ORIGINAL PAPER

MONOTONE SEQUENCES IN METRIC SPACES

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Manuscript received: 15.11.2024; Accepted paper: 27.12.2024;
Published online: 30.03.2025.

Abstract. *The aim of this paper is to introduce the notion of monotone sequence in metric spaces. Some properties and examples are given.*
Keywords: *metric space; sequences; monotonicity.*

1. INTRODUCTION

Recall that a *metric space* is a non-empty set X endowed with a *distance* (or a *metric*), that is an application $f: X \to [0, \infty)$, satisfying the following axioms:
a) $d(x,x) = 0$, for every $x \in X$;
b) $d(y,x) = d(x,y)$, for every $x, y \in X$;
c) $d(x,y) + d(y,z) \geq d(x,z)$, for every $x, y, z \in X$.

The third property is called *the triangle inequality*.

The notion of *metric space* was introduced by the French mathematician Maurice Fréchet in 1906 and the German mathematician Felix Hausdorff in 1914. Since then, many properties and applications have been stated.

Usually, the notion of *monotonicity* is defined in spaces Y endowed with an ordering relation " $\leq$ ". More exactly, we say that a sequence

$$(y_n)_{n\geq 1} \subset (Y, \leq)$$

is *increasing* (or *decreasing*) if for every integer $n \geq 1$, it holds:

$$y_n \leq y_{n+1} \text{ (or } y_n \geq y_{n+1}\text{)}.$$

Moreover, we say that a sequence $(y_n)_{n\geq 1} \subset (Y, \leq)$ is monotone if it is either increasing or decreasing. Although there is no ordering relation in a general metric space, such a order was introduced by Menger [1], who defined the notion of points that are in between two points. More precisely, if points $a, b, x \in (X, d)$, then it is said that x is between a and b, if

$$d(a,x) + d(x,b) = d(a,b).$$

A kind of monotonicity was defined by Mud [2] with respect to a fixed element chosen in the respective metric space. We will introduce in this paper a notion of monotone sequences in metric spaces, not involving elements outside the given sequence.

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https://doi.org/10.46939/J.Sci.Arts-25.1-a01

2. THE RESULTS

As it is often done, we will start with the case of the metric space of the real numbers. In this sense, remark that if $(x_n)_{n\geq 1}$ is a monotone (say increasing) sequence of real numbers, then for every positive integers $m \leq n \leq p$, we have $x_m \leq x_n \leq x_p$ and consequently:

$$|x_m - x_n| + |x_n - x_p| = |x_m - x_p|. \quad (1)$$

This equality holds true also for any decreasing sequence $(x_n)_{n\geq 1} \subset \mathbb{R}$.

If we consider the (usual) metric space $(\mathbb{R}, d)$, with $d(x, y) = |x - y|$, then (1) can be rewritten in terms of distance d.

This fact entitles us to introduce the following:

Definition 1. Let (X, d) be a metric space, where $\mathrm{X} \neq \emptyset$. We will say that a sequence $(\mathrm{x_n})_{\mathrm{n}\geq 1} \subset \mathrm{X}$ is monotone if for every positive integers $\mathrm{m} \leq \mathrm{n} \leq \mathrm{p}$, we have:

$$d(x_m, x_n) + d(x_n, x_p) = d(x_m, x_p). \quad (2)$$

In the sequel we will characterize the monotone sequences in some classical metric spaces.

2.1. THE USUAL REAL METRIC SPACE

In the (usual) metric space $(\mathbb{R}, d)$, with $d(x, y) = |x - y|$, relation (2) becomes:

$$|x_m - x_n| + |x_n - x_p| = |x_m - x_p|,$$

where $1 \leq m \leq n \leq p$. If we take $n = m + 1$ and $p = m + 2$, we deduce that:

$$|x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| = |x_m - x_{m+2}|.$$

By denoting $a = x_m - x_{m+1}$ and $b = x_{m+1} - x_{m+2}$, we get: $|a| + |b| = |a + b|$. But this is true only if and only if a and b have the same sign.

It follows that the quantity $x_m - x_{m+1}$ has a constant sign, for every integer $m \geq 1$.

This means that the sequence $(x_n)_{\mathrm{N}\geq 1}$ is monotone with respect to the usual ordering on $\mathbb{R}$.

In conclusion, in the (usual) metric space $(\mathbb{R}, d)$, the monotonicity given by Definition 1 is the same as the monotonicity with respect to the natural ordering on $\mathbb{R}$.

2.2. THE K-DIMENSIONAL TAXICAB METRIC SPACE

Let us consider the k-dimensional real space $\mathbb{R}^k = \{X \mid X = (x_1, \dots, x_k)\ ,\ x_1 \in \mathbb{R}, \dots, x_k \in \mathbb{R}\}$ endowed with the so-called Taxicab metric:

$$d(X,Y) = \sum_{i=1}^{k} |x_i - y_i| \,,$$

for every $X, Y \in \mathbb{R}^k$, $X = (x_1, \dots, x_k)$, $Y = (y_1, \dots, y_k)$.

