ORIGINAL PAPER

A REMARKABLE CONTRIBUTION TO SOFT INT-GROUP THEORY VIA A COMPREHENSIVE VIEW OF SOFT COSETS

ASLIHAN SEZGİN[1], ALEYNA İLGİN[2], FATIMA ZEHRA KOCAKAYA[2], ZEYNEP HARE BAŞ[2], BEYZA ONUR[2], FİLİZ ÇITAK[3]

Manuscript received: 02.08.2024; Accepted paper: 25.11.2024;
Published online: 30.12.2024.

Abstract. *This paper aims to expand soft int-group theory by analyzing its many aspects and structural properties regarding soft cosets and soft quotient groups, which are crucial concepts of the theory. All the characteristics of soft cosets are given in accordance with the properties of classical cosets in abstract algebra, and many interesting analogous results are obtained. It is proved that if an element is in the e-set, then its soft left and right cosets are the same and equal to the soft set itself. The main and remarkable contribution of this paper to the theory is that the relation between the e-set and the normality of the soft int-group is obtained, and it is proved that if the e-set has an element other than the identity of the group, then the soft int-group is normal. Based on this significant fact, it is revealed that if the soft set is not normal, then there do not exist any equal soft left (right) cosets. These relations are quite striking for the theory, since based on these facts, we show that the normality condition on the soft int-group is unnecessary in many definitions, propositions, and theorems given before. Furthermore, we come up with a fascinating result, unlike classical algebra that to construct a soft quotient group and to hold the fundamental homomorphism theorem, the soft int-group needs not to be normal. It is also demonstrated that the soft int-group is an abelian (normal) int-group if and only if the soft quotient group of G relative to the soft group is abelian. Finally, the torsion soft-int group and $\wp$-soft int-group are introduced, and we show that soft int-group f_G is a torsion soft-int group ($\wp$-soft int-group) if and only if the soft quotient group $G/_{f_G}$ is a torsion ($\wp$-group), respectively.*

Keywords: *Soft set; soft int-groups; normal soft int-groups; e-sets; soft cosets; soft quotient groups.*

1. INTRODUCTION

Researchers are unable to properly address issues in many fields, including engineering, economics, environmental and health sciences, and science, due to the presence of various forms of uncertainty. There are three well-known fundamental theories-probability theory, fuzzy set theory, and interval mathematics-used as mathematical tools to cope with uncertainty. However, due to the limitations of each of these theories, Molodtsov [1], in 1999, presented Soft Set Theory as a mathematical tool to get beyond this uncertainty. Since then,

[1] Amasya University, Department of Mathematics and Science Education, 05100 Amasya, Turkey.
E-mail: aslihan.sezgin@amasya.edu.tr
[2] Amasya University, Department of Mathematics, 05100 Amasya, Turkey. E-mails: aleynailgiin@gmail.com; fzkcky05@gmail.com; Zeynephare05@gmail.com; beyzaonur133@gmail.com.
[3] Tokat Gaziosmanpaşa University, Department of Mathematics, 60000 Tokat, Turkey.
E-mail: filiz.citak@gop.edu.tr.

https://doi.org/10.46939/J.Sci.Arts-24.4-a13

this theory has been used in a variety of fields, such as measurement theory, operations research, game theory, optimization theory, information systems, and decision-making.

In terms of soft-set operations, the initial contributions were provided by Maji et al. [2] and Pei and Miao [3]. Then, Ali et al. [4] defined some new restricted and extended soft set operations, and Sezgin and Atagün [5] investigated more about the properties of these operations and defined restricted symmetric differences. Qin and Hong [6] and Ali et al. [7] studied the algebraic structures of soft sets. Sezgin et al. [8] and Stojanovic [9] defined extended and symmetric differences of soft sets, respectively. Studies on soft set operations [10-18] have been of interest since the soft set's inception. Some drastic changes were made to the definition and operations of soft set by Çağman and Enginoğlu [19], and then the soft set theory has been the subject of intense research, with several significant theoretical and practical findings.

Soft algebraic structures are studied by many researchers, as well as soft set operations, with the study of Aktas and Çağman [20]. A parametrized family of a group's subgroups is what Aktas and Çağman [20] defined as a soft group. In other words, if and only if every value in the set-valued function is a subgroup of the group, then a soft set is referred to as a soft group over the group. In this regard, normalistic soft groups were studied in [21]. Aslam et al. [22] introduced normal soft groups, abelian soft groups, and abelian soft subgroups of a soft group, and their properties were checked out. Also, cyclic soft groups, cyclic soft subgroups, soft cosets of a soft subgroup of a soft group, order of a soft group, soft index, and partition of a soft set over a group were studied, and soft maximal normal subgroups, simple soft groups, and factor soft groups were obtained. Aktaş and Özlü [23] introduced the order of the soft groups, power of the soft sets, power of the soft groups, and cyclic soft groups on a group and investigated the relationship between cyclic soft groups and classical groups. In 2019, Nazmal [24] handled the homomorphic image and preimage of soft groups under soft mappings. Alajlan and Alghamdi [25] studied the center of the soft group, the kernel of soft homomorphisms and soft automorphisms, characteristic soft subgroups of a given soft group. In all these studies, the soft group structure is the one defined by Aktaş and Çağman [20]. Ghosh and Samanta [26] came up with a new idea of soft groups using the concept of soft elements, and Yaylalı et al. [27] extended the study on binary operations on a soft set by using the soft element definition, and Weldetekle et al. [28] proposed a new definition for soft groups based on soft binary operations.

Çağman et al. [29] constructed a novel class of soft groups on soft sets, which they call soft intersection groups, or simply "soft int-groups." This novel idea is very useful in that it enhances soft set theory in connection to group structure since it is founded on the inclusion relation and intersection of sets and unifies set theory, group theory, and soft set theory. This kind of soft algebraic structure inspired many studies in soft set literature [30-35]. In [29], Çağman et al. provided the notions of soft int-subgroup and normal soft int-subgroup based on the definition of soft int-group. Also, abelian soft sets, soft cosets, e-sets, soft images, and soft preimages were described, and their applications to group theory concerning the soft int-group were given. In [36], some additional properties of soft sets and soft int-groups were provided, and some significant relationships between soft products, soft inclusions, and soft int-groups were obtained. In [37], normal soft int-groups were presented in detail, and their related properties were investigated. Some relations on alfa-inclusion, soft product, and normal soft int-group were obtained. In addition, normalizer, quotient group, and other concepts concerning these were given. In [38], more on the normal soft groups, soft Cayley and Lagrance Theorems were handled. In [39], studies on the normalizer of a soft set and the normal soft set on a groupoid were given. In [40], commutative soft sets and commutative soft intersection groups were investigated. In [41,42], normal soft int-subgroups of a soft int-group were defined, several relations concerning them were given and homomorphism and isomorphism theorems were applied to the soft int-groups. In [43], the soft quotient subgroup and quotient dual soft subgroup were defined, their algebraic properties were obtained, the fundamental isomorphism theorems in soft subgroups analogous to group theory were investigated, and the second type nilpotent soft subgroups were as introduced as new concepts in [44]. Cosets, normal groups, and quotient groups are very significant concepts in abstract algebra. Lagrange Theorem, the elegant and important theorem, comes from the simple counting of cosets and the number of elements in each coset. Normal subgroups play an important role in determining both the structure of G and the nature of homomorphisms with domain G. Quotient groups are helpful as they allow us to examine the group in simpler forms using quotient groups. That is to say, a group may occasionally be too complex to investigate in its entirety; however, quotient groups typically have a simpler structure than the entire group. Thus, when we examine the quotient group, we can learn more about the initial group structure. To provide a concrete example of this let's think $Z(G)$, the center of the group G. If the quotient group $G/_{Z(G)}$ is cyclic, then the group G is abelian itself. This fact may be used to demonstrate the abelian nature of every group of order p^2 (p prime). Also, every homomorphism with domain G gives rise to a quotient group $G/_{H}$, and every quotient group $G/_{H}$ gives rise to a homomorphism mapping G into $G/_{H}$. Thus, homomorphism and quotient groups are closely related [45].

In soft int-group theory, the concept of soft cosets was first defined by Çağman et al. in [29]; however, in this study, only the definition, an example, and a few theorems regarding the relation of e-set and cosets were given. In [37,38], normal soft int-groups, the basic concepts for soft quotient groups, were handled, and soft quotient groups were constructed; but many related concepts are missing and need to be corrected.

In this paper, to contribute to soft int-group theory and soft set literature, we deal with the crucial concepts of soft cosets and soft quotient groups. By exploring and evaluating its various structural characteristics, particularly in relation to the theory's key concepts of soft cosets and soft quotient groups, this study seeks to further the field of soft int-group theory. First of all, the designation of soft coset is revised to conform to the standard notation of soft set theory. Then, all of the properties of soft sets are obtained in line with the classical cosets in abstract algebra, and several fascinating analogs are found. We show that when G is abelian, soft left and right cosets coincide, but the group G being abelian is not a necessary condition for the right and left cosets to be equal. Soft group itself is shown to be soft left and right coset, and the left (right) coset is not soft int-group, in general. We also prove that if an element is in the e-set, then its soft left and right cosets coincide, and they are equal to the soft set itself. The main contribution of this paper is that we give the relation between the e-set and the normality of the soft int-group, and we demonstrate that if the e-set has an element other than identity, then the soft int-group is normal, and so if the soft set is not normal, then there do not exist any equal soft left (right) cosets. This relation is very important in the theory, because based on this fact, it is proved that in many definitions, propositions, and theorems in [29,37,38] the condition of normality on the soft group is unnecessary. Furthermore, unlike classical algebra and what is quite interesting is that for constructing a soft quotient group and to hold the isomorphism $G/_{f_G} \cong G/_{G_{f_G}}$, the soft int-group f_G needs not be normal. It is also presented that if each soft left (right) coset is equal to its inverse, then the soft int group is normal. Moreover, it is obtained that there is a one-to-one correspondence between the set of all soft left cosets and the set of all soft right cosets. Furthermore, it is observed that f_G is an abelian (normal) soft int-group if and only if $G/_{f_G}$ is abelian. This theorem is crucial, since by commenting on the abelian property of $G/_{G_{f_G}}$, we can deduce about the normality of the soft set f_G. Finally, torsion soft-int group, p-soft int-group are introduced, and we prove that f_G is a torsion soft-int group (p-soft int-group) if and only if the soft quotient group $G/_{f_G}$ is a torsion (p-group), respectively. In this regard, this paper is an overall study of the soft cosets. Moreover, since the soft set is a generalization of fuzzy sets, fuzzy group theory may be revisited by taking into account the outstanding results in this study about normality and quotient groups.

2. PRELIMINARIES

In this section, we remind the basic concepts regarding soft sets and soft int-groups.

Definition 2.1. Let U be the universal set, E be the parameter set, $P(U)$ be the power set of U and $A \subseteq E$. A soft set f_A over U is a set-valued function such that $f_A: E \to P(U)$ such that for all $x \notin A$, $f_A(x) = \emptyset$. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)): x \in E, f_A(x) \in P(U)\}$$

[1,19]. Throughout this paper, the set of all the soft sets over U is designated by $S_E(U)$.

Definition 2.2. Let $f_A, f_B \in S_E(U)$. If $f_A(x) \subseteq f_B(x)$ for all $x \in E$, then f_A is a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$ [19].

Definition 2.3. Let $f_A, f_B \in S_E(U)$. If $f_A(x) = f_B(x)$ for all $x \in E$, then f_A is called soft equal to f_B and denoted by $f_A = f_B$. If $f_A \neq f_B$, then we say that f_A and f_B are different soft sets [19].

From now on, G is a group with the operation "$.$" and identity "e" and for all $x, y \in G$; we prefer to use "xy" instead of all "$x.y$". Also, all the soft sets are the elements of the set $S_G(U)$, and H is a subgroup of G otherwise specified. For all undefined concepts as regards groups, normal subgroups, cosets, quotient groups, and homomorphism, we refer to [44,45].

Definition 2.4. Let f_G be a soft set over U. Then, f_G is called soft int-group over U if

$$f_G(xy) \supseteq f_G(x) \cap f_G(y) \text{ and } f_G(x^{-1}) = f_G(x) \text{ for all } x, y \in G \text{ [29].}$$

Theorem 2.5. f_G is a soft int-group over U if $f_G(xy^{-1}) \supseteq f_G(x) \cap f_G(y)$ for all $x, y \in G$ [29].

Proposition 2.6. Let f_G be a soft int-group over U. Then, $f_G(e) \supseteq f_G(x)$ for all $x \in G$ [29].

Definition 2.7. Let f_G be a soft int-group over U. Then, e-set of f_G, denoted by G_{f_G}, is defined as

$$G_{f_G} = \{x \in G \mid f_G(x) = f_G(e)\} \text{ [29].}$$

Definition 2.8. Let f_G be a soft int-group over U. f_G. Then, the soft set f_G is called an abelian soft set over U if $f_G(xy) = f_G(yx)$, for all $x, y \in G$. It is obvious that if G is abelian, then f_G is abelian [37].

