'(x)}} \right)^{\frac{1}{r^- - p^0}}, 1 \right\}.$$

If we fix $0 < \delta < t_1$ and we put

$$\varrho = \delta^{p^0} - C_1 \frac{\lambda}{r^-} |K|_{\theta'(x)} \delta^{r^-},$$

then, for all $\lambda > 0$, we have

$$J_\lambda(u) \geq \varrho > 0 \text{ for all } u \in E \text{ with } \| u \| = \delta.$$

The proof of Lemma 3.1 is now completed.

Lemma 3.2. Assume that we are under hypothesis of Theorem 3.1. If $E_1 \subset E$ is a finite dimensional subspace, then the set $S = \{u \in E_1; J_\lambda(u) \geq 0\}$ is bounded in E.

Proof. From the definition of p^0, we have,

$$\frac{\varphi(t)}{\Phi(t)} \leq \frac{p^0}{t}, \forall t > 0.$$

So for all $\sigma > 1$, we have

$$\log(\Phi(t)) - \log(\Phi(t/\sigma)) = \int_{t/\sigma}^{t} \frac{\varphi(s)}{\Phi(s)} ds \leq \int_{t/\sigma}^{t} \frac{p^0}{s} ds = \log\left(\sigma^{p^0}\right).$$

Therefore, for all $t > 1$, we get

$$\Phi(\sigma t) \leq \sigma^{p^0}\Phi(t). \tag{3.4}$$

Hence, if $u \in E_1$ with $\| u \| > 1$. Then, we have

$$\begin{aligned}\int_\Omega \Phi(|\nabla u(x)|)dx &= \int_\Omega \Phi\left(\| u \| \frac{|\nabla u(x)|}{\| u \|}\right)dx,\\ &\leq \| u \|^{p^0}\int_\Omega \Phi\left(\frac{|\nabla u(x)|}{\| u \|}\right)dx,\\ &\leq \| u \|^{p^0}.\end{aligned} \tag{3.5}$$

On the other hand, from Proposition 2.1 and the embedding from E into $L^{q(x)}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega \frac{g(x)}{q(x)}|u(x)|^{q(x)}dx &\leq \frac{\| g \|_\infty}{q^-}\int_\Omega |u(x)|^{q(x)}dx,\\ &\leq C_1\frac{\| g \|_\infty}{q^-}\left(\| u \|^{q^-} + \| u \|^{q^+}\right),\\ &= C_1\left(\| u \|^{q^-} + \| u \|^{q^+}\right).\end{aligned} \tag{3.6}$$

Now, by combining equations (3.5), and (3.6) with the hypothesis (H_2) and the fact that all norms are equivalent for a finite-dimensional space, we obtain

$$\begin{aligned}J(u) &= \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx + \int_\Omega \frac{g(x)}{q(x)}|u(x)|^{q(x)}dx - \lambda\int_\Omega F\big(x, u(x)\big)dx,\\ &\leq \| u \|^{p^0} + C_1\left(\| u \|^{q^-} + \| u \|^{q^+}\right) - \lambda C_2\int_\Omega |u|^{r(x)}dx,\\ &\leq \| u \|^{p^0} + C_1\left(\| u \|^{q^-} + \| u \|^{q^+}\right) - \lambda C_2 \min\left(\| u \|^{r^-}, \| u \|^{r^+}\right),\\ &\leq \| u \|^{p^0} + C_1\left(\| u \|^{q^-} + \| u \|^{q^+}\right) - \lambda C_2 \| u \|^{r^-}.\end{aligned}$$

Hence, for all $u \in S$ with $\| u \| > 1$, we have

$$\| u \|^{r^-} \leq C_1 \| u \|^{p^0} + C_2\left(\| u \|^{q^-} + \| u \|^{q^+}\right).$$

Since $1 < p^0 < q^+ < r^-$, then, we conclude that S is bounded in E. The proof of Lemma 3.2 is completed.