Let $(X_n)_{\mathrm{N} \geq 1} \subset \mathbb{R}^k$, say

$$\mathrm{X_N} = \left(\mathrm{X_N^1}, \dots, \mathrm{X_N^k}\right), \quad \mathrm{n} \geq 1,$$

be a monotonic sequence in the sense of Definition 1.

For every positive integers $m \leq n \leq p$, we have $d(X_m, X_n) + d\left(X_n, X_p\right) = d(X_m, X_p)$, or

$$\sum_{i=1}^{k} \left|X_m^i - X_n^i\right| + \sum_{i=1}^{k} \left|X_n^i - X_p^i\right| = \sum_{i=1}^{k} \left|X_m^i - X_p^i\right|. \tag{3}$$

By summing the inequalities:

$$\left|X_m^i - X_n^i\right| + \left|X_n^i - X_p^i\right| \geq \left|X_m^i - X_p^i\right|, \quad 1 \leq i \leq k, \tag{4}$$

we get the general inequality:

$$\sum_{i=1}^{k} \left|X_m^i - X_n^i\right| + \sum_{i=1}^{k} \left|X_n^i - X_p^i\right| \geq \sum_{i=1}^{k} \left|X_m^i - X_p^i\right|.$$

In our case (3), this inequality holds true with equality, and consequently, the same with inequalities (4):

$$\left|X_m^i - X_n^i\right| + \left|X_n^i - X_p^i\right| = \left|X_m^i - X_p^i\right|, \quad 1 \leq i \leq k.$$

As we proceeded in the previous example, it results that the numbers $X_m^i - X_n^i$ and $X_n^i - X_p^i$ have the same sign, for every $m \leq n \leq p$ and $1 \leq i \leq k$. This means that for all $1 \leq i \leq k$, the sequence $\left(X_n^i\right)_{n \geq 1} \subset \mathbb{R}$ is monotone, in the usual sense.

Now we are in a position to give the following.

Theorem 1. A sequence $(\mathrm{X_n})_{\mathrm{n} \geq 1}$, $\mathrm{X_N} = \left(\mathrm{X_n^1}, \dots, \mathrm{X_N^k}\right)$, of elements from the k-dimensional Taxicab metric space is monotone in the sense of Definition 1 if and only if each real sequence $(\mathrm{X_n^1})_{\mathrm{n} \geq 1}$, $(\mathrm{X_n^2})_{\mathrm{n} \geq 1}$, …, $\left(\mathrm{X_n^k}\right)_{\mathrm{n} \geq 1}$ is monotone in the usual sense (possibly of different monotonicity).

2.3. THE K-DIMENSIONAL EUCLIDIAN METRIC SPACE

Let us consider the k-dimensional real space $\mathbb{R}^k = \{X \mid X = (x_1, \dots, x_k)\,,\ x_1 \in \mathbb{R}, \dots, x_k \in \mathbb{R}\}$ endowed with the Euclidian metric:

$$\mathrm{D(X,Y)} = \sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} (\mathrm{X_i} - \mathrm{Y_i})^2}\ ,$$

for every $X, Y \in \mathbb{R}^k$, $X = (x_1, \dots, x_k)$, $Y = (y_1, \dots, y_k)$.

Let $(X_n)_{\mathrm{N}\geq 1} \subset \mathbb{R}^n$, say

$$\mathrm{X_N} = \left(\mathrm{X_N^1}, \dots, \mathrm{X_N^k}\right),\ \ \mathrm{n} \geq 1,$$

be a monotone sequence in the sense of Definition 1.

For every positive integers $m \leq n \leq p$, we have $d(X_m, X_n) + d\left(X_n, X_p\right) = d(X_m, X_p)$, or

$$\sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} (\mathrm{X_m^i} - \mathrm{X_n^i})^2} + \sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} \left(\mathrm{X_n^i} - \mathrm{X_p^i}\right)^2} = \sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} \left(\mathrm{X_m^i} - \mathrm{X_p^i}\right)^2}. \tag{5}$$

By denoting $a_i = \mathrm{X_m^i} - \mathrm{X_n^i}$, $\mathrm{b_i} = \mathrm{X_n^i} - \mathrm{X_p^i}$ (for arbitrary positive integers $\mathrm{m, n, p}$), we get:

$$\sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{A_I^2}} + \sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{b_i^2}} = \sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} (\mathrm{a_i} + \mathrm{b_i})^2}\ .$$

But the following inequality holds true, for every real numbers $a_i, b_i, 1 \leq i \leq n$:

$$\sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{A_I^2}} + \sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{b_i^2}} \geq \sqrt{\sum_{\mathrm{i=1}}^{\mathrm{k}} (\mathrm{a_i} + \mathrm{b_i})^2}. \tag{6}$$