Definition 2.9. Let f_G be a soft int-group over U. f_G, Then, f_G is called soft normal in G (or normal soft int-group), if f_G is an Abelian soft set [37].

Definition 2.10. Let H be a subgroup of G and f_G be a soft int-group over U and f_H be a nonempty soft subset of f_G over U. If f_H itself is a soft int-group over U, then f_H is called a soft int-subgroup of f_G over U [29].

Definition 2.11. Let f_G be a soft int-group over U and f_N be a soft int-subgroup of f_G. Then, f_N is called a normal soft int-subgroup over U, if it is an abelian soft subset of f_G over U [29].

Theorem 2.12. Let f_G be a soft int-group over U. Then, the following conditions are equivalent:

i) f_G is a normal soft int-group, that is, $f_G(xy) = f_G(yx)$ for all $x, y \in G$.
ii) f_G is an abelian soft int-group.
iii) f_G is constant in the conjugate class of G, that is, $f_G(xyx^{-1}) = f_G(y)$ for all $x, y \in G$.
iv) $f_G(xyx^{-1}) \supseteq f_G(y)$ for all $x, y \in G$.
v) $f_G(xyx^{-1}) \subseteq f_G(y)$ for all $x, y \in G$ [37].

Definition 2.13. Let f_G be a soft int-group over U and $a \in G$. Then, soft left coset of f_G, denoted by af_G, is defined by the approximation function $af_G = f_G(a^{-1}x)$ for all $x \in G$; and soft right cosets of f_G, denoted by $f_G a$, is defined by $f_G a = f_G(xa^{-1})$ for all $x \in G$ [29].

Theorem 2.14. f_G is a normal soft int-group if and only if $af_G = f_G a$ for all $a \in G$ [37,41].
We refer to [29,36-43] for all undefined concepts as regards soft int-groups, normal soft int-groups, and soft quotient (or factor) groups, to [47] regarding the possible applications of network analysis and graph applications on soft sets, and to [48,49] regarding the picture fuzzy soft sets.

Since in this paper, we correct some problematic cases in [29,37,38] and prove that in most of the theorems, propositions, and definitions, some of the assumptions are not necessary, we are of the opinion that it is more appropriate to remind these theorems, propositions, and definitions in Section 3 when the occasion arises.

3. RESULTS AND DISCUSSION

In this section, first, we revise the presentation of the definition of soft left (right) coset defined first in [29], since soft left (right) coset defined itself is a soft set; hence, it is more appropriate to give the definition of soft left (right) coset in the general and classical format of soft set as follows:

Definition 3.1. Let f_G be a soft int-group over U and $a \in G$. Then, the soft set af_G defined by

$$af_G = \{(x, f_G(a^{-1}x)), x \in G\}$$

is called a soft left coset of f_G. Similarly, the soft set $f_G a$ defined by

$$f_G a = \{(x, f_G(xa^{-1})), x \in G\}$$

is called a soft right coset of f_G [29].

In abstract algebra, if G is an abelian group, then every left coset of G is a right coset of G, i.e., aH = Ha for all. $a \in G$. As an analogy, we have the following:

Note 3.2. If G is an abelian group, then each soft left and soft right coset of f_G always coincide with each other, i.e., $af_G = f_G a$ for all $a \in G$, since $f_G(a^{-1}x) = f_G(xa^{-1})$ for all $x, a \in G$. In [37], it is stated that if f_G is a normal soft int-group, then every soft left coset is equal to a soft right coset. Anyway, if G is an abelian group, then f_G is a normal soft int-group is obvious.

Example 3.3. Consider the Klein-4 group $G = \{e, a, b, c\}$ defined by the following table:

Table 1. Cayley table of binary operation.

.	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $U = \mathbb{Z}$ be the universal set and f_G be a soft set over U defined by as follows:

$$f_G = \{(e, \{-3, -2, -1, 0\}), (a, \{-2, -1\}), (b, \{-2, -1, 0\}), (c, \{-3, -2, -1\})\}.$$

One can easily show that f_G is soft int-group over $\mathbb{Z}$. Then, we can calculate the soft left cosets of f_G, the soft left cosets are as follows:

$$ef_G = \{(e, \{-3, -2, -1, 0\}), (a, \{-2, -1\}), (b, \{-2, -1, 0\}), (c, \{-3, -2, -1\})\}$$

$$af_G = \{(e, \{-2, -1\}), (a, \{-3, -2, -1, 0\}), (b, \{-3, -2, -1\}), (c, \{-2, -1, 0\})\}$$

$$bf_G = \{(e, \{-2, -1, 0\}), (a, \{-3, -2, -1\}), (b, \{-3, -2, -1, 0\}), (c, \{-2, -1\})\}$$

$$cf_G = \{(e, \{-3, -2, -1\}), (a, \{-2, -1, 0\}), (b, \{-2, -1\}), (c, \{-3, -2, -1, 0\})\}$$

Since G is an abelian group, $f_G(a^{-1}x) = f_G(xa^{-1})$ for all $a, x \in G$; thus the soft left cosets are the soft right cosets of f_G at the same time. Thus, the soft right cosets are as follows:

$$f_G e = \{(e, \{-3, -2, -1, 0\}), (a, \{-2, -1\}), (b, \{-2, -1, 0\}), (c, \{-3, -2, -1\})\}$$

$$f_G a = \{(e, \{-2, -1\}), (a, \{-3, -2, -1, 0\}), (b, \{-3, -2, -1\}), (c, \{-2, -1, 0\})\}$$

$$f_G b = \{(e, \{-2, -1, 0\}), (a, \{-3, -2, -1\}), (b, \{-3, -2, -1, 0\}), (c, \{-2, -1\})\}$$

$$f_G c = \{(e, \{-3, -2, -1\}), (a, \{-2, -1, 0\}), (b, \{-2, -1\}), (c, \{-3, -2, -1, 0\})\}$$

Example 3.4. Let $U = \mathbb{Z}$ be the universal set and $G = D_3 = \{< x, y >: x^3 = y^2 = e,\ xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ be the set of parameters defined as following table:

Table 2. Cayley table of binary operation.

$*$	e	x	x^2	y	yx	yx^2
e	e	x	x^2	y	yx	yx^2
x	x	x^2	e	yx^2	y	yx
x^2	x^2	e	x	yx	yx^2	y
y	y	yx	yx^2	e	x	x^2
yx	yx	yx^2	y	x^2	e	x
yx^2	yx^2	y	yx	x	x^2	e

We define the soft set f_G over U as follows:

$$f_G = \{(e,\{-2,-1,0,1,2\}),(x,\{-2,-1\}),(x^2,\{-2,-1\}),(y,\{-2,0\}),(yx,\{-2,2\}),(yx^2,\{-2,1\})\}$$

One can easily show that f_G is soft int-group over $\mathbb{Z}$ and different soft left cosets of f_G are as follows:

$$ef_G = \{(e,\{-2,-1,0,1,2\}),(x,\{-2,-1\}),(x^2,\{-2,-1\}),(y,\{-2,0\}),(yx,\{-2,2\}),(yx^2,\{-2,1\})\}$$

$$xf_G = \{(e,\{-2,-1\}),(x,\{-2,-1,0,1,2\}),(x^2,\{-2,-1\}),(y,\{-2,2\}),(yx,\{-2,1\}),(yx^2,\{-2,0\})\}$$

$$x^2f_G = \{(e,\{-2,-1\}),(x,\{-2,-1\}),(x^2,\{-2,-1,0,1,2\}),(y,\{-2,1\}),(yx,\{-2,0\}),(yx^2,\{-2,2\})\}$$

$$yf_G = \{(e,\{-2,0\}),(x,\{-2,2\}),(x^2,\{-2,1\}),(y,\{-2,-1,0,1,2\}),(yx,\{-2,-1\}),(yx^2,\{-2,-1\})\}$$

$$yxf_G = \{(e,\{-2,2\}),(x,\{-2,1\}),(x^2,\{-2,0\}),(y,\{-2,-1\}),(yx,\{-2,-1,0,1,2\}),(yx^2,\{-2,-1\})\}$$

$$yx^2f_G = \{(e,\{-2,1\}),(x,\{-2,0\}),(x^2,\{-2,2\}),(y,\{-2,-1\}),(yx,\{-2,-1\}),(yx^2,\{-2,-1,0,1,2\})\}$$

When we can calculate the soft right cosets of f_G similarly, different soft right cosets of f_G are as follows:

$$f_Ge = \{(e,\{-2,-1,0,1,2\}),(x,\{-2,-1\}),(x^2,\{-2,-1\}),(y,\{-2,0\}),(yx,\{-2,2\}),(yx^2,\{-2,1\})\}$$

$$f_Gx = \{(e,\{-2,-1\}),(x,\{-2,-1,0,1,2\}),(x^2,\{-2,-1\}),(y,\{-2,1\}),(yx,\{-2,0\}),(yx^2,\{-2,2\})\}$$

$$f_Gx^2 = \{(e,\{-2,-1\}),(x,\{-2,-1\}),(x^2,\{-2,-1,0,1,2\}),(y,\{-2,2\}),(yx,\{-2,1\}),(yx^2,\{-2,0\})\}$$

$f_G y$
$= \{(e,\{-2,0\}),(x,\{-2,1\}),(x^2,\{-2,2\}),(y,\{-2,-1,0,1,2\}),(yx,\{-2,-1\}),(yx^2,\{-2,-1\})\}$

$f_G yx$
$= \{(e,\{-2,2\}),(x,\{-2,0\}),(x^2,\{-2,1\}),(y,\{-2,-1\}),(yx,\{-2,-1,0,1,2\}),(yx^2,\{-2,-1\})\}$

$f_G yx^2$
$= \{(e,\{-2,1\}),(x,\{-2,2\}),(x^2,\{-2,0\}),(y,\{-2,-1\}),(yx,\{-2,-1\}),(yx^2,\{-2,-1,0,1,2\})\}$

Note 3.5. In Example 3.3, although each left coset is a soft right coset at the same time; in Example 3.4, it is seen that $af_G \neq f_G a$ for all $a \in G$-{e}. Hence, by Theorem 2.14, the soft set f_G in Example 3.3. is a normal soft int-group; whereas the soft set f_G in Example 3.4 is not.

The converse of Note 3.2 holds when the soft set is injective, as shown in the following proposition:

Proposition 3.6. If f_G is an injective function and $af_G = f_G a$ for all $a \in G$, then G is an abelian group.

Proof: Let $af_G = f_G a$ for all $a \in G$. Then, $\forall x,y \in G$;

$$f_G(xy) = (x^{-1}f_G)(y) = (f_G x^{-1})(y) = f_G(y(x^{-1})^{-1}) = f_G(yx)$$

Since f_G is an injective function;xy=yx for all x,y ∈G. Thus, G is an abelian group.

Note 3.7. In Note 3.2, it is stated that if G is an abelian group, then $af_G = f_G a$ for all $a \in G$. However, even when G is not an abelian group, it is possible that $af_G = f_G a$ for all $a \in G$. That is, the property of abelian for the group G is not a necessary condition for the equality of all soft left and soft right cosets. We have the following example:

Example 3.8. Let $U = \mathbb{Z}$ be the universal set and $G = D_3 = \{< x,y >: x^3 = y^2 = e,\ xy = yx^2\} = \{e,x,x^2,y,yx,yx^2\}$ be the set of parameters. We define the soft set f_G over U as follows:

$$f_G = \{(e,\mathbb{Z}),(x,\mathbb{Z}),(x^2,\mathbb{Z}),(y,\{-2,1\}),(yx,\{-2,1\}),(yx^2,\{-2,1\})\}$$

One can easily show that f_G is a soft int-group over $\mathbb{Z}$. Then, different soft left cosets of f_G are as follows:

$$ef_G = xf_G = x^2 f_G = \{(e,\mathbb{Z}),(x,\mathbb{Z}),(x^2,\mathbb{Z}),(y,\{-2,1\}),(yx,\{-2,1\}),(yx^2,\{-2,1\})\}$$
$$yf_G = yxf_G = yx^2 f_G = \{(e,\{-2,1\}),(x,\{-2,1\}),(x^2,\{-2,1\}),(y,\mathbb{Z}),(yx,\mathbb{Z}),(yx^2,\mathbb{Z})\}\}$$

When we calculate the soft right cosets of f_G, different soft right cosets are as follows:

$$f_G e = f_G x = f_G x^2 = \{(e,\mathbb{Z}),(x,\mathbb{Z}),(x^2,\mathbb{Z}),(y,\{-2,1\}),(yx,\{-2,1\}),(yx^2,\{-2,1\})\}$$
$$f_G y = f_G yx = f_G yx^2 = \{(e,\{-2,1\}),(x,\{-2,1\}),(x^2,\{-2,1\}),(y,\mathbb{Z}),(yx,\mathbb{Z}),(yx^2,\mathbb{Z})\}$$

Hence, $ef_G = f_G e,\ xf_G = f_G x,\ x^2 f_G = f_G x^2,\ yf_G = f_G\, y,\ yxf_G = f_G yx,\ \ yx^2 f_G = f_G yx^2$. We see that even though the group $G = D_3$ is not an abelian group; $af_G = f_G a$ for all $a \in G$, that is, f_G is a normal soft int-group by Theorem 2.14.