Lemma 3.3. Let (u_n) be a bounded sequence in E. Assume that there exists a subsequence still denoted by (u_n), such that $u_n \rightharpoonup \tilde{u}$, weakly in E. Then the following statements hold true:

$$\lim_{n\to\infty}\int_\Omega |u_n|^{q(x)-2}u_n(u_n-\tilde{u})dx = 0, (i)$$
$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-\tilde{u})dx = 0. (ii)$$

Proof. (i) Using the Hölder inequality and Proposition 2.2, we have

$$\int_\Omega |u_n|^{q(x)-2}u_n(u_n-\tilde{u})dx \leq ||u_n|^{q(x)-2}u_n|_{\frac{q(x)}{q(x)-1}}|u_n-\tilde{u}|_{q(x)},$$
$$\leq C_1 \max\left(|u_n|_{q(x)}^{q^- -1}, |u_n|_{q(x)}^{q^+ -1}\right)|u_n-\tilde{u}|_{q(x)}.$$

By letting n tends to infinity, we deduce the first part of Lemma 3.3.

(ii) From the Hölder inequality and hypothesis (H_2), we have

$$\int_\Omega f(x,u_n)(u_n-\tilde{u})dx \leq \int_\Omega K(x)|u_n|^{r(x)-2}u_n(u_n-\tilde{u})dx,$$
$$\leq C_1 |K(x)|u_n|^{r(x)-1}|_{\frac{r(x)}{r(x)-1}}|u_n-\tilde{u}|_{r(x)},$$
$$\leq C_1 |K|_{\theta'(x)}\left||u_n|^{r(x)-1}\right|_{\frac{\theta(x)}{\theta(x)r'(x)-1}}|u_n-\tilde{u}|_{r(x)},$$
$$\leq C_1 |K|_{\theta'(x)}\max\left(|u_n|^{r^+-1}_{\frac{\theta(x)(r(x)-1)}{\theta(x)r'(x)-1}}, |u_n|^{r^--1}_{\frac{\theta(x)(r(x)-1)}{\theta(x)r'(x)-1}}\right)|u_n-\tilde{u}|_{r(x)},$$
$$\leq C_1 |K|_{\theta'(x)}\max\left(C_2^{r^+-1}\|u_n\|^{r^+-1}, C_2^{r^--1}\|u_n\|^{r^--1}\right)|u_n-\tilde{u}|_{r(x)},$$
$$\leq C_1 |K|_{\theta'(x)}\max\left(\|u_n\|^{r^+-1}, \|u_n\|^{r^--1}\right)|u_n-\tilde{u}|_{r(x)}.$$

By letting n tends to infinity, we obtain the second part of Lemma 3.3.

Proposition 3.1. Assume that we have: $1 < p^0 < q^+ < r^+ < p_0$. If $\{u_n\} \subset E$ is a sequence, such that $J_\lambda(u_n)$ is bounded and $dJ_\lambda(u_n)$ tends to zero as n tends to infinity, then, $\{u_n\}$ has a convergent subsequence.

Proof. First, we show that $\{u_n\}$ is bounded in E. Assume by contradiction the contrary. Then, there exists a subsequence, still denoted by $\{u_n\}$, such that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, without loss of generality, we can assume that $\|u_n\| > 1$, for any integer n.

From the fact that $dJ_\lambda \to 0$, we deduce the existence of $N_1 > 0$, such that for any $n > N_1$, we have
$$\|dJ_\lambda(u_n)\| \leq 1.$$

On the other hand, for any $n > N_1$, the application $v \to \langle dJ_\lambda(u_n), v\rangle$ is linear and continuous. So we get

$$|\langle dJ_\lambda(u_n), v\rangle| \leq \|dJ_\lambda(u_n)\| \parallel v \parallel \leq \parallel v \parallel, \quad \forall v \in E.$$

By replacing v with u_n in the above inequality, we obtain

$$-\|u_n\| \leq \langle dJ(u_n), u_n\rangle \leq \|u_n\|.$$

Therefore,

$$-\|u_n\| - \langle dI(u_n), u_n\rangle + \lambda \int_\Omega f\big(x, u_n(x)\big)u_n(x)dx \leq \int_\Omega g(x)|u_n(x)|^{q(x)}dx. \tag{3.7}$$

