ORIGINAL PAPER

RECTIFYING CURVES IN THE LIGHTLIKE CONE

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Abstract. *A curve is called a rectifying curve if its position vector field always lies in its rectifying plane. In the present paper, we investigate the rectifying curves in the lightlike cone* Q^2*. For this firstly, we determine the curvatures of the rectifying curve in* Q^2 *. Then, with the help of these curvatures, we obtain the parametric representations of all rectifying curves in* Q^2*. Finally, we give various examples to support the theoretical results and show their images in the lightlike cone.*

Keywords: *Special curves; Minkowski space; special surfaces; frame fields.*

1. INTRODUCTION

In semi-Riemannian manifolds, vectors are classified into three types based on their causal properties: spacelike, timelike, and lightlike (null). Similarly, surfaces in these manifolds are categorized as spacelike, timelike, or lightlike, depending on the nature of their normal vector field. A surface is called lightlike if its normal vector is lightlike at every point, distinguishing it from spacelike and timelike surfaces. While the induced metric tensor field on spacelike and timelike surfaces is non-degenerate, lightlike surfaces are characterized by a degenerate metric, making their geometry unique. This distinction has led to extensive research into the geometric and physical properties of lightlike surfaces, with numerous applications in physics and mathematics [1-3].

Lightlike curves, which differ significantly from classical curves, have become a focal point in studies due to their relevance in various physical phenomena. For instance, the structure of light cones and the curves contained within them have been examined from a geometric perspective, highlighting their role in understanding black hole horizons, cosmic structures, and electromagnetic fields [4-7]. The study of these structures has intensified as they have proven effective in addressing numerous problems in theoretical and applied physics, particularly those involving null points, null curves, and their interplay with the spacetime fabric [8-14].

Among the notable curves in this context are rectifying curves**,** which are defined by the property that their position vectors lie entirely within their rectifying plane. These curves have been studied extensively in various dimensions and manifold settings, revealing remarkable geometric behaviors and applications [15-21]. One particularly important setting for such curves is the lightlike cone**,** a structure central to the general theory of relativity, where light propagates along null directions.

In relativity, the lightlike cone represents the causal structure of spacetime, representing the trajectories of light and other physical phenomena. This geometric structure not only provides a framework for understanding the universe's causal order but also plays a

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critical role in modeling particle trajectories near black holes and the motion of light in curved spacetime. Consequently, the study of rectifying curves within the lightlike cone has gained attention, offering insights into both mathematical theory and physical applications [22-27]. In addition, the behavior of basic geometric structures and their relationships are also discussed [28-32].

The aim of this study is to investigate rectifying curves within the lightlike cone by deriving their parametric representations. Using the curvatures obtained from the Frenet and Darboux frames formulations, we develop the parametric equations for these curves and present illustrative examples demonstrating their geometric properties. This work contributes to the study of lightlike structures and their importance in physics and geometry, opening new directions for future research on null geometries and their wider applications.

2. PRELIMINARIES

A semi-Riemannian manifold (M, g) is given with a non-degenerate metric tensör g with signature $(-,+,+)$ defined by

$$g(x, y) = -x_1y_1 + x_2y_2 + x_3y_3 \tag{1}$$

for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \chi(M)$. A vector is called space-like if $g(x, x) > 0$, time-like if $g(x, x) < 0$, and lightlike if $g(x, x) = 0$. If $g(x, y) = 0$, the vectors x and y are said to be orthogonal. Two null vectors are orthogonal if and only if they are linearly dependent.

Let γ be a smooth curve with the immersion $\iota : \gamma \to M$ in the semi-Riemannian manifold (M, g). Assume that s represents the matching local parameter and U is a coordinate neighborhood on C. The map that provides γ,

$$\gamma \ : \ I \to M, s \to \gamma(s) \tag{2}$$

where I represents an open interval in IR. The tangent vector field of γ on U is

$$T = \frac{d\gamma}{ds}. \tag{3}$$

Definition 2.1. In E_1^3, M be a lightlike surface. Its normal vector is denoted by Z, and T represents the tangent vector of a spacelike curve γ, which is parameterized by the arc-length parameter and lies on this surface. Additionally, let Y be a lightlike vector that satisfies the following equations:

$$g(Z, Z) = g(Y, Y) = g(Z, T) = g(Y, T) = 0, g(Z, Y) = g(T, T) = 1. \tag{4}$$

The Darboux frame along a spacelike curve γ\gammaγ in the lightlike surface MMM is defined as follows:

$$\begin{aligned} T' &= \quad k_z Z + k_y Y \\ Z' &= \quad -k_y T + \tau_r Z \\ Y' &= \quad -k_z T - \tau_r Y, \end{aligned} \tag{5}$$

where the Darboux frame curvatures of γ in the lightlike surface M are indicated by the notations k_y, k_z, τ_r [8].