Note 3.9. By Note 3.2 and Note 3.7, we can conclude that if G is abelian, then f_G is a normal soft int-group; however, if G is not abelian, then f_G may be a normal soft int-group (see Example 3.8) or f_G may not be a normal soft int-group (see Example 3.3) as well.

In classical algebra, H itself is a left as well as right coset as eH=He=H. As an analogy, we have the following:

Proposition 3.10. Let f_G be soft set over U. Then, f_G itself is a left as well as right coset, that is, ef_G = f_Ge =f_G.

Proof: Let f_G be soft set over U. Then $\forall x \in G$;

$$(\mathrm{e}f_G)(x) = f_G(\mathrm{e}^{-1}x) = f_G(x) = f_G(x\mathrm{e}^{-1}) = (f_G\mathrm{e})(x)$$

Thus, the proof is completed.

In classical algebra, $a \in H \Leftrightarrow aH = H$. As an analogy, we have the following:

Theorem 3.11. Let f_G be a soft int-group over U. Then,

$$a \in G_{f_G} \Leftrightarrow af_G = f_G = f_G a$$

where G_{f_G}={x ∈G | $f_G(x)$= $f_G(e)$}.

Proof: Let f_G be a soft int-group over U and $a \in G_{f_G}$. Then, $f_G(a)$= $f_G(e)$. Thus,

$$af_G(x) = f_G(a^{-1}x) \supseteq f_G(a^{-1}) \cap f_G(x) = f_G(a) \cap f_G(x) = f_G(e) \cap f_G(x) = f_G(x)$$

Hence, $af_G \,\tilde{\supseteq}\, f_G$. Moreover, $\forall x \in G$;

$$f_G(x) = f_G(aa^{-1}x) \supseteq f_G(a) \cap f_G(a^{-1}x) = f_G(e) \cap f_G(a^{-1}x) = f_G(a^{-1}x) = (af_G)(x)$$

Thus, $f_G \,\tilde{\supseteq}\, af_G$. So, $af_G = f_G$. Similarly, $\forall x \in G$;

$$(f_G a)(x) = f_G(xa^{-1}) \supseteq f_G(x) \cap f_G(a^{-1}) = f_G(x) \cap f_G(a) = f_G(x) \cap f_G(e) = f_G(x)$$

Hence, $f_G a \,\tilde{\supseteq}\, f_G$. Also, $\forall x \in G$;

$$f_G(x) = f_G(xa^{-1}a) \supseteq f_G(xa^{-1}) \cap f_G(a) = f_G(xa^{-1}) \cap f_G(e) = f_G(xa^{-1}) = f_G a(x)$$

Thus, $f_G \,\tilde{\supseteq}\, f_G\, a$, implying that $f_G = f_G a$. We obtain that $a \in G_{f_G} \Longrightarrow af_G = f_G a = f_G$.

Conversely, let $af_G = f_G a = f_G$. Then, $\forall x \in G$; $af_G(x) = f_G a(x) = f_G(x)$. When we choose $x = a$, it yields

$$af_G(a) = f_G a(a) = f_G(a) \Rightarrow f_G(a^{-1}a) = f_G(aa^{-1}) = f_G(a) \Rightarrow f_G(e) = f_G(a) \Rightarrow a \in G_{f_G}$$

Note 3.12. In [38], it is proved that $a \in G_{f_G} \Leftrightarrow af_G = f_G$. However, Theorem 3.11 is more comprehensive. And it is of great importance for us, since it shows which soft left cosets and soft right cosets are equal to f_G, as well as it shows us which left and right cosets coincide with each other even if G is not abelian and/or f_G is not a normal soft int-group.

Moreover, we can conclude from Theorem 3.11 that when the cardinality of G_{f_G} is greater than 1, some soft left (right) cosets are the same, that is, they coincide with each other.

Example 3.13. Let $U = D_2$=$\{< x, y >: x^2 = y^2 = e,\ xy = yx\} = \{e, x, y, yx\}$ be the universal set and $G = S_3$ be the set of parameters. We define the soft set f_G over U as follows:

$$f_G = \{((1), D_2), ((12), \{e, y\}), ((13), \{e, y\}), ((23), \{e, y\}), ((123), D_2), ((132), D_2)\}$$

One can easily show that f_G is a soft int-group over D_2. Since

$$G_{f_G}=\{(1), (123), (132)\},$$

by Theorem 3.11 $(1)f_G$=$f_G(1) = f_G$, $(123)\ f_G$= $f_G(123) = f_G$ and $(132)\ f_G$= $f_G(132) = f_G$. In fact, $(1)f_G$=$f_G(1) = f_G$ is obvious by Proposition 3.10. Moreover,

$(123)f_G$= $\{(1, D_2)$, $((12),\{ e, y\})$, $((13),\{ e, y\})$, $((23), \{ e, y\})$, $((123), D_2), ((132), D_2)\}$

and

$f_G(123)$ = $\{(1, D_2)$, $((12),\{ e, y\})$, $((13),\{ e, y\})$, $((23), \{ e, y\})$, $((123), D_2), ((132), D_2)\}$.

Hence,

$$(123)\ f_G = f_G(123) = f_G$$

One can similarly show that $(132)f_G = f_G(132) = f_G$. Hence,

$$(1)f_G = f_G(1) = (123)f_G = f_G(123) = (132)f_G = f_G(132) = f_G$$

Similarly, in Example 3.8, G_{f_G}=$\{e, x, x^2\}$ and hence, $ef_G = xf_G = x^2f_G = f_G$ and $f_Ge = f_Gx = f_Gx^2$=f_G, thus

$$ef_G = f_G\mathrm{e} = xf_G = f_Gx = x^2f_G = f_Gx^2 = f_G$$

In classical algebra, left (right) cosets are not subgroups in general. It is obvious that when aH=H (Ha=H), left (right) cosets are subgroups. As an analogy, we have the following:

Proposition 3.14. Soft left (right) cosets are not soft int-groups, in general.

Proof: In Example 3.3, consider the left coset af_G. Since,

$$af_G = \{(e, \{-2, -1\}), (a, \{-3, -2, -1,0\}), (b, \{-3, -2, -1\}), (c, \{-2, -1,0\})\}$$

and $af_G(bb) = af_G(e) = \{ -2, -1\} \not\supseteq af_G(b) \cap af_G(b) = \{-3, -2, -1\}$, af_G is not a soft int-group; however as $ef_G = f_G$, ef_G is obviously a soft int-group.

Similarly, in Example 3.8, since $ef_G = \mathrm{x}f_G = x^2f_G$=$f_G$, each of them is a soft int-group; however, it is obvious that yf_G(and hence $y\mathrm{x}f_G, yx^2f_G$ as $yf_G = y\mathrm{x}f_G = yx^2f_G$) is not a soft int-group.

In classical algebra, $aH = bH \Leftrightarrow a^{-1}b \in H$ and $Ha = Hb \Leftrightarrow ab^{-1} \in H$. As an analogy, we have the following:

Proposition 3.15. Let f_G be a soft int-group over U. Then,

$$af_G = bf_G \Leftrightarrow aG_{f_G} = bG_{f_G} \quad (f_G a = f_G b \Leftrightarrow G_{f_G}a = G_{f_G}b)$$

for all $a, b \in$ G [29].

Note 3.16. Let f_G be a soft int-group over U. Then, by Proposition 3.15,

$$af_G = \text{b}f_G \Leftrightarrow aG_{f_G} = bG_{f_G} \Leftrightarrow a^{-1}b \in G_{f_G} \quad (f_G a = f_G b \Leftrightarrow G_{f_G}a = G_{f_G}b \Leftrightarrow ab^{-1} \in G_{f_G})$$

for all $a, b \in G$. Proposition 3.15 is of great importance in terms of showing which soft left (right) cosets are equal to each other.

Example 3.17. Let $U = \mathbb{Z}$ be the universal set and $G = \{\text{e}, x^2, \text{y}, \text{y}x^2\}$, the subgroup of $D_4 = \{< x, y >: x^4 = y^2 = e,\ xy = yx^3\} = \{e, x, x^2, x^3, y, yx, yx^2, yx^3\}$, be the set of parameters. We define the soft set f_G over U as follows:

$$f_G = \{(e, \mathbb{Z}), (x^2, \mathbb{Z}^-), (y, \mathbb{Z}^-), (\text{y}x^2, \mathbb{Z})\}$$

One can easily show that f_G is a soft int-group over $\mathbb{Z}$. The soft left cosets of f_G are:

$\text{e}f_G = \{(\text{e}, \mathbb{Z}), (\text{x}^2, \mathbb{Z}^-), (\text{y}, \mathbb{Z}^-), (\text{yx}^2, \mathbb{Z})\}$, $\text{x}^2f_G = \{(\text{e}, \mathbb{Z}^-), (\text{x}^2, \mathbb{Z}), (\text{y}, \mathbb{Z}), (\text{yx}^2, \mathbb{Z}^-)\}$,
$\text{y}f_G = \{(\text{e}, \mathbb{Z}^-), (\text{x}^2, \mathbb{Z}), (\text{y}, \mathbb{Z}), (\text{yx}^2, \mathbb{Z}^-)\}$, $\text{yx}^2f_G = \{(\text{e}, \mathbb{Z}), (\text{x}^2, \mathbb{Z}^-), (\text{y}, \mathbb{Z}^-), (\text{yx}^2, \mathbb{Z})\}$.

It is seen that $\text{e}f_G = \text{yx}^2f_G$ and $\text{x}^2f_G = \text{y}f_G$. In fact; since $G_{f_G} = \{e, yx^2\}$, it is expected by Proposition 3.10 that $\text{e}f_G = \text{yx}^2f_G = f_G$. In fact; e G_{f_G}= e.{e, yx²} = {e, yx²}, $\text{yx}^2G_{f_G}$= yx².{e, yx²} = {yx², e} and $\text{x}^2G_{f_G}$= x².{e, yx²} = {x², y}, $\text{y}G_{f_G}$= y.{e, yx²} = {y, x²}. Thus, $\text{e}G_{f_G} = \text{yx}^2 G_{f_G}$ and $\text{e}f_G = \text{yx}^2f_G$. Also, $\text{x}^2G_{f_G} = \text{y}G_{f_G}$ and $\text{x}^2f_G = \text{y}f_G$.

Proposition 3.18. Let f_G be a soft int-group over U and $a, b \in$ G. If $f_G(ab^{-1}) = f_G(e)$, then $f_G(a) = f_G(b)$ [36].

Corollary 3.19. Let f_G be a soft int-group over U and $a, b \in$ G. If $ab^{-1} \in G_{f_G}$, then $f_G(a) = f_G(b)$.

Proposition 3.20. Let f_G be a soft int-group over U and $a, b \in$ G. Then,

i. $ab \in G_{f_G} \Rightarrow f_G(a) = f_G(b)$
ii. $a^{-1}b \in G_{f_G} \Rightarrow f_G(a) = f_G(b)$
iii. $a^{-1}b^{-1} \in G_{f_G} \Rightarrow f_G(a) = f_G(b)$

Proof: Let f_G be a soft int-group over U and $a, b \in$ G.

(i) If $ab \in G_{f_G}$, then $f_G(ab) = f_G(e)$. Thus,

$$f_G(a) = f_G(abb^{-1}) \supseteq f_G(ab) \cap f_G(b) = f_G(e) \cap f_G(b) = f_G(b)$$

Therefore, $f_G(a) \supseteq f_G(b)$. Similarly,

$$f_G(b) = f_G(a^{-1}ab) \supseteq f_G(a^{-1}) \cap f_G(ab) = f_G(a) \cap f_G(e) = f_G(a).$$

Hence, $f_G(b) \supseteq f_G(a)$. Thus, $f_G(a) = f_G(b)$.

(ii) If $a^{-1}b \in G_{f_G}$, then $f_G(a^{-1}b) = f_G(e)$. Thus,

$$f_G(a) = f_G(a^{-1}) = f_G(a^{-1}bb^{-1}) \supseteq f_G(a^{-1}b) \cap f_G(b) = f_G(e) \cap f_G(b) = f_G(b)$$

Therefore, $f_G(a) \supseteq f_G(b)$. Similarly,

$$f_G(b) = f_G(aa^{-1}b) \supseteq f_G(a) \cap f_G(a^{-1}b) = f_G(a) \cap f_G(e) = f_G(a).$$

Hence, $f_G(b) \supseteq f_G(a)$. Thus, $f_G(a) = f_G(b)$.