It follows that

$$\begin{aligned}
J_\lambda(u_n) &= \int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]dx - \lambda \int_\Omega F(x, u_n(x))dx + \int_\Omega \frac{g(x)}{q(x)}|u_n(x)|^{q(x)}dx,\\
&\geq \int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]dx\\
&-\frac{1}{q^+}\int_\Omega (\alpha(|\nabla u_n(x)|)|\nabla u_n(x)|^2 + \alpha(|u_n(x)|)|u_n(x)|^2)dx\\
&-\lambda\int_\Omega \left(F(x, u_n(x)) - \frac{1}{q^+}f(x, u_n(x))u_n(x)\right)dx - \frac{1}{q^+}\|u_n\|,\\
&= \int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]dx\\
&-\frac{1}{q^+}\int_\Omega (\varphi(|\nabla u_n(x)|)|\nabla u_n(x)| + \varphi(|u_n(x)|)|u_n(x)|)dx\\
&-\lambda\int_\Omega \left(F(x, u_n(x)) - \frac{1}{q^+}f(x, u_n(x))u_n(x)\right)dx - \frac{1}{q^+}\|u_n\|.
\end{aligned}$$

From (2.3) we have $t\varphi(t) \leq p^0\Phi(t)$, which implies that

$$D(u_n) - \tfrac{1}{q^+}\textstyle\int_\Omega \varphi(|\nabla u_n(x)|)|\nabla u_n(x)| + \varphi(|u_n(x)|)|u_n(x)|dx \geq \left(1 - \tfrac{p^0}{q^+}\right)D(u_n).$$

Using the above relations, Proposition 2.2, the Hölder inequality and hypothesis (H_2), we get

$$\begin{aligned}
J_\lambda(u_n) &> \left(1 - \frac{p^0}{q^+}\right)D(u_n) - \frac{\lambda C_3}{r^-}|K|_{\theta'(x)}\|u_n\|^{r^+} + \frac{C_0}{q^+}\int_\Omega |u_n|^{r(x)}dx - \frac{1}{q^+}\|u_n\|,\\
&\geq \left(1 - \frac{p^0}{q^+}\right)\|u_n\|^{p_0} - \frac{\lambda C_3}{r^-}|K|_{\theta'(x)}\|u_n\|^{r^+} - \frac{1}{q^+}\|u_n\| + \frac{C_0}{q^+}\max\big(\| u \|^{r^+}, \| u \|^{r^-}\big),\\
&\geq \left(1 - \frac{p^0}{q^+}\right)\|u_n\|^{p_0} - \frac{\lambda C_3}{r^-}|K|_{\theta'(x)}\|u_n\|^{r^+} - \frac{1}{q^+}\|u_n\| + \frac{C_0}{q^+}\| u \|^{r^+}.
\end{aligned}$$

Since $1 < p^0 < q^+ < r^+ < p_0$, then, by letting n tends to infinity, we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E.

Now, from the fact that E is a reflexive Banach space, up to a subsequence, there exists $\tilde{u}$, such that

$$u_n \rightharpoonup \tilde{u}, \text{ weakly in } E.$$

Since $dJ_\lambda(u_n) \to 0$ and u_n is bounded in E, then, we get

$$|\langle dJ(u_n), u_n - \tilde{u}\rangle| \leq |\langle dJ(u_n), u_n\rangle| + |\langle dJ_\lambda(u_n), \tilde{u}\rangle|,$$
$$\leq \|dJ_\lambda(u_n)\|\|u_n\| + \|dJ(u_n)\| \parallel \tilde{u} \parallel.$$

So

$$\lim_{n\to\infty} \langle dJ(u_n), u_n - \tilde{u}\rangle = 0.$$

Hence, from Lemma 3.3 and the last relation, we obtain

$$\lim_{n\to\infty} \langle dI(u_n), u_n - \tilde{u}\rangle = 0. \tag{3.8}$$

On the other hand, the convexity of Φ, implies that I is convex. So

$$\limsup_{n\to\infty} I(u_n) \leq I(\tilde{u}). \tag{3.9}$$

By combining (3.9), with the weakly lower semi continuity of I, we deduce

$$\lim_{n\to\infty} I(u_n) = I(\tilde{u}). \tag{3.10}$$

That is

$$\lim_{n\to\infty} \int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]dx = \int_\Omega [\Phi(|\nabla \tilde{u}(x)|) + \Phi(|\tilde{u}(x)|)]dx. \tag{3.11}$$