A specific example of the surface M, which is a lightlike cone, is obtained when the normal vector Z of the lightlike surface is taken as $\gamma(s)$. The lightlike cone Q^2 in E_1^3 is defined as the set of all lightlike vectors, satisfying the following condition:

$$Q^2 = \{x = (x_1, x_2, x_3) \in E_1^3 \ : \ x_1^2 + x_2^2 + x_3^2 \neq 0\}. \tag{6}$$

The normals of a lightlike cone are its generators. One of these generators lies on the tangent plane of the cone, and the other vectors on this tangent plane are orthogonal to the generator, and have a spacelike quality [3].

Let Q^2 denote a lightlike cone, and γ a regular curve on it. If we take $Z = \gamma$, where Z is the normal vector of this cone, and $Y = y$ (in the frame previously mentioned), we get the following equalities,

$$g(\gamma,\gamma) = g(y,y) = g(\gamma,T) = g(y,T) = 0, g(\gamma,y) = g(T,T) = 1. \tag{7}$$

The position vector of a lightlike cone serves as the normal vector Z for any lightlike surface. If the lightlike surface Q^2 is considered instead of the surface M, then the Darboux frame associated with Q^2 can be obtained, as defined by Liu [5,8].

Theorem 2.1. $\gamma : I \to M \subset E_1^3$ be the curve in the lightlike cone Q^2 and $\{T, \gamma, y\}$ be the Darboux frame along the curve γ. If the derivative of the unit tangent vector field T' is a spacelike vector, then the Frenet frame curvatures and Darboux frame curvatures satisfy the following relation:

$$\kappa(s) = \sqrt{-k_z}, \tau(s) = -\frac{k_z'}{2k_z} \tag{8}$$

where $k_z < 0$. If the derivative of the unit tangent vector field T' is a timelike vector, then the Frenet frame curvatures and Darboux frame curvatures satisfy the following relation:

$$\kappa(s) = \sqrt{k_z}, \tau(s) = -\frac{k_z'}{2k_z} \tag{9}$$

where $k_z > 0$. If the derivative of the unit tangent vector field T' is a lightlike vector, then the Frenet frame curvatures $\kappa(s)$ can not defined [8].

3. RECTIFYING CURVES IN THE LIGHT CONE

The primary results of this article include the following theorems, which classify all rectifying curves on a lightlike cone. In this section, the lightlike cone is denoted by Q^2, and its Darboux frame along γ is represented as $\{T, \gamma, y\}$.

Theorem 3.1. Let $\gamma : I \to Q^2 \subset E_1^3$ be a spacelike curve in Q^2. If γ'' is a spacelike vector, then γ has the following curvature and torsion:

$$\kappa(s) = \frac{2}{s(2c+s)}, \tau = -\frac{(2c+2s)}{s(2c+s)} \tag{10}$$

where c is a constant.

Proof: If γ is a spacelike rectifying curve, then the curvature and torsion of the curve γ satisfy

$$\frac{\tau}{\kappa} = c + s. \tag{11}$$

However, we are aware that in the case where *T*` is a spacelike vector, then we have

$$(s) = \sqrt{-k_z}, \tau(s) = -\frac{k_z'}{2k_z} \tag{12}$$

If we consider with eq. (11) and eq. (12) we obtain

$$k_z'^2 + 8(c + s)2k_z^3 = 0. \tag{13}$$

The solution of the eq. (13) obtained as

$$k_z(s) = \frac{4}{2s^2(2c+s)^2 - 2i\sqrt{2}\,ks^2 - k^2}.$$

Next, the differential equation's actual solution for $k = 0$ is computed,

$$k_z(s) = -\frac{2}{s^2(2c+s)^2} \tag{14}$$

where c is a constant.