(iii) If $a^{-1}b^{-1} \in G_{f_G}$, then $f_G(a^{-1}b^{-1}) = f_G(e)$. Thus,

$$f_G(a) = f_G(a^{-1}) = f_G(a^{-1}b^{-1}b) \supseteq f_G(a^{-1}b^{-1}) \cap f_G(b) = f_G(e) \cap f_G(b) = f_G(b)$$

Therefore, $f_G(a) \supseteq f_G(b)$. Similarly,
$$f_G(b) = f_G(b^{-1}) = f_G(aa^{-1}b^{-1}) \supseteq f_G(a) \cap f_G(a^{-1}b^{-1}) = f_G(a) \cap f_G(e) = f_G(a).$$

Hence, $f_G(b) \supseteq f_G(a)$. Thus, $f_G(a) = f_G(b)$.

Proposition 3.21. Let f_G be a soft int-group over U and $a, b \in$ G. Then,

$$af_G = bf_G \Leftrightarrow f_G(a^{-1}b) = f_G(b^{-1}a) = f_G(e)$$

Proof: Let f_G be a soft int-group over U, $a, b \in$ G and $af_G = bf_G$. Then,

$$af_G = bf_G \Rightarrow aG_{f_G} = bG_{f_G} \Rightarrow a^{-1}b \in G_{f_G},\ b^{-1}a \in G_{f_G} \Rightarrow f_G(a^{-1}b) = f_G(b^{-1}a) = f_G(e)$$

Let $f_G(a^{-1}b) = f_G(b^{-1}a) = f_G(e)$. Then, for all $x \in G$;

$$af_G(x) = f_G(a^{-1}x) = f_G(a^{-1}bb^{-1}x) \supseteq f_G(a^{-1}b) \cap f_G(b^{-1}x) = f_G(b^{-1}x) = bf_G(x)$$
$$bf_G(x) = f_G(b^{-1}x) = f_G(b^{-1}aa^{-1}x) \supseteq f_G(b^{-1}a) \cap f_G(a^{-1}x) = f_G(a^{-1}x) = af_G(x)$$

Thus, $af_G = bf_G$.

Theorem 3.22. Let f_G be a soft int-group over U, and f_N be a normal soft int-subgroup of f_G over U, $a, b \in$ G. Then, $af_G = bf_G \Rightarrow f_N(a) = f_N(b)$ [29].

Note 3.23. In [29,37], Theorem 3.22 is given with the condition that f_N is a normal soft int-subgroup of soft int-group of f_G over U and in the proof, the normality is used to prove the theorem. However, in the following theorem, we revise Theorem 3.22 [29,37] by removing the condition of normality.

Theorem 3.24. Let f_G be a soft int-group over U, $a, b \in$ G. Then, $af_G = bf_G \Rightarrow f_G(a) = f_G(b)$ $(f_G a = f_G b \Rightarrow f_G(a) = f_G(b))$

Proof: Let f_G be a soft int-group over U, $a, b \in$ G and $af_G = bf_G$. Then,

$$af_G = bf_G \Rightarrow aG_{f_G} = bG_{f_G} \Rightarrow a^{-1}b \in G_{f_G} \Rightarrow f_G(a) = f_G(b)$$

Note 3.25. The converse of Theorem 3.24, that is $f_G(a) = f_G(b) \Rightarrow af_G = bf_G$ needs not be true, as seen from the following example.

Example 3.26. Let $U = \mathbb{Z}$ be the universal set and $G = D_3 = \{< x, y >: x^3 = y^2 = e,\ xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ be the set of parameters. We define the soft set f_G over U as follows:

$$f_G = \{(e, \mathbb{Z}), (x, \{0,1,2,3\}), (x^2, \{0,1,2,3\}), (y, \{0,1\}), (yx, \{0,1\}), (yx^2, \{0,1\})\}$$

One can easily show that f_G is soft int-group over $\mathbb{Z}$. Soft left cosets xf_G and $x^2 f_G$ are as follows:

xf_G = { (e, {0, 1, 2, 3}), $(x, \mathbb{Z})$, (x^2, {0, 1, 2, 3}), (y, {0,1}), (yx, {0, 1}), (yx^2, {0, 1})}
$x^2 f_G$= { (e, {0, 1, 2, 3}), (x, {0, 1, 2, 3}), (x^2,$\mathbb{Z}$), (y, {0, 1}), (yx, {0,1}), (yx^2, {0, 1})}

It is obvious that f_G(x) = $f_G(x^2)$; however $xf_G \neq x^2 f_G$.

Theorem 3.27. Let f_G be a soft int-group over U, $x, y \in$ G. Then, $f_G(x) = f_G(e) \Leftrightarrow f_G(xy) = f_G(y)$ for all $y \in G$ [38].

Proposition 3.28. Let f_G be a soft int-group over U, $x, y \in$ G. Then, $f_G(x) = f_G(e) \Longrightarrow f_G(xy) = f_G(yx)$ for all $y \in G$.

Proof: Let f_G be a soft int-group over U, $x, y \in$ G. Then, from Theorem 3.27, if $f_G(x) = f_G(e)$, then $f_G(xy) = f_G(y)$ for all $y \epsilon G$. Thus,

$$f_G(yx) \supseteq f_G(y) \cap f_G(x) = f_G(y) \cap f_G(e) = f_G(y) \text{ and}$$
$$f_G(y) = f_G(yxx^{-1}) \supseteq f_G(yx) \cap f_G(x) = f_G(yx) \cap f_G(e) = f_G(yx)$$

Thus, $f_G(yx) = f_G(y)$. By Theorem 3.27, $f_G(x) = f_G(e) \Longrightarrow f_G(xy) = f_G(yx)$ for all $y \in G$.

Corollary 3.29. Let f_G be a soft int-group over U, $x, y \in$ G. Then, $x \in G_{f_G} \Longrightarrow f_G(xy) = f_G(yx)$ for all $y \in G$.

Theorem 3.30. Let f_G be a soft int-group over U. Then, G_{f_G} is a subgroup of G [29].

Corollary 3.31. Let G be an abelian group and f_G be a soft int-group over U. Then, G_{f_G} is a normal subgroup of G. [37].

Proposition 3.32. Let f_G be a normal soft int-group over U. Then, G_{f_G} is a normal subgroup of G [37].

Proof: By Theorem 3.30, G_{f_G} is a subgroup of G when f_G is a soft int-group over U. Therefore, it is enough to show that when f_G is a normal soft int-group over U, G_{f_G} is a normal subgroup of G. Let $x \in G$ and $n \in G_{f_G}$. Then,

$$f_G(xnx^{-1}) = f_G(xx^{-1}n) = f_G(n) = f_G(e)$$

Thus, $xnx^{-1} \in G_{f_G}$ and G_{f_G} is a normal subgroup of G. Here it is obvious that if G is abelian, then f_G is a normal soft int-group over U and G_{f_G} is a normal subgroup of G.
It is obvious by the definition of G_{f_G} that $\{e\} \in G_{f_G}$ and so $G_{f_G} \neq \emptyset$. What if $G_{f_G} \neq \{e\}$? Now, we are ready to give our main theorem, which gives the relation between G_{f_G} and the normality of f_G.

Theorem 3.33. Let f_G be a soft int-group over U. Then,

$$G_{f_G} \neq \{e\} \Rightarrow f_G \text{ is a normal soft int-group.}$$

Proof: Let f_G be a soft int-group over U and $G_{f_G} \neq \{e\}$. That is to say, there is at least one more element in G_{f_G} different from the identity element "e" of G. We need to show one of the equivalent conditions in Theorem 2.12.

1. CASE: Suppose $a \in G_{f_G}, b \in G_{f_G}$. Then, $ab \in G_{f_G}$. It follows that $f_G(a) = f_G(b) = f_G(ab) = f_G(e)$. By Proposition 3.28, for all $a, b \in G$, $f_G(ab) = f_G(ba)$.

2. CASE: Suppose $a \in G_{f_G}$, $b \notin G_{f_G}$. Then, $ab \notin G_{f_G}$. It follows that $f_G(a) = f_G(e)$. By Proposition 3.28, for all $a, b \in G$; $f_G(ab) = f_G(ba)$.

3. CASE: Suppose $a \notin G_{f_G}$, $b \in G_{f_G}$. Then, $ab \notin G_{f_G}$. It follows that $f_G(b) = f_G(e)$. By Proposition 3.28, for all $a \in G, f_G(ab) = f_G(ba)$.

4. CASE: Now suppose $a \notin G_{f_G}$, $b \notin G_{f_G}$. Then, either $ab \in G_{f_G}$ or $ab \notin G_{f_G}$.

i. Suppose $a \notin G_{f_G}$, $b \notin G_{f_G}$ then $ab \in G_{f_G}$. It follows that $f_G(ab) = f_G(e)$. By Proposition 3.20 (i), $f_G(a) = f_G(b)$. Thus, $f_G(aba^{-1}) \supseteq f_G(a) \cap f_G(b) \cap f_G(a^{-1}) = f_G(b) \cap f_G(b) \cap f_G(b) = f_G(b)$. By Theorem 2.12, f_G is a normal soft int-group.

ii.Suppose $a \notin G_{f_G}$, $b \notin G_{f_G}$ then $ab \notin G_{f_G}$. It is well known that
$[a \notin G_{f_G} \ \wedge \ b \notin G_{f_G} \Longrightarrow ab \notin G_{f_G}] \cong [ab \in G_{f_G} \Longrightarrow a \in G_{f_G} \ \vee \ b \in G_{f_G}]$

Let $ab \in G_{f_G} \Rightarrow a \in G_{f_G} \wedge b \notin G_{f_G}$. It follows that $f_G(ab) = f_G(e)$ and so $f_G(a) = f_G(b)$. But this is not a case under this cirtumtance, since $a \in G_{f_G} \wedge b \notin G_{f_G}$.
Let $ab \in G_{f_G} \Rightarrow a \notin G_{f_G} \wedge b \in G_{f_G}$. It follows that $f_G(ab) = f_G(e)$ and so $f_G(a) = f_G(b)$. But this is not a case under this circumstance, since $a \notin G_{f_G} \wedge b \in G_{f_G}$.
Let $ab \in G_{f_G} \Rightarrow a \in G_{f_G} \wedge b \in G_{f_G}$. It follows that $f_G(ab) = f_G(e)$ and so $f_G(a) = f_G(b)$, and since $a \in G_{f_G}$ and $b \in G_{f_G}$. $f_G(a) = f_G(b) = f_G(e)$. By Proposition 3.28, for all $a, b \in G$; $f_G(ab) = f_G(ba)$.

Example 3.34. In Example 3.8, $G_{f_G} = \{e, x, x^2\} \neq \{e\}$ and f_G is a normal soft int-group (even though $G = D_3$ is not abelian)

Note 3.35. The converse of Theorem 3.33, that is "f_G is a normal soft int-group $\Rightarrow G_{f_G} \neq \{e\}$" needs not be true is seen in Example 3.3. In fact, in Example 3.3, f_G is a normal soft int-group; but $G_{f_G} = \{e\}$.

Corollary 3.36. Let f_G be a soft int-group over U. Then, if f_G is not a normal soft int-group, $G_{f_G} = \{e\}$.
Now, we can revise Proposition 3.32 as follows:

Proposition 3.37. Let f_G be soft int-group (need not to be normal) over U. Then, G_{f_G} is a normal subgroup of G.

Proof: i) Let f_G be a normal soft int-group over U. Then, G_{f_G} is a normal subgroup of G by Proposition 3.32.
ii) Now let f_G be not a normal soft int-group over U. Then, $G_{f_G} = \{e\}$ by Corollary 3.36. Since $G_{f_G} = \{e\}$ is a trivial normal subgroup, the proof is completed.

Example 3.38. In Example 3.4, f_G is not a normal soft int-group and G_{f_G}=$\{e\}$ and, G_{f_G} is a normal subgroup of G.

Note 3.39. The converse of Proposition 3.37, that is "$G_{f_G} = \{e\} \Rightarrow f_G$ is not a normal soft int-group" needs not be true as seen from Example 3.3. In Example 3.3, $G_{f_G} = \{e\}$; however, f_G is a normal soft int-group.

Proposition 3.40. Let f_G be a soft int-group over U and $a \in$ G. Then,

$$af_G = a^{-1}f_G \Leftrightarrow a^2 \in G_{f_G}$$

Proof: Let f_G be a soft int-group over U, $a \in$ G. Then,

$$af_G = a^{-1}f_G \Leftrightarrow aG_{f_G} = a^{-1}G_{f_G} \Leftrightarrow a^{-1}a^{-1} \in G_{f_G} \Leftrightarrow (a^{-1})^2 \in G_{f_G} \Leftrightarrow (a^2)^{-1} \in G_{f_G} \Leftrightarrow a^2 \in G_{f_G}$$

Example 3.41. In Example 3.8, $x^{-1} = x^2$, $xf_G = x^2f_G$, and $x^2 \in G_{f_G}$ and $x^2f_G = f_G = f_Gx^2$

Proposition 3.42. Let f_G be a soft int-group over U and $a \in G$. For all $a \in G$;

$$f_G(a^2) = f_G(e) \Rightarrow f_G \text{ is a normal soft int-group.}$$

Proof: Let f_G be a soft int-group over U and $\forall a \in G; f_G(e) = f_G(a^2)$. Let $a, b \in G$, then $ab \in G$. Thus,

$$f_G(e) = f_G((ab)^2) = f_G(abab) = f_G(ab(a^{-1}a^2)b) = f_G((aba^{-1})(a^2b))$$

We obtain that $f_G((aba^{-1})\ (a^2b)) = f_G(e)$. By Proposition 3.20 (i), $f_G(aba^{-1}) = f_G(a^2b)$

Therefore, $f_G(aba^{-1}) = f_G(a^2b) \supseteq f_G(a^2) \cap f_G(b) = f_G(b)$. By Theorem 2.12, f_G is a normal soft int-group.