It is also called the Minkovski inequality. As it is equivalent to the Cauchy-Buniakovski-Schwarz inequality (see, e.g., [1, 3-5]), inequality (6) holds with equality if and only if

$$\frac{\mathrm{A_1}}{\mathrm{B_1}} = \frac{\mathrm{A_2}}{\mathrm{B_2}} = \cdots = \frac{\mathrm{A_N}}{\mathrm{B_N}}, \tag{7}$$

with the convention that if a denominator vanishes, then the corresponding numerator vanishes, too. If we denote

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_k \end{pmatrix},\ \ b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_k \end{pmatrix},$$

then (7) can be written in the form $a = \lambda b$, for some $\lambda \in \mathbb{R}$.

Coming back to (5), and taking $m = 1$ and $n = 2$, we get: $X_p - X_2 = \lambda_p(X_2 - X_1)$, or:

$$X_p = \lambda_p(X_2 - X_1) + X_2,$$

where $\lambda_p \in \mathbb{R}$, and $\lambda_1 = -1$, $\lambda_2 = 1$. For these forms, the vectors: $X_m - X_n$, $X_n - X_p$, and $X_m - X_p$ are proportional:

$$X_m - X_n = (\lambda_m - \lambda_n)(X_2 - X_1)$$
$$X_n - X_p = (\lambda_n - \lambda_p)(X_2 - X_1)$$
$$X_m - X_p = (\lambda_m - \lambda_p)(X_2 - X_1).$$

Now, these are satisfied with a monotone sequence $(\lambda_n)_{n\geq 1}$. We are in a position to give the following

Theorem 2. A sequence $(X_n)_{n\geq 1}$, $X_N = (X_n^1, \dots, X_N^k)$, of elements from the k-dimensional Euclidian metric space is monotone in the sense of Definition 1 if and only if there exists a monotone sequence $(\lambda_n)_{n\geq 1} \in \mathbb{R}$, such that, for every integer $n \geq 1$, it holds:

$$X_n = \lambda_n(X_2 - X_1) + X_2.$$

3. A THEOREM OF WEIERSTRASS TYPE

We establish in this section a theorem of Weierstrass type in general metric spaces, with respect to the monotonicity defined in the first section of this paper.

In the sequel, (X, d) will be again a general metric space.

Recall that a sequence $(x_n)_{n\geq 1} \subset (X, d)$ is called *a Cauchy sequence* if for every $\varepsilon > 0$, there exists an integer $n(\varepsilon)$ (depending on ε) such that $d(x_m, x_n) < \varepsilon$, for every $m, n > n(\varepsilon)$.

Moreover, we say that the sequence $(x_n)_{n\geq 1} \subset (X, d)$ converges to $x \in X$, if for every $\varepsilon > 0$, there exists an integer $n(\varepsilon)$ (depending on ε) such that $d(x_n, x) < \varepsilon$, for every $n > n(\varepsilon)$.

It is well-known that a convergent sequence is a Cauchy sequence, but the reciprocal part it is not necessarily true.

A metric space is called *complete* if every Cauchy sequence is convergent.

We also use next the notion of *a bounded* sequence $(x_n)_{n\geq 1} \subset (X, d)$, that is $d(x_n, a) < M$, for every integer $n \geq 1$, with some $a \in X$ and $M > 0$. Note that if the sequence $(x_n)_{n\geq 1}$ is bounded, then for every other $a' \in X$, there exists $M' > 0$ such that $d(x_n, a') < M'$, for every positive integer n.

We give the following

Theorem 4. Every montone and bounded sequence

$$(x_n)_{n\geq 1} \subset (X, d)$$

is a Cauchy sequence.

Proof: Let

$$M = \sup_{n\geq 1} d(x_1, x_n).$$

For every $\varepsilon > 0$ (arbitrarily fixed), there exists $k \geq 1$ such that $d(x_1, x_k) > M - \varepsilon$. Then for every $m, n \geq k$, we have:

$$d(x_m, x_n) < \varepsilon, \tag{8}$$

meaning that $(x_n)_{n\geq 1}$ is a Cauchy sequence. This can be proven as follows.

Assume, without loss of generality, that $m \leq n$. We have:

$$d(x_1, x_m) = d(x_1, x_k) + d(x_k, x_m) \geq d(x_1, x_k) > M - \varepsilon,$$

so $d(x_1, x_m) > M - \varepsilon$. Using also $d(x_n, x_1) \leq M$, we deduce:

$$d(x_m, x_n) = d(x_1, x_n) - d(x_1, x_m) < M - (M - \varepsilon) = \varepsilon.$$

In conclusion, relation (8) is true, and the proof is completed.

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