Theorem 3.2. Consider the spacelike curve $\gamma : I \to Q^2 \subset E_1^3$ in Q^2. If γ'' is a spacelike vector, then γ has one of the following parameterizations:

$$\begin{aligned}\gamma(s) = {} & a_1 s(s+2c)\left(\frac{s}{2c+s}\right)^{\frac{\sqrt{(c-1)(c+1)}}{c}} + a_2 s(s+2c)\left(\frac{2c+s}{s}\right)^{\frac{\sqrt{(c-1)(c+1)}}{c}} \\ & + a_3 s(s+2c)\left(\frac{s}{2c+s}\right)^{\frac{\sqrt{(c-1)(c+1)}}{2c}}\left(\frac{2c+s}{s}\right)^{\frac{\sqrt{(c-1)(c+1)}}{c}} \\ \gamma(s) = {} & b_1 s(s\pm 2) + b_2 s(s\pm 2) ln\frac{s\pm 2}{s} + b_3 s(s\pm 2) ln\frac{s\pm 2^2}{s}, c = \pm 1 \\ \gamma(s) = {} & c_1 s^2 cos\left(\frac{1}{s}\right)^2 + c_2 s^2 cos\left(\frac{1}{s}\right) sin\left(\frac{1}{s}\right) - c_3 s^2 cos\left(\frac{1}{s}\right)^2 sin\left(\frac{1}{s}\right)^2, c = 0\end{aligned} \tag{15}$$

where $a_i, b_i, c_i \in E_1^3, i = 1,2,3$.

Proof: Consider the spacelike curve $\gamma : I \to Q^2 \subset E_1^3$ in Q^2. Next, we have

$$\begin{aligned} T' &= k_z\gamma - y \\ \gamma' &= T \\ y' &= -k_z T \end{aligned} \tag{16}$$

This gives the following differential equation

$$\gamma''' - 2k_z\gamma' - k_z' = 0. \tag{17}$$

If we use the equations in eq. (17), we obtain a third-order differential equation. The solution of this equation gives the parametric representations of the rectifying curves γ, which can be expressed in one of the following forms:

$$\begin{aligned} \gamma(s) = {} & a_1 s(s+2c)\left(\frac{s}{2c+s}\right)^{\frac{\sqrt{(c-1)(c+1)}}{c}} + a_2 s(s+2c)\left(\frac{2c+s}{s}\right)^{\frac{\sqrt{(c-1)(c+1)}}{c}} \\ & + a_3 s(s+2c)\left(\frac{s}{2c+s}\right)^{\frac{\sqrt{(c-1)(c+1)}}{2c}}\left(\frac{2c+s}{s}\right)^{\frac{\sqrt{(c-1)(c+1)}}{c}} \end{aligned}$$

$$\gamma(s) = b_1 s(s\pm 2) + b_2 s(s\pm 2)ln\frac{s\pm 2}{s} + b_3 s(s\pm 2)ln\frac{s\pm 2^2}{s}, c = \pm 1$$

$$\gamma(s) = c_1 s^2 cos\left(\frac{1}{s}\right)^2 + c_2 s^2 cos\left(\frac{1}{s}\right) sin\left(\frac{1}{s}\right) - c_3 s^2 cos\left(\frac{1}{s}\right)^2 sin\left(\frac{1}{s}\right)^2, c = 0$$

where $a_i, b_i, c_i \in E_1^3, i = 1,2,3$.

Theorem 3.3. Let $\gamma : I \to Q^2 \subset E_1^3$ be a spacelike curve in Q^2. If γ'' is a timelike vector, then γ has following curvature and torsion:

$$\kappa(s) = \frac{2}{\sqrt{4 \pm \sqrt{2}cks + 2s^2(2c+s)^2 \pm 2\sqrt{2}ks^2 + k^2}} \tag{19}$$

$$\tau(s) = \frac{2(c+s)}{\sqrt{4 \pm \sqrt{2}cks + 2s^2(2c+s)^2 \pm 2\sqrt{2}ks^2 + k^2}} \tag{20}$$

where c, k are constants.