In classical group theory, if $a^2 = e$ (that is $a^{-1} = a$) for all $x \in G$, then G is an abelian group. As an analogy, we have the following:

Theorem 3.43. Let f_G be a soft int-group over U and $a \in$ G. If $\forall a \in G;\ af_G = a^{-1}f_G$, then f_G is a normal soft int-group.

Proof: Let f_G be a soft int-group over U and $a \in G$. If $af_G = a^{-1}f_G$ for all $a \in G$; then $a^2 \in G_{f_G}$ by Proposition 3.40. Thus, $f_G(a^2) = f_G(e)$ for all $a \in G$. Hence, f_G is a normal soft int-group by Proposition 3.42.

Theorem 3.44. Let f_G be a soft int-group over U and $a, b \in G$. Then,

$$a \sim b \Leftrightarrow f_G(ab^{-1}) = f_G(e)$$

is an equivalence relation on G.

Proof: Let f_G be a soft int-group over U and $a, b \in G$.

Reflexive: For all $a \in G$; $f_G(aa^{-1}) = f_G(e)$.

Symmetry: Suppose that $a \sim b$, we need to show that $b \sim a$. $a \sim b \Rightarrow f_G(ab^{-1}) = f_G(e) \Rightarrow f_G(ba^{-1}) = f_G(e) \Rightarrow b \sim a$.

Transitive: Let $a \sim b$ and $b \sim c$. We need to show that $a \sim c$. $a \sim b \Rightarrow f_G(ab^{-1}) = f_G(e)$ and $b \sim c \Rightarrow f_G(bc^{-1}) = f_G(e)$. Hence, $f_G(ac^{-1}) = f_G(ab^{-1}bc^{-1}) \supseteq f_G(ab^{-1}) \cap f_G(bc^{-1}) = f_G(e) \cap f_G(e) = f_G(e)$

This implies that $a \sim c$. Since $\sim$ is an equivalence relation on G, we can find the equiavalence classes for all $a \in$ G.

$$\bar{a} = \{b \mid f_G(ab^{-1}) = f_G(e)\} = \{b \mid ab^{-1} \in G_{f_G}\} = \{b \mid ba^{-1} \in G_{f_G}\} = \{b \mid b \in G_{f_G}a\} = G_{f_G}a$$

That is $\bar{a} = G_{f_G}a$. If $a \sim b \Leftrightarrow f_G(a^{-1}b) = f_G(e)$, then $\bar{a} = aG_{f_G}$.

In classical algebra, there is a one-to-one correspondence between the set of all left cosets of H in G and the set of right cosets of H in G. As an analogy, we have the following:

Theorem 3.45. There is a one-to-one correspondence between the set of all soft left cosets and the set of all soft right cosets of f_G.

Proof: Let f_G be a soft int-group over U and A= $\{f_G a: a \in$G$\}$ and B= $\{af_G: a \in$ G$\}$ be the set of soft right and soft left cosets, respectively. Define a map from A to B as follows:

$$\varphi : A \longrightarrow B$$
$$f_G a \longrightarrow \varphi(f_G a) = a^{-1}f_G$$

First, let us show φ is a well-defined function. Let $a, b \in G$ and $f_G a = f_G$b. We need to show that $a^{-1}f_G = b^{-1}f_G$.

$$f_G a = f_G b \Rightarrow G_{f_G}a = G_{f_G}b \Rightarrow a\, b^{-1} \in G_{f_G} \Rightarrow (a^{-1})^{-1}b^{-1} \in G_{f_G} \Rightarrow a^{-1}G_{f_G} = b^{-1}G_{f_G} \Rightarrow a^{-1}f_G = b^{-1}f_G$$

Hence, φ is a well-defined function.

Now, we show that φ is an injective function. Let $a, b \in G$ and $\varphi(f_G a) = \varphi(f_G b)$. We need to show that $f_G a = f_G$b.

$\varphi(f_G\ a) = \varphi(f_Gb) \Rightarrow a^{-1} f_G =$ b$^{-1} f_G \Rightarrow a^{-1}\ G_{f_G} =$ b$^{-1} G_{f_G} \Rightarrow ab^{-1} \in G_{f_G} \Rightarrow G_{f_G} a = G_{f_G}b\Rightarrow f_G a = f_G$b.

Hence, that φ is an injective function.

Finally, we need to show that φ is a surjective function. Since for all $af_G \in$ B, there exists $f_G a^{-1} \in$ A such that $\varphi(f_G a^{-1}) = af_G$ (since G is a group, every element has an inverse), φ is a surjective function. Therefore, φ is a bijective function.

Definition 3.46. The number of (different) soft right cosets of f_G is equal to the number of different soft left cosets and is known as the soft index of f_G in G, denoted by $[G: f_G]$.
In other words, when we consider the set $G/_{f_G} = \{ xf_G : x \in G\}$, then the cardinality of the set $G/_{f_G}$ is the soft index of f_G in G [38].

Example 3.47. In Example 3.3, there are 4 different soft left cosets and 4 different soft right cosets; in Example 3.4, there are 6 different soft left cosets and 6 different soft right cosets; in Example 3.8 there are 2 different soft left cosets and 2 different soft right cosets.

In fact, In Example 3.3, $G/_{f_G} = \{ef_G, af_G, bf_G, cf_G\}$ and so [G: f_G]=4; in Example 3.4,

$G/_{f_G} = \{ef_G, xf_G, x^2 f_G, yf_G, yxf_G, yx^2 f_G\}$ and so [G: f_G]=6; in Example 3.8, $G/_{f_G} = \{ef_G, yf_G\}$ and [G: f_G]=2 .

Corollary 3.48. By Theorem 3.45, we can conclude that $f_G a = f_G b \Leftrightarrow a^{-1} f_G = b^{-1} f_G$ ($af_G = bf_G \Leftrightarrow f_G a^{-1} = f_G b^{-1}$) for all a, b ∈ G. In Example 3.8, $f_G x = f_G x^2$, $x^{-1} = x^2$ and $x^2 f_G = xf_G$.

In abstract algebra, "Quotient Group" has a vital role in group theory. The concept of "Quotient Group" (which we say soft quotient group) relative to the normal soft int-group f_G was first constructed and defined by Kaygısız [37] as follows:

Definition 3.49. Let f_G be a normal soft int-group. Consider the set $G/_{f_G}$, where $G/_{f_G} = \{ xf_G : x \in G\}$. Define an operation of composition on $G/_{f_G}$ as follows:

$$*: G/_{f_G} \times G/_{f_G} \longrightarrow G/_{f_G}$$

$$(xf_G, yf_G) = (xf_G) * (yf_G) = xyf_G$$

for all $x, y \in G$. Then $\left(G/_{f_G}, *\right)$ forms a group under the composition and is called the (soft) quotient (or factor) group of G relative to the normal soft int-group f_G [37].

Theorem 3.50. Let f_G be a normal soft int group. Then, ${}^{G}/_{f_G} \cong {}^{G}/_{G_{f_G}}$[37].

Proof: Since f_G is a normal soft int group by Theorem 3.32 and G_{f_G} is a normal subgroup of G. Hence, ${}^{G}/_{G_{f_G}}$ is a quotient group. In addition, ${}^{G}/_{G_{f_G}}$ is a group [37]. Now, define a map

$$\psi: {}^{G}/_{f_G} \longrightarrow {}^{G}/_{G_{f_G}}$$

$$xf_G \longrightarrow xG_{f_G}$$

Firstly, ψ is a homomorphism since, for all xf_G, $yf_G \in {}^{G}/_{f_G}$;

$$\psi\,((xf_G)*(yf_G)) = \psi\,(xyf_G) = xyG_{f_G} = (xG_{f_G})(yG_{f_G}) = \psi\,(xf_G)\,\psi\,(yf_G)$$

On the other hand, $xG_{f_G} = yG_{f_G} \Longrightarrow xf_G = yf_G$, and so ψ is injective. For all $xG_{f_G} \in {}^{G}/_{G_{f_G}}$, there exists at least one $xf_G \in {}^{G}/_{f_G}$ such that $\psi(xf_G) = xG_{f_G}$. Thus, ψ is bijective and so ${}^{G}/_{f_G} \cong {}^{G}/_{G_{f_G}}$.

This theorem shows that G_{f_G} plays a key role in the analysis of the number of different soft left (right) cosets.

Example 3.51. In Example 3.3, G= $\{e,a,b,c\}$ and G_{f_G}=$\{e\}$. Thus, ${}^{G}/_{G_{f_G}} = \{eG_{f_G}, aG_{f_G}, bG_{f_G}, cG_{f_G}\}$ and so ${}^{G}/_{f_G} = \{ef_G, af_G, bf_G, cf_G\}$. In Example 3.4, G= $\{e,x,x^2,y,yx,yx^2\}$, G_{f_G}=$\{e\}$. So, ${}^{G}/_{G_{f_G}} = \{eG_{f_G}, xG_{f_G}, x^2G_{f_G}, yG_{f_G}, yxG_{f_G}, yx^2G_{f_G}\}$ and so ${}^{G}/_{f_G} = \{ef_G, xf_G, x^2f_G, yf_G, yxf_G, yx^2f_G\}$. In Example 3.8, G= $\{e,x,x^2,y,yx,yx^2\}$ and G_{f_G}=$\{e,x,x^2\}$. Therefore, ${}^{G}/_{G_{f_G}} = \{eG_{f_G}, yG_{f_G}\}$ and so ${}^{G}/_{f_G} = \{ef_G, yf_G\}$.

Note 3.52. By Proposition 3.15 and Theorem 3.50,

$$[G: G_{f_G}] = [G: f_G]$$

This equality shows us that when we know the set G_{f_G}, we can calculate the number of distinct soft left or soft right cosets of G_{f_G} in G, that is we can obtain the index of G_{f_G} in G, and so the soft index of f_G in G.

Example 3.53. In Example 3.3 [G: G_{f_G}]=4; in Example 3.4 [G: G_{f_G}]=6; in Example 3.8 [G: G_{f_G}]=2. Thus, in Example 3.3 [G: f_G]=4; in Example 3.4 [G: f_G]= 6 and in Example 3.8 [G: f_G]=2.

Proposition 3.54. Let G_{f_G}=$\{e\}$. Then, $af_G = bf_G \Leftrightarrow a = b$

Proof: Let G_{f_G}=$\{e\}$. Then, ${}^{G}/_{G_{f_G}} \cong$ G and so ${}^{G}/_{f_G} \cong$ G. Since [G: G_{f_G}]= [G: f_G], the number of different soft left (right) cosets is the cardinality of G. This means that if G_{f_G}=$\{e\}$, then there is not any soft left (right) coset that is equal to each other. That is, af_G =bf_G $\Leftrightarrow$ $a = b$.

Example 3.55. In Example 3.3 and Example 3.4, G_{f_G}=$\{e\}$ and there is not any soft left (right) coset such that $a \neq b$, and $af_G = bf_G$, where $a, b \in G$.
In fact; in Example 3.3. G_{f_G}=$\{e\}$ and $G/_{f_G} = \{ef_G,\ af_G, bf_G, cf_G\}$. That is, there are 4 different soft (left) right cosets, where 4 is the cardinality of G.
And in Example 3.4. G_{f_G}=$\{e\}$ and $G/_{f_G} = \{ef_G,\ xf_G, x^2f_G, yf_G, yxf_G, yx^2f_G\}$. That is, there are 6 different soft (left) right cosets, where 6 is the cardinality of G.

Proposition 3.56. Let f_G be not a normal soft int-group and $a, b \in G$. Then, $af_G = bf_G \Leftrightarrow a = b$

Proof: By Corollary 3.36, if f_G is not a normal soft int-group, then $G_{f_G} = \{e\}$. By Proposition 3.54, $af_G = bf_G \Leftrightarrow a = b$.

Example 3.57. In Example 3.4**,** the soft set f_G is not a normal soft int-group and

$$G/_{f_G} = \{\ \mathrm{e}f_G, xf_G, x^2f_G, yf_G,\ yxf_G, yx^2f_G\}$$

We see that for $a, b \in G$, af_G=bf_G $\Leftrightarrow a = b$.