Since Φ is increasing and convex, then, for all $x \in \Omega$, we have

$$\Phi\left(\frac{1}{2}|\nabla u_n(x) - \nabla\tilde{u}(x)|\right) + \Phi\left(\frac{1}{2}|u_n(x) - \tilde{u}(x)|\right)$$
$$\leq \Phi\left(\frac{1}{2}(|\nabla u_n(x)| + |\nabla\tilde{u}(x)|)\right) + \Phi\left(\frac{1}{2}(|u_n(x)| + |\tilde{u}(x)|)\right),$$
$$\leq \frac{1}{2}(\Phi(|\nabla u_n(x)|) + \Phi(|\nabla\tilde{u}(x)|)) + \frac{1}{2}(\Phi(|u_n(x)|) + \Phi(|\tilde{u}(x)|)).$$

By integrating the above inequalities over Ω, and using Lemma 2.2, we get

$$0 \leq 2\int_\Omega \Phi\left(\frac{1}{2}|\nabla(u_n - \tilde{u})(x)|\right) + \Phi\left(\frac{1}{2}|(u_n - \tilde{u})(x)|\right) dx,$$
$$\leq \int_\Omega \Phi(|\nabla u_n(x)|)dx + \int_\Omega \Phi(|\nabla\tilde{u}(x)|)dx + \int_\Omega \Phi(|u_n(x)|)dx + \int_\Omega \Phi(|\tilde{u}(x)|)dx,$$
$$\leq \int_\Omega \Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)dx + \int_\Omega \Phi(|\nabla\tilde{u}(x)|) + \Phi(|\tilde{u}(x)|)dx,$$
$$\leq \max\left(\|u_n\|^{p_0}, \|u_n\|^{p^0}\right) + \max\left(\parallel \tilde{u} \parallel^{p_0}, \parallel \tilde{u} \parallel^{p^0}\right).$$

Since $\{u_n\}$ is bounded in E, then, there exists $M > 0$, such that

$$0 \leq \int_\Omega \left[\Phi\left(\frac{1}{2}|\nabla(u_n - \tilde{u})(x)|\right) + \Phi\left(\frac{1}{2}|(u_n - \tilde{u})(x)|\right)\right] dx \leq M. \tag{3.12}$$

On the other hand, since $\{u_n\}$ converges weakly to $\tilde{u}$ in E. Then, Theorem 2.1 in [33] implies that

$$\int_\Omega \frac{\partial u_n}{\partial x_i} v dx \to \int_\Omega \frac{\partial \tilde{u}}{\partial x_i} v dx, \forall v \in L_{\Phi^*}(\Omega), i = 1 \dots, N.$$

In particular, the above result holds for all $v \in L^\infty(\Omega)$. Hence

$$\frac{\partial u_n}{\partial x_i} \rightharpoonup \frac{\partial \tilde{u}}{\partial x_i}, \text{weakly in } L^1(\Omega), \text{for all } i = 1, \dots, N.$$

Which yields to

$$\nabla u_n(x) \to \nabla \tilde{u}(x) \quad \text{a.e. } x \in \Omega. \tag{3.13}$$

By combining Equations (3.11), (3.12) and (3.13) with the Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega \left[\Phi\left(\frac{1}{2}|\nabla(u_n - \tilde{u})(x)|\right) + \Phi\left(\frac{1}{2}|(u_n - \tilde{u})(x)|\right)\right] dx = 0. \tag{3.14}$$

On the other hand, assumption (Φ_0), implies that (Φ) satisfies Δ_2- condition. So from (3.14), we get

$$\lim_{n\to\infty} \|u_n - \tilde{u}\| = 0.$$

The *Proof* of Proposition 3.1 is now complete.

Proof of Theorem 3.1. It is clear that the functional J_λ is even, and $J_\lambda(0) = 0$. Moreover, Proposition 3.1, implies that J_λ satisfies the Palais-Smale condition. Finally, from Lemmas 3.1 and 3.2, we can apply Theorem 2.2 to the functional J_λ. So we conclude that the problem (P_λ) has infinitely many weak solutions in E.

CONCLUSION

In this paper, we establishes the existence of multiple solutions for a non-homogeneous problem in the framework of Orlicz-Sobolev spaces by using variational techniques that rely, especially, critical point theory, and the Z2-symmetry version for the well known Mountain Pass theorem.

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