If γ is a spacelike rectifying curve, then the curvature and torsion of the curve γ satisfy the following relation:

$$\frac{\tau}{\kappa} = c + s. \tag{21}$$

On the other hand, we know that if T` is a timelike vector, then we have

$$\kappa(s) = \sqrt{2k_z}, \tau(s) = -\frac{k_z'}{2k_z} \tag{22}$$

Using the eq. (21) and eq. (22) we derive the differential equation that follows.

$$k_z'^2 + 8(c + s)2k_z^3 = 0. \tag{23}$$

The differential equation's solution is computed as

$$k_z(s) = \frac{4}{4 \pm \sqrt{2}cks + 2s^2(2c+s)^2 \pm 2\sqrt{2}ks^2 + k^2} \tag{24}$$

where c, k are constants.

Theorem 3.4. Let $\gamma : I \to Q^2 \subset E_1^3$ be a spacelike rectifying curve in Q^2. If γ'' is a timelike vector, then γ has the following parametric representation:

$$\begin{aligned}
\gamma(s) = {} & d_1(\pm 2\sqrt{2}k - 8cs - 4s^2)\left(\frac{s}{2c+s}\right)^{\frac{\sqrt{4c^2 \pm 2\sqrt{2}k} + \sqrt{\pm 2\sqrt{2}k + 4c^2 + 4}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}} \\
& \left(\frac{-2s - 2c + \sqrt{4c^2 \pm 2\sqrt{2}k}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}\right)^{-\frac{2\sqrt{\pm 2\sqrt{2}k + 4c^2 + 4}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}} \\
& + d_2\left(2s + 2c + \sqrt{4c^2 \pm 2\sqrt{2}k}\right)^{\frac{\sqrt{4c^2 \pm 2\sqrt{2}k} - \sqrt{\pm 2\sqrt{2}k + 4c^2 + 4}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}} \\
& \left(-2s - 2c + \sqrt{4c^2 \pm 2\sqrt{2}k}\right)^{\frac{\sqrt{4c^2 \pm 2\sqrt{2}k} - \sqrt{\pm 2\sqrt{2}k + 4c^2 + 4}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}} \\
& + d_3(\pm 2\sqrt{2}k - 8cs - 4s^2)^{\frac{\sqrt{4c^2 \pm 2\sqrt{2}k} + \sqrt{\pm 2\sqrt{2}k + 4c^2 + 4}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}} \\
& \left(\frac{-2s - 2c + \sqrt{4c^2 \pm 2\sqrt{2}k}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}\right)^{-\frac{2\sqrt{\pm 2\sqrt{2}k + 4c^2 + 4}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}} \\
& \left(2s + 2c + \sqrt{4c^2 \pm 2\sqrt{2}k}\right)^{\frac{\sqrt{4c^2 \pm 2\sqrt{2}k} - \sqrt{\pm 2\sqrt{2}k + 4c^2 + 4}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}} \\
& \left(-2s - 2c + \sqrt{4c^2 \pm 2\sqrt{2}k}\right)^{\frac{\sqrt{4c^2 \pm 2\sqrt{2}k} - \sqrt{\pm 2\sqrt{2}k + 4c^2 + 4}}{\sqrt{4c^2 \pm 2\sqrt{2}k}}}
\end{aligned} \tag{25}$$

and

$$\gamma(s) = e_1 s^2 \cosh\left(\frac{1}{s}\right)^2 + e_2 s^2 \cosh\left(\frac{1}{s}\right)\sinh\left(\frac{1}{s}\right) + e_3 s^2 \sinh\left(\frac{1}{s}\right)^2, c = 0, k = 0. \tag{26}$$

where $d_i, e_i, \in E_1^3, i = 1,2,3$.

Proof. Consider the spacelike curve $\gamma : I \to Q^2 \subset E_1^3$ in Q^2. Next, we have

$$\begin{aligned}
T' &= k_z\gamma - y \\
\gamma' &= T \\
y' &= -k_z T
\end{aligned} \tag{27}$$

This gives the following differential equation

$$\gamma''' - 2k_z\gamma' - k_z' = 0. \tag{28}$$

If k_z is written in the eq. (28) and the equation is solved, we obtain the parametric representations of the rectifying curves in the form eq. (25) and eq. (26).

Remark. Let $\gamma : I \to Q^2 \subset E_1^3$ be a spacelike curve in Q^2. If γ'' is a lightlike vector, then $k_z = 0$, in this case, there is not definition of the curvature of γ.