It is well-known in abstract algebra that for the set $G/_H$, H need not be a normal subgroup; however in order to construct a quotient group $G/_H$, H must be a normal subgroup of group G; otherwise, the composition defined on the set $G/_H$ is not well-defined. As an example, let $G = D_3 = \{< x, y >: x^3 = y^2 = e,\ xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ be the set of parameters and $H = \{e, y\}$ be the subgroup of G (Take care that H is not a normal subgroup of G), then

$$xH = yx^2H \text{ and } x^2H = yxH$$

Although $(xH,\ x^2H) = \{yx^2H, yxH)$; $(xh)(x^2H) = x^3H = eH$ and $(yx^2H)(yxH) = (yx^2)(yx)H = x^2H$, so

$$(xH)(x^2H) \neq (yx^2H)(yxH)$$

So the composition is not well-defined when H is not a normal subgroup.

Now we show that, to a big surprise, to construct a soft quotient group relative to f_G, f_G need not be a normal soft int-group. We show that when f_G is not a normal soft int-group, the composition operation defined on the set $G/_{\mathrm{H}}$ is again well-defined, as the operation is independent of the choice of representatives. This is very interesting, as we do not encounter such a case in classical abstract algebra.

In order to show this, first of all, we update the definition of soft quotient group relative to the soft int-group f_G and then show that the composition defined is well-defined.

Definition 3.58. Let f_G be soft int-group (needs not be normal). Consider the set $G/_{f_G}$, where $G/_{f_G} = \{\ xf_G\colon x \in G\ \}$ is the set of all the soft left (right) cosets of f_G. Define an operation of composition on $G/_{f_G}$ as follows:

$$*: {}^{G}/_{f_G} \text{x} \, {}^{G}/_{f_G} \longrightarrow {}^{G}/_{f_G}$$

$$(xf_G, yf_G) = (xf_G) * (yf_G) = xyf_G$$

for all $x, y \in G$. Then, $\left({}^{G}/_{f_G}, *\right)$ forms a group under the composition and is called the soft quotient (or factor) group of G relative to the soft int-group f_G.

Theorem 3.59. Let f_G be a soft-int group of G. Then, the set ${}^{G}/_{f_G}$ of all the soft left (right) cosets of f_G in G forms a group under the composition defined by $(xf_G) * (yf_G) = xyf_G$.

Proof: Let f_G be a soft-int group of G and consider the set

$${}^{G}/_{f_G} = \{xf_G : x \in G\}$$

for xf_G, $yf_G \in {}^{G}/_{f_G}$. Define the composition on ${}^{G}/_{f_G}$ as $(xf_G) * (yf_G) = xyf_G$. To show that the above composition is well-defined, let $a_1, a_2, b_1, b_2 \in G$ and

$$(a_1 f_G, b_1 f_G) = (a_2 f_G, b_2 f_G)$$

We need to show that

$$a_1 f_G * b_1 f_G = a_2 f_G * b_2 f_G$$

i) Let f_G be not a normal soft int-group and $(a_1 f_G, b_1 f_G) = (a_2 f_G, b_2 f_G)$. Thus, $a_1 f_G = a_2 f_G$ and $b_1 f_G = b_2 f_G$. By Proposition 3.56, $a_1 = a_2$ and $b_1 = b_2$. Thus, $a_1 b_1 f_G = a_2 b_2 f_G$ and, so $a_1 f_G * b_1 f_G = a_2 f_G * b_2 f_G$.

ii) Let f_G be a normal soft int-group and $(a_1 f_G, b_1 f_G) = (a_2 f_G, b_2 f_G)$. Thus, $a_1 f_G = a_2 f_G$ and $b_1 f_G = b_2 f_G$. Let $x \in G$. Then,

$$(a_1\, b_1) f_G(\text{x}) = f_G((a_1 b_1)^{-1} x) = f_G(({b_1}^{-1} {a_1}^{-1}) x) = (\, b_1 f_G)({a_1}^{-1} x) = (b_2 f_G)({a_1}^{-1} x) = (f_G b_2)({a_1}^{-1} x) = (f_G)({a_1}^{-1} x {b_2}^{-1}) = (\, a_1 f_G)(x {b_2}^{-1}) = (\, a_2 f_G)(x {b_2}^{-1}) = (f_G a_2)(\text{x} {b_2}^{-1}) = f_G(x {b_2}^{-1} {a_2}^{-1}) = f_G(x (a_2 b_2)^{-1}) = (a_2\, b_2) f_G(\text{x})$$

Thus, $a_1\, b_1 f_G = a_2 b_2 f_G$ and so $a_1 f_G * b_1 f_G = a_2 f_G * b_2 f_G$. This shows that the above composition is well-defined, that is this operation is independent of the choice of representatives xf_G and $\text{y}f_G$, although f_G is not a normal soft int-group.

In order to show that $({}^{G}/_{f_G}, *)$ is a group, that is the closure, associative, identity, and inverse property are satisfied, we refer to Kaygısız [37].

Example 3.60. Consider the soft set

$$f_G = \{(e, \{-2, -1, 0, 1, 2\}), (x, \{-2, -1\}), (x^2, \{-2, -1\}), (y, \{-2, 0\}), (yx, \{-2, 2\}), (yx^2, \{-2, 1\})\}$$

in Example 3.4. We know that f_G is a soft int-group over $\mathbb{Z}$, but it is not a normal soft int-group. Soft left cosets of f_G are as follows:

$$G/_{f_G} = \{ ef_G , xf_G , x^2f_G, yf_G, yxf_G, yx^2f_G\}$$

We see that there is no soft left (right) coset that is equal to each other, that is, $af_G = bf_G \Leftrightarrow a$=b. When we construct the soft quotient group of G relative to the soft int-group f_G, the Cayley group table of the soft quotient group is as follows:

Table 3. Cayley table of binary operation.

$*$	ef_G	xf_G	x^2f_G	yf_G	yxf_G	yx^2f_G
ef_G	ef_G	xf_G	x^2f_G	yf_G	yxf_G	yx^2f_G
xf_G	xf_G	x^2f_G	ef_G	yx^2f_G	yf_G	yxf_G
x^2f_G	x^2f_G	ef_G	xf_G	yxf_G	yx^2f_G	yf_G
yf_G	yf_G	yxf_G	yx^2f_G	ef_G	xf_G	x^2f_G
yxf_G	yxf_G	yx^2f_G	yf_G	x^2f_G	ef_G	xf_G
yx^2f_G	yx^2f_G	yf_G	yxf_G	xf_G	x^2f_G	ef_G

One can easily show that $*$ is a well-defined operation on $G/_{f_G}$ and $\left(G/_{f_G},*\right)$ is a group; in fact, the soft quotient group is isomorphic to D_3.

Now, we are ready to revise Theorem 3.50 as follows:

Theorem 3.61. Let f_G be a soft int group (needs not be normal). Then, $G/_{f_G} \cong G/_{G_{f_G}}$.

Proof: By Proposition 3.37, G_{f_G} is a normal group when f_G is normal or not. The rest of the proof is obvious by Theorem 3.50 and by Proposition 3.56.

Moreover, in [38, 41], for the canonical homomorphism, the condition of normality on f_G is given, but this condition is unnecessary, so we revise the canonical homomorphism by removing the normality condition on f_G.

Theorem 3.62. Let f_G be soft int-group, Then, for all x$\in G$,

$$\phi: G \longrightarrow G/_{f_G}$$
$$x \longrightarrow xf_G$$

is a homomorphism with the Kernel G_{f_G} (a Canonical (natural) homomorphism).

Proof: Let $x, y \in G$, and $x = y$. Then, it is obvious that $xf_G = yf_G$. Hence, ϕ is a well-defined function. Moreover,

$$\phi\text{ (xy)= xy}f_G\text{=(x}f_G)(\text{y}f_G) = \phi\text{ (x)}\phi\text{(y)}$$

Hence, ϕ is a homomorphism. Let us find the Kernel of the homomorphism:

Kernel ϕ={x$\in G : \phi(x) = f_G$}={x$\in G : xf_G = f_G$}=G_{f_G} (By Theorem 3.11)

Here note that if f_G is not normal, then $G/_{f_G}$ is still a group; so the normality of $f_G$can be removed.

In abstract, algebra it is well-known that the group homomorphism $\phi\colon G \longrightarrow G$ is injective if and only if the Kernel $\phi = \{e_G\}$. Now, we have the following corollary:

Corollary 3.63. Let f_G be a not normal soft int-group. Then,

$$\phi\colon G \longrightarrow {}^{G}\!/_{f_G}$$
$$x \longrightarrow xf_G$$

is a monomorphism.

Proof: By Theorem 3.62, the function $\phi\colon G \longrightarrow {}^{G}\!/_{f_G}$ such that $\phi(x) \longrightarrow xf_G$ for all $x \in G$ is a homomorphism. Let f_G be a not normal soft int-group. Then, $G_{f_G} = \text{Kernel}\ \phi = \{e_G\}$. Thus, ϕ is injective, and so ϕ is a monomorphism. Moreover, from this fact, we can deduce that $\phi(x) = \phi(y) \Longrightarrow x = y$. Thus, $xf_G = yf_G$ implies that $x = y$ when f_G is a not normal soft int-group as shown in Proposition 3.56.

In abstract algebra, if G is an abelian group, then so is ${}^{G}\!/_{f_G}$. As an analogy, we have the following:

Theorem 3.64. If G is an abelian group, then so is $\left({}^{G}\!/_{f_G}, *\right)$ [37].

In abstract algebra, if ${}^{G}\!/_{H}$ is abelian, then G needs not be an abelian group. (Consider the quotient group ${}^{S_3}\!/_{A_3}$, whose cardinality is 2, and so abelian; but the symmetric group S_3 is not abelian.) As an analogy, we have the following example showing that the converse of Theorem 3.64 is not true in general.

Example 3.65. Let $U = \mathbb{Z}$ be the universal set and $G = D_4 = \{< x, y >: x^4 = y^2 = e,\ xy = yx^3\} = \{e, x, x^2, x^3, y, yx, yx^2, yx^3\}$ be the set of parameters. We define the soft set f_G over U as follows:

$$f_G = \{(e, \mathbb{Z}), (x, \{0\}), (x^2, \mathbb{Z}), (x^3, \{0\}), (y, \{0\}), (yx, \{0\}), (yx^2, \{0\}), (yx^3, \{0\})\}$$

One can easily show that f_G is soft int-group over $\mathbb{Z}$. Since $G_{f_G} = \{e, x^2\}$, by Theorem 3.33 f_G is a normal soft int-group over G and there are 4 different left(right) cosets as following: $ef_G = x^2 f_G,\ xf_G = x^3 f_G,\ yf_G = yx^2 f_G, yxf_G = yx^3 f_G$ and so ${}^{G}\!/_{f_G} = \{ef_G, xf_G, yf_G, yxf_G\}$. When we construct the soft quotient group of G relative to the soft int-group f_G, the Cayley group table of the quotient group is as follows:

Table 4. Cayley table of binary operation.

$*$	ef_G	xf_G	yf_G	yxf_G
ef_G	ef_G	xf_G	yf_G	yxf_G
xf_G	xf_G	ef_G	yxf_G	yf_G
yf_G	yf_G	yxf_G	ef_G	xf_G
yxf_G	yxf_G	yf_G	xf_G	ef_G

It is obvious that $\left({}^{G}/_{f_G},*\right)$ is abelian (Klein 4-group); however and $G = D_4$ is not abelian. To satisfy the converse of Theorem 3.64, we have the following:

Theorem 3.66. If $\left({}^{G}/_{f_G},*\right)$ is abelian and f_G is injective, then G is abelian.

Proof: Let $\forall a, b \in G$ and $af_G * bf_G = bf_G * af_G$. Then,

$$af_G * bf_G = bf_G * af_G \Rightarrow abf_G = baf_G \Rightarrow abG_{f_G} = baG_{f_G} \Rightarrow (ab)^{-1}(ba) \in G_{f_G} \Rightarrow (b^{-1}a^{-1})(ba) \in G_{f_G} \Rightarrow f_G\big((b^{-1}a^{-1})(ba)\big) = f_G(e) \Rightarrow f_G((ab)^{-1}(ba)) = f_G(e) \Rightarrow f_G(ba) = f_G(ab) \Rightarrow ab = ba$$

Thus, G is abelian.

Theorem 3.67. f_G is an abelian (normal) soft int-group if and only if ${}^{G}/_{f_G}$ is abelian.

Proof: Let f_G be an abelian (normal) soft int-group. Then, $(xyx^{-1}y^{-1}) = f_G(e)$ for all $x, y \in G$; f_G. Thus,

$$f_G((xy)(yx)^{-1}) = f_G(e) \Rightarrow f_G((yx)^{-1}(xy)) = f_G(e) \Rightarrow (yx)^{-1}(xy) \in G_{f_G} \Rightarrow yxG_{f_G} = xyG_{f_G} \Rightarrow yxf_G = xyf_G \Rightarrow yf_G * xf_G = xf_G * yf_G.$$

Thus, ${}^{G}/_{f_G}$ is abelian.