Example 3.1. If we choose $a_1 = (0,1,0), a_2 = \left(\frac{1}{2}, 0, \frac{1}{2}\right), a_3 = \left(\frac{1}{2}, 0, -\frac{1}{2}\right), c = 2$ we get following rectifying curve in the lightlike cone Q^2 parameterized by

$$\gamma_1(s) = (\frac{1}{2}s^{1-\frac{\sqrt{3}}{2}}(s+1)^{1+\frac{\sqrt{3}}{2}}, s(s+4), \frac{1}{2}s^{1+\frac{\sqrt{3}}{2}}\left(s+1)^{1-\frac{\sqrt{3}}{2}}\right)$$

The curvatures of the rectifying curve calculated as

$$\kappa(s) = \frac{2}{s(4+s)}, \tau(s) = \frac{(4+2s)}{s(4+s)} \tag{29}$$

Then the image of the rectifying curve γ_1 is illustrated in Fig. 1.

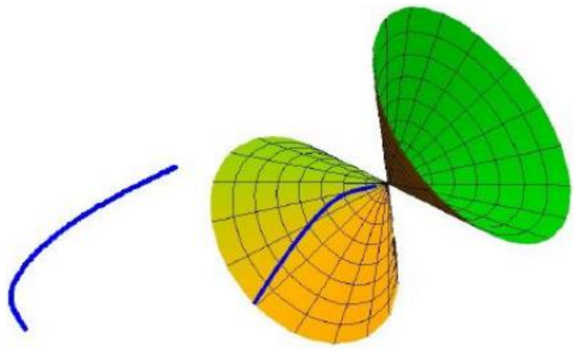

Figure 1. Rectifying curve γ_1 and image of this curve on the lightlike cone.

Example 3.2. If we choose $b_1 = (\frac{1}{2}, 0, -\frac{1}{2}), b_2 = (0,1,0), b_3 = (\frac{1}{2}, 0, \frac{1}{2}), c = 1$ we get following rectifying curve in the lightlike cone Q^2 parameterized by

$$\gamma_2(s) = s(s+2)(\frac{1}{2}ln(\frac{s\pm2}{s})^2 + \frac{1}{2}, ln(\frac{s\pm2}{s})^2, \frac{1}{2}ln\left(\frac{s\pm2}{s})^2 - \frac{1}{2}\right)$$

The curvatures of the rectifying curve calculated as

$$\kappa(s) = \frac{2}{s(2+s)}, \tau(s) = \frac{(2+2s)}{s(2+s)} \tag{30}$$

Then the image of the rectifying curve γ_2 is illustrated in Fig. 2.

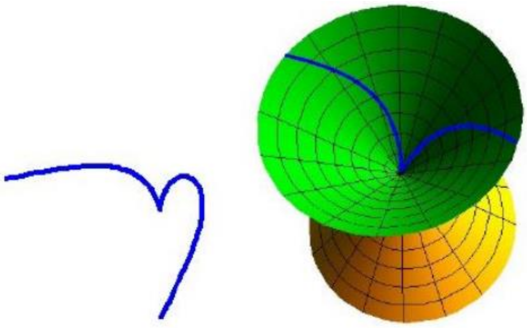

Figure 2. Rectifying curve γ_2 and image of this curve in the lightlike cone.

Example 3.3. If we choose $c_1 = (\frac{1}{2}, \frac{1}{2}, 0), c_2 = (0,0,1), c_3 = (\frac{1}{2}, -\frac{1}{2}, 0), c = 0, k = 0$, we get following rectifying curve in the lightlike cone Q^2 parameterized by

$$\gamma_3(s) = (\frac{s^2}{2}, \frac{s^2}{2} cos \frac{2}{s}, \frac{s^2}{2} sin \frac{2}{s})$$

The curvatures of the rectifying curve calculated as

$$\kappa(s) = \frac{2}{s^2} \tau(s) = \frac{2}{s} \tag{31}$$

Then the image of the rectifying curve γ_3 is illustrated in Figs. 3 and 4.

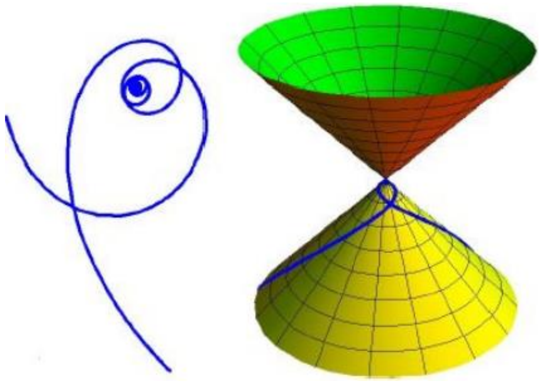

Figure 3. Rectifying curve γ_3 and image of this curve in the lightlike cone.

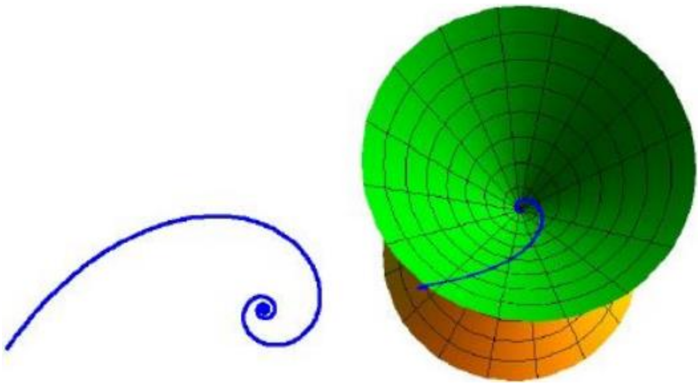

Figure 4. Rectifying curve γ_3 and image of this curve in the lightlike cone with different parameter range.

Example 3.4. If we choose $d_1 = (\frac{1}{2}, \frac{1}{2}, 0), d_2 = (0,0,1), d_3 = (\frac{1}{2}, -\frac{1}{2}, 0), c = 0, k = 0$, we get following rectifying curve in the lightlike cone Q^2 parameterized by

$$\gamma_4(s) = (\frac{s^2}{2} cosh \frac{2}{s}, \frac{s^2}{2}, \frac{s^2}{2} sinh \frac{2}{s})$$

The curvatures of the rectifying curve are calculated as

$$\kappa(s) = \frac{2}{\sqrt{2}s^2} \quad (32)$$

$$\tau(s) = \frac{2s}{\sqrt{2}s^2} \quad (33)$$

Then the image of the rectifying curve γ_4 is illustrated in Fig. 5.

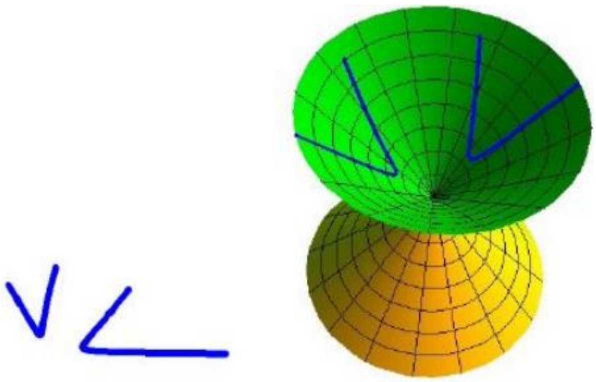

Figure 5. Rectifying curve γ_4 and image of this curve in the lightlike cone.

4. CONCLUSIONS

The lightlike cone is an important concept in physics. Keeping track of it in a given physical situation is a remarkably important tool and technique for experiencing many physical conditions. For instance, [6] provides a mathematical comparison between light cones in Minkowski spacetime and the principles of black hole thermodynamics. It is demonstrated in [7] that the conformal partner of the Bertotti-Robinson spacetime's Killing horizon structure can be the light cones' con-formal Killing horizon structure in Minkowski spacetime. Furthermore, the near-extremal and extremally charged black holes are encoded by the near horizon geometry. This fact is explained by the disappearance of the Weyl tensor at the horizon of an extremal Reissner-Nordstrom black hole. The near-horizon geometry becomes conformally flat as a result [14]. The nature of light cone thermodynamics is effectively described by this finding [6]. Unquestionably, bends in the lightlike cone have a significant impact on our understanding of the events linked by motion at the speed of light. Therefore, research in this area is aided by the examination of the unique curves in the lightlike cone. We identify every rectifying curve in the lightlike cone for this reason. In addition, we present the photographs of these curves. The nature of light cone thermodynamics is effectively described by this finding [6].

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