Conversely, let ${}^{G}/_{f_G}$ be an abelian group. Then, $xf_G * yf_G = yf_G * xf_G \Rightarrow xyf_G = yxf_G \Rightarrow xyG_{f_G} = yxG_{f_G} \Rightarrow (xy)^{-1}(yx) \in G_{f_G} \Rightarrow f_G((xy)^{-1}(yx)) = f_G(e) \Rightarrow f_G(xy) = f_G(yx)$ for all $x, y \in G$. Thus, f_G is an abelian (normal) soft int-group.

Corollary 3.68. f_G is an abelian (normal) soft int-group if and only if ${}^{G}/_{G_{f_G}}$ is abelian. This result is of great importance since by looking at the abelian property of ${}^{G}/_{G_{f_G}}$, we can comment on the normality of f_G. Also, f_G is not an abelian (normal) soft int-group if and only if ${}^{G}/_{G_{f_G}}$ is not abelian.

Furthermore, in classical algebra it is well-known that if the index of H in G is 2, then H is normal. Similarly, if the soft index of f_G in G is 2, then f_G is normal in G. In fact, let [G: f_G]=2. Then, the cardinality of the soft quotient group ${}^{G}/_{f_G}$ is 2, and hence ${}^{G}/_{f_G}$ is an abelian group (Since 2 is a prime number). Hence, by Theorem 3.67, f_G is an abelian (normal) soft int-group. Here remember that ${}^{G}/_{f_G}$ is a group whether f_G is normal or not. Similarly, if the soft index of f_G in G is p or p^2, where p is a prime number, then f_G is normal in G, too. In Example 3.65, [G: f_G]=4=2^2, and f_G is normal (Note that in Example 3.65, $G_{f_G} \neq$\{e\}). Moreover, if [G: G_{f_G}]= p or p^2, then f_G is normal in G since [G: f_G]= [G: G_{f_G}].

Example 3.69. In Example 3.4, f_G is not an abelian (normal) soft int-group and ${}^{G}/_{f_G} = \{ef_G, xf_G, x^2f_G, yf_G, yxf_G, yx^2f_G\}$ is not abelian, too; since $(x^2f_G)(yf_G) = yxf_G$ and $(yf_G)(x^2f_G) = yx^2f_G$, and $yxf_G \neq yx^2f_G$,

Definition 3.70. Let f_G be a soft int-group over U. If there exists a positive integer n such that

$$f_G(x^n) = f_G(e)$$

then x is said to have finite order and the smallest such positive n such that $f_G(x^n) = f_G(e)$ is called the order of x with respect to f_G, and is denoted by $SOf_G(x)$. If there does not exist such a positive integer, then x is said to have infinite order [38].

Definition 3.71. If $SOf_G(x)$ is finite for all x∈ G, then f_G is called a torsion soft-int group.

Definition 3.72. If $SOf_G(x)$ is a power of a prime number p for all x∈ G, f_G is called p-soft int-group.

Example 3.73. In Example 3.4, $SOf_G(e)$=1, $SOf_G(x)$= $SOf_G(x^2)$=3, $SOf_G(y)$= $SOf_G(yx)$= $SOf_G(yx^2)$=2. Thus, f_G is torsion soft-int group and p-soft int-group.

The following definitions, order of a soft left(right) coset in the quotient group and the generator of the soft quotient group, are given in a classical way as in abstract algebra:

Definition 3.74. Let $G/_{f_G}$ be a quotient group and $xf_G \in G/_{f_G}$. If there exists a positive integer n such that $(xf_G)^n = f_G$ then xf_G is said to have finite order and the smallest such positive n such that $(xf_G)^n = f_G$ is called the order of xf_G is denoted by $SO(xf_G)$.

Definition 3.75. Let $G/_{f_G}$ be a quotient group. If there exists $xf_G \in G/_{f_G}$ such that

$$< (xf_G) >= \{(xf_G)^n, n \in \mathbb{Z}\} = G/_{f_G}$$

then $G/_{f_G}$ is called a cyclic soft quotient group and xf_G is called the generator of $G/_{f_G}$.

Example 3.76. Consider $G = \mathbb{Z}_{10}{}^* = \{\overline{1}, \overline{3}, \overline{7}, \overline{9}\}$ as the group and let $U = \mathbb{Z}$ be the universal set and f_G be a soft set over U defined by as follows:

$$f_G = \{(\overline{1}, \{-3,-2,-1,0\}), (\overline{3}, \{-2,-1\}), (\overline{7}, \{-2,-1\}), (\overline{9}, \{-3,-2,-1\})\}.$$

One can easily show that f_G is a soft int-group over $\mathbb{Z}$. Then, since $G = \mathbb{Z}_{10}{}^*$ is abelian, soft left and right cosets of f_G are as follows:

$$\overline{1}f_G = f_G\overline{1} = f_G = \{(\overline{1}, \{-3,-2,-1,0\}), (\overline{3}, \{-2,-1\}), (\overline{7}, \{-2,-1\}), (\overline{9}, \{-3,-2,-1\})\}$$

$$\overline{3}f_G = f_G\overline{3} = \{(\overline{1}, \{-2,-1\}), (\overline{3}, \{-3,-2,-1,0\}), (\overline{7}, \{-3,-2,-1\}), (\overline{9}, \{-2,-1\})\}$$

$$\overline{7}f_G = f_G\overline{7} = \{(\overline{1}, \{-2,-1\}), (\overline{3}, \{-3,-2,-1\}), (\overline{7}, \{-3,-2,-1,0\}), (\overline{9}, \{-2,-1\})\}$$

$$\overline{9}f_G = f_G\overline{9} = \{(\overline{1}, \{-3,-2,-1\}), (\overline{3}, \{-2,-1\}), (\overline{7}, \{-2,-1\}), (\overline{9}, \{-3,-2,-1,0\})\}$$

Since $<\bar{3}f_G> = \{\bar{3}f_G,\ \bar{9}f_G,\ \bar{7}f_G,\ f_G)\} = G/_{f_G}$, $G/_{f_G}$ is cyclic and $\bar{3}f_G$ is generator of $G/_{f_G}$. Similarly, one can show that $<\bar{7}f_G> = \{\bar{7}f_G,\ \bar{9}f_G,\ \bar{3}f_G,\ f_G)\} = G/_{f_G}$ and so $\bar{7}f_G$ is generator of $G/_{f_G}$.

Theorem 3.77. If G is a cyclic group then so is $G/_{f_G}$.

Proof: Let $x \in G,\ G =< x >$. Since $G/_{f_G} = \{gf_G = (x^n)f_G,\ n \in \mathbb{Z}\}$. Since

$$(x^n)f_G = (xx\ldots x)f_G = (xf_G)(xf_G)\ldots(xf_G) = (xf_G)^n$$

Thus, $G/_{f_G} = \{(xf_G)^n,\ n \in \mathbb{Z}\}$, so $G/_{f_G}$ is cyclic.

The converse of Theorem 3.77 is not true in general as shown in the following example.

Example 3.78. In Example 3.8, $G/_{f_G} = \{ef_G, yf_G\}$ and $G/_{f_G} =< yf_G >$; that is to say $G/_{f_G}$ is cyclic; but $G = S_3$ is not cyclic.

Theorem 3.79. f_G is a torsion soft-int group $\Leftrightarrow$ $G/_{f_G}$ is a torsion group.

Proof: Let f_G be a torsion soft-int group. Then, for any x$\in G$, there exists a positive integer n such that $f_G(x^n) = f_G(e)$. Thus,

$$f_G(x^n) = f_G(e) \Longrightarrow x^n \in G_{f_G} \Longrightarrow x^n f_G = f_G = ef_G \Longrightarrow (xf_G)^n = ef_G$$

Thus, $G/_{f_G}$ is a torsion group. Conversely, let $G/_{f_G}$ be a torsion group. Then, for any x$\in G$, there exists a positive integer n such that $(xf_G)^n = ef_G$. Thus,

$$(xf_G)^n = ef_G \Longrightarrow x^n f_G = ef_G = f_G \Longrightarrow x^n \in G_{f_G} \Longrightarrow f_G(x^n) = f_G(e)$$

Therefore, f_G is a torsion soft-int group.

Theorem 3.80. f_G is a *p*-soft-int group $\Leftrightarrow$ $G/_{f_G}$ is a *p*-group.

Proof: Let f_G be a *p*-soft-int group. Then, for any x$\in G$, there exist nonnegative integers k and prime *p* such that $f_G\left(x^{p^k}\right) = f_G(e)$. Then,

$$f_G(x^{p^k}) = f_G(e) \Longrightarrow x^{p^k} \in G_{f_G} \Longrightarrow x^{p^k} f_G = ef_G = f_G \Longrightarrow (xf_G)^{p^k} = ef_G$$

Thus, $G/_{f_G}$ is a *p*-group. Conversely, let $G/_{f_G}$ is a *p*-group. Then, for any x$\in G$, there exist nonnegative integers k and prime *p* such that $(xf_G)^{p^k} = ef_G$. Thus,

$$(xf_G)^{p^k} = ef_G \Longrightarrow x^{p^k} f_G = ef_G = f_G \Longrightarrow x^{p^k} \in G_{f_G} \Longrightarrow f_G(x^{p^k}) = f_G(e)$$

Hence, f_G is a p-soft-int group.

Proposition 3.81. Let f_G be a soft int-group over U and $a \in G$. For all $a \in G$, $f_G(a^2) = f_G(e) \Rightarrow {}^{G}/_{f_G}$ is an abelian 2-group.

Proof: Let f_G be a soft int-group over U and $a \in G$. By Proposition 3.42, if $f_G(a^2) = f_G(e)$ for all $a \in G$, then f_G is an abelian (normal) soft int-group over U. Then, by Theorem 3.67, ${}^{G}/_{f_G}$ is an abelian group. Now, we need to show that ${}^{G}/_{f_G}$ is a a 2-group. Let $f_G(a^2) = f_G(e)$ for all $a \in G$. Thus,

$$f_G(a^2) = f_G(e) \Rightarrow a^2 \in G_{f_G} \Rightarrow a^2 f_G = f_G \Rightarrow aaf_G = (af_G)(af_G) = f_G \Rightarrow (af_G)^2 = f_G$$

Thus, for all $a \in G$, ord(af_G)≤ 2, where ord(af_G) is the order of af_G. Therefore, for all $a \in G$, ord(af_G)= 1 or ord(af_G)=2. Hence, ${}^{G}/_{f_G}$ is a 2-group.

Corollary 3.82. Let f_G be a soft int-group over U and $a \in$ G. If for all $a \in G$; $af_G = a^{-1}f_G$, then ${}^{G}/_{f_G}$ is an abelian 2-group.

Proof: The proof is obvious by Proposition 3.40, Theorem 3.43 and Proposition 3.81.

Theorem 3.83. Let a, b ∈ G and f_G be a soft int-group over U. $f_G a = f_G b \Leftrightarrow f_G a^{-1} = f_G b^{-1}$ $(af_G = bf_G \Leftrightarrow a^{-1}f_G = b^{-1}f_G)$ for all a, b ∈ G.

Proof: Let a, b ∈ G and $f_G\, a = f_G\, b$. We need to show that $f_G a^{-1} = f_G\, b^{-1}$. If $f_G$is not a normal soft int-group and $f_G\, a = f_G\, b$, then $a = b$ by Proposition 3.58), and so $a^{-1} = b^{-1}$. Therefore, $f_G a^{-1} = f_G b^{1}$. If f_G is a normal soft int-group, then $f_G a = f_G\, b \Rightarrow G_{f_G}a = G_{f_G}b \Rightarrow a\, b^{-1} \in G_{f_G} \Rightarrow b\, a^{-1} \in G_{f_G} \Rightarrow a^{-1}\, b \in G_{f_G} \Rightarrow G_{f_G}a^{-1} = Gf_G b^{-1} \Rightarrow f_G a^{-1} = f_G b^{-1}$. Here note that if $f_G$is normal and $ba^{-1} \in G_{f_G} \Rightarrow f_G(\, ba^{-1}) = f_G(a^{-1}b)$= $f_G(e)$, thus $a^{-1}b \in G_{f_G}$.
Now, let $f_G a^{-1} = f_G b^{-1}$. We need to show that $f_G a = f_G\, b$.

$$f_G a^{-1} = f_G b^{-1} \Rightarrow G_{f_G}a^{-1} = G_{f_G}b^{-1} \Rightarrow a^{-1}b \in G_{f_G} \Rightarrow b\, a^{-1} \in G_{f_G} \Rightarrow a\, b^{-1} \in G_{f_G} \Rightarrow G_{f_G}\, a = G_{f_G}b \Rightarrow f_G\, a = f_G\, b.$$

Here note that when $a^{-1}b \in G_{f_G} \Rightarrow f_G(a^{-1}b) = f_G(ba^{-1})$= $f_G(e)$, thus $ba^{-1} \in G_{f_G}$.
In fact, in Example 3.8, $f_G x = f_G x^2$, $x^{-1} = x^2$ and $f_G x^2 = f_G x$.
In classical algebra, when G is a group and H and K are subsets of G, then H= $K \Leftrightarrow xH = xK$ (H= $K \Leftrightarrow H$x = Kx) for ∀x∈ G. As an analogy, we have the following:

Proposition 3.84. Let f_H and f_K be soft int-subgroups of f_G and $a \in G$. Then,

i. $f_H = f_K \Leftrightarrow af_H = af_K$
ii. $f_H = f_K \Leftrightarrow f_H a = f_K\, a$

Proof: We give the proof of (i), as (ii) can be shown similarly. Let $f_H = f_K$. Then, ∀ x∈ G;

$$(af_H)(\mathrm{x}) = f_H\, (a^{-1}\mathrm{x}) = f_K(a^{-1}\mathrm{x}) = (af_K)(\mathrm{x})$$

Hence, $af_H = af_K$. Conversely, let $af_H = af_K$. Then, for all x∈ G,

$$f_H(x)= f_H(a^{-1}\, a\text{ x})= (af_H)(\,a\text{x})= (af_K)(\,a\text{x}) = f_K(a^{-1}\, a\text{ x}) = f_K(\text{ex}) = f_K(x)$$

Thus, $f_H = f_K$.
In classical algebra, when G is a group and H and K are subsets of G, then $H \subseteq K \Leftrightarrow xH \subseteq xK$ ($H \subseteq K \Leftrightarrow H\text{x} \subseteq Kx$) for all x,y ∈G. As an analogy, we have the following:

Proposition 3. 85. Let f_H and f_K be soft int-subgroups of f_G and $a \in G$. Then,

i. $f_H \tilde{\subseteq} f_K \Leftrightarrow af_H \tilde{\subseteq} af_K$
ii. $f_H \tilde{\subseteq} f_K \Leftrightarrow f_H a \tilde{\subseteq} f_K a$

Proof: We give the proof of (i), as (ii) can be shown similarly. Let $f_H \tilde{\subseteq} f_K$. Then, ∀ x∈ G;

$$af_H(\text{x})= f_H(a^{-1}\text{x})\subseteq f_K(a^{-1}\text{x})=af_K(\text{x})$$

Thus, $af_H \tilde{\subseteq} af_K$. Conversely, let $af_H \tilde{\subseteq} af_K$. Then, for all x ϵ G,

$$f_H(\text{x})= f_H(\text{ex})= f_H\ (a^{-1}a\text{x})= a\text{f}_\text{H}(a\text{x}) \subseteq af_K(a\text{x})= f_K(a^{-1}a\text{x}) =f_K\ (\text{ex})=f_K(\text{x}).$$

Thus, $f_H \tilde{\subseteq} f_K$.

In classical algebra, when G is a group and H is a subset of G, then, $x(y\text{H})=(xy)\text{H}$ for all $x, y \in$ G. As an analogy, we have the following:

Proposition 3.86. Let f_G be a soft int-group and $a \in G$. Then,

i. $a(bf_G) = abf_G$
ii. $(f_G a)\ b = f_G ab$

Proof: We give the proof of (i), as (ii) can be shown similarly. For all x∈ G;

$$(a(bf_G))(\text{x})=bf_G(a^{-1}\text{x})= f_G(b^{-1}(\ a^{-1}\text{x}))= f_G(b^{-1}a^{-1}\text{x})=f_G(ab)^{-1}\text{x})=((a\ b)f_G)(\text{x})$$

Thus, $a\ (bf_G) = abf_G$.

In classical algebra, when G is a group and H and K are subsets of G, then $xH = yK \Leftrightarrow H= x^{-1}yK$ for all $x, y \in G$. As an analogy, we have the following:

Proposition 3.87. Let f_H and f_K be soft int-subgroups of f_G and $a, b \in G$. Then,

i. $af_G=bf_H \Leftrightarrow f_G= a^{-1}bf_H$
ii. $f_G\ a =f_H\ b \Leftrightarrow f_G= f_H ba^{-1}$

Proof: We give the proof of (i), as (ii) can be shown similarly. Let $a, b \in G$ and $af_G=bf_H$. Then, $\forall x \in G$;

$$(a^{-1}bf_H)(\text{x})=f_H((a^{-1}b)^{-1}x)=f_H(b^{-1}ax)=f_H(b^{-1}(ax))=bf_H(a\text{ x})=af_G(a\text{ x})=f_G(a^{-1}ax)=f_G(\text{x})$$

Hence, $\text{f}_\text{G} =a^{-1}b\,\text{f}_\text{H}$.Conversely, let and $a, b \in G$ and $f_G=a^{-1}bf_H$. Then, for all x∈ G;

$$(af_G)(\text{x})= f_G(a^{-1}\text{x})=(a^{-1}\text{b}f_H)(\ a^{-1}\text{x})=(bf_H)(a\ a^{-1}\text{x})=(bf_H)(\text{ex})=(bf_H)(\text{x}).$$

Hence, af_G=bf_H.

4. CONCLUSIONS

In this paper, we have dealt with the crucial concepts of the soft int-group theory, the soft cosets, and soft quotient groups, by examining and assessing their numerous structural features, especially in connection to classical abstract algebra. We showed that when G is abelian, soft left and right cosets coincide; but when the group G is not abelian, right and left cosets can be equal as well. That is to say, the soft int group may be normal. We focused on the key concept of e-set, and we proved that if an element is in the e-set, then its soft left and right cosets coincide, and also they are equal to the soft set itself. This paper's outstanding contribution is that we obtained the relation between the e-set and the normality of the soft int-group. We proved that if the e-set has an element other than the identity of the group, then the soft int-group is normal, and thus if the soft set is not normal, then there does not exist any equal soft left (right) cosets. This relation is quite vital in the theory, since based on this fact, we realize that in many definitions, propositions, and theorems in [29, 37, 38], the condition of normality of the soft group is not necessary and when we reveal this condition, the theorems still hold. Also, we came up with quite an interesting result, for constructing a soft quotient group and holding the isomorphism ${}^{G}\!/_{f_G} \cong {}^{G}\!/_{G_{f_G}}$, the soft group f_G needs not to be normal. We obtained some more characterization as regards the normality of the soft group such as if each soft left (right) cosets is equal to its inverse, then the soft int group is normal. Also, it was observed that f_G is an abelian (normal) soft int-group if and only if ${}^{G}\!/_{f_G}$ is abelian. Besides, the torsion soft-int group and p-soft int-group were introduced and we proved that f_G is a torsion soft-int group (p-soft int-group) if and only if the soft quotient group ${}^{G}\!/_{f_G}$ is a torsion (p-group), respectively. In this regard, this paper is an overall study of soft cosets, and we hope this paper will contribute to the theory both as a theoretical and practical aspect. In future studies, soft middle cosets, pseudo cosets, and double pseudo cosets can be studied in detail and their relations between soft cosets can be obtained. Furthermore, fuzzy group theory can be handled again as regards the normality and fuzzy quotient group considering the remarkable findings in this paper, since the soft set is a generalization of the fuzzy set.

REFERENCES

[1] Molodtsov, D., *Computers and Mathematics with Applications,* **37**(4-5), 19, 1999.

[2] Maji, P. K., Biswas, R., Roy, A. R., *Computers and Mathematics with Applications,* **45**, 555, 2003.

[3] Pei, D., Miao, D., *IEEE International Conference on Granular Computing,* **2**, 617, 2005.

[4] Ali, M. I., Feng, F., Liu, X., Min, W.K., Shabir, M., *Computers Mathematics with Applications,* **57**(9), 1547, 2009.

[5] Sezgin, A., Atagün, A.O., *Computers and Mathematics with Applications,* **61**(5), 1457, 2011.

[6] Qin, K., Hong, Z., *Journal of Computational and Applied Mathematics,* **234**(5), 1347, 2010
[7] Ali, M. I., Shabir, M., Naz, M., *Computers and Mathematics with Applications,* **61**(9), 2647, 2011.
[8] Sezgin, A., Ahmad, S., Mehmood, A., *Journal of New Theory,* **27**, 33, 2019.
[9] Stojanovic, N. S., *Military Technical Courier,* **69**(4), 779, 2021.
[10] Sezgin, A., Sarıalioğlu, M., *Journal of Kadirli Faculty of Applied Sciences* **4**(2), 325, 2024.
[11] Sezgin, A., Aybek, F.N., *Matrix Science Mathematic,* **7**(1), 27, 2023.
[12] Sezgin, A., Aybek, F.N., Atagün, A.O., *Black Sea Journal of Engineering and Science,* **6**(4), 330, 2023.
[13] Sezgin, A., Aybek, F.N., Güngör, N.B., *Acta Informatica Malaysia,* **7**(1), 3, 2023.
[14] Sezgin, A., Demirci, A.M., *Ikonion Journal of Mathematics,* **5**(2), 24, 2023
[15] Sezgin, A., Dagtoros, K., *Scientific Journal of Mehmet Akif Ersoy University,* **6**(2), 31, 2023.
[16] Sezgin, A., Yavuz, E., *Necmettin Erbakan University Journal of Science and* Engineering, **5**(2), 189, 2023.
[17] Sezgin, A., Çağman, N., *Osmaniye Korkut Ata University Journal of the Institute of Science and Technology,* **7**(1), 58, 2024.
[18] Sezgin, A., Çalışıcı, H., *Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B-Teorik Bilimler,* **12**(1), 32, 2024.
[19] Çağman, N., Enginoğlu, S., *European Journal of Operational Research,* **207**(2), 848, 2010.
[20] Aktaş, H., Çağman, N., *Information Sciences,* **177**(1), 2726, 2017.
[21] Sezgin, A., Atagün, A.O., *Computers and Mathematics with Applications,* **62**(2), 685, 2011.
[22] Aslam, M., Qurashi, S.M., *Annals of Fuzzy Mathematics and Informatics,* **4**, 177, 2012.
[23] Aktaş, H., Özlü, Ş., *Scientific World Journal*, **2014**(1), 1, 2014.
[24] Nazmal, S., *Palestina Journal of Mathemaics,* **8**, 189, 2019.
[25] Alajlan, A.I., Alghamdi, A.M., *Symmetry,* **15**(7), 1450, 2023.
[26] Ghosh, D.M.J., Samanta, T.K., *Jordan Journal of Mathematics and Statistics*, **9**(2), 141, 2016.
[27] Yaylalı, G., Polat, N.Ç., Tanay, B., *Journal of Intelligent and Fuzzy Systems,* **36**(3), 2963, 2019.
[28] Weldetekle, T.D., Belayneh, B.B., Wale, Z.T., Addis, G.M., *Research in Mathematics,* **11**(1), 1, 2024.
[29] Çağman, N., Çıtak, F., Aktaş, H., *Neural Computing and Applications,* **2,** 151, 2012.
[30] Sezgin, A., *Algebra Letters*, **3**, 3, 2016.
[31] Atagün, A., Sezgin, A., *Mathematical Sciences Letters,* **7**(1), 37, 2018.
[32] Sezgin, A., *Filomat,* **32**(8), 2995, 2018.
[33] Sezer, A.S., Çağman, N., Atagün, A.O., *Annals of Fuzzy Mathematics and Informatics,* **9**(2), 235, 2015.
[34] Sezgin, A., Çağman, N., Atagün, A. O., *Applied Soft Computing*, 54, 366, 2017.
[35] Sezgin, A., Atagün, A.O., Çağman, N., Demir, H., *New Mathematics and Natural Computation*, **18**(2), 495, 2022.
[36] Kaygısız, K., *Annals of Fuzzy Mathematics and Informatics,* **4**(2), 365, 2012.
[37] Kaygısız, K., Normal Soft int-Group, arXiv:1209.3157, 2012.
[38] Yalçın, T., *Amasya University the Graduate School of Natural and Applied Sciences (MSc Thesis),* Amasya, 58, 2013.

[39] Şimşek, İ., Kaygısız, K., Çağman, N., *Contemporary Analysis and Applied Mathematics*, **2**(2), 58, 2024.
[40] Şimşek, İ., Kaygısız, K., Çağman, N., *Journal of New Theory*, **6**, 66, 2015.
[41] Kaygısız, K., *Afrika Mathematica*, **29**, 641, 2018.
[42] Rasuli, R., *Annals of Mathematics and Computer Science*, **9,** 41, 2022.
[43] Çağman, N., Barzegar, R., Hosseini, S.B., *Soft Computing*, **24**, 11841, 2020.
[44] Barzegar, R., Hosseini, S.B., Çağman, N., *Afrika Mathematika*, **34**(17), 1, 2023.
[45] Fraleigh J.B., *A First Course in Abstract Algebra*, Addison-Wesley, Boston, 520, 2003.
[46] Hungerford, T.W., *Algebra, Graduate Texts in Mathematics*, Springer Science & Business Media, New York, 504, 2003.
[47] Pant, S., Dagtoros, K., Kholil, M.I., Vivas, A., *Optimum Science Journal*, **1,** 1, 2024.
[48] Memiş, S., *Journal of New Theory*, **38**, 1, 2022.
[49] Naeem, K., Memiş, S., *Granular Computing*, **8**(2), 397, 2023.