ORIGINAL PAPER

GEOMETRIC PERSPECTIVE OF BERRY'S PHASE ACCORDING TO ALTERNATIVE ORTHOGONAL MODIFIED FRAME

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Abstract. *It is very important to determine the Frenet vector fields to obtain the characterizations of curves in 3-dimensional Euclidean space. However, it may not be possible to define Frenet vector fields at some points of the curve, or the Frenet frame may not be sufficient to determine the curve's characterization. New frames can be defined as alternatives to the Frenet frame in these situations. Lots of work has been done for these cases such as Bishop frame, $\{d_2,C,W\}$- frame, orthogonal frame, etc. In this study, we produce the magnetic curves related to an alternative modified frame that can be used in cases where the Frenet frame and, $\{d_2,C,W\}$- frame cannot be defined at some points. It is known that the trajectories of a charged particle in a magnetic field are called magnetic curves. The adapted frame vectors defined on the curve are critical when examining these trajectories. In this context, the new alternative orthogonal* $\{\mathbb{N},\mathbb{C},\mathbb{W}\}$ *frame described in this article will be beneficial as it is defined at every point of the magnetic curve. Thus, we obtain the characterization of the magnetic and electromagnetic curves associated with this frame. We found that magnetic and electromagnetic curves are circular helices or constant precession slant helices. Also, we showed that the electric field parallel transported with the Fermi Walker parallel transportation rule along a curve with the* $\{\mathbb{N},\mathbb{C},\mathbb{W}\}$ *frame. Furthermore, we provide some examples and visualize them using the Mathematica and Maple programs.*

Keywords: *Frame fields; magnetic flows; vector fields; electromagnetic curves; ordinary differential equations.*

1. INTRODUCTION

Curves in differential geometry have significant applications in mathematics and physics. The characterization of curves can explain many geometric and physical phenomena. A frame defined at every point on a curve provides a more precise approach to describing such situations. In three-dimensional spaces, a divergence-free vector field generates a magnetic field.

When a charged particle enters a magnetic field, the field influences the particle's motion. As a result, the particle follows a new trajectory known as a *magnetic curve*. Concerning the magnetic field **B**, the Lorentz force equation is given by:

$$\varphi(X) = \boldsymbol{B} \wedge X \quad (1)$$

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The following equations define the magnetic curves associated with the Lorentz force equations:

$$\varphi(d_1) = \boldsymbol{B} \wedge d_1 = \nabla_{d_1} d_1, \varphi(d_2) = \boldsymbol{B} \wedge d_2 = \nabla_{d_1} d_2, \varphi(d_3) = \boldsymbol{B} \wedge d_3 = \nabla_{d_1} \boldsymbol{B}, \\ \varphi(E) = \boldsymbol{B} \wedge E = \nabla_{d_1} E. \tag{2}$$

These magnetic curves are called d_1 -magnetic, d_2-magnetic, d_3-magnetic, and electromagnetic curves, respectively [1-4]. Magnetic curves are widely applied in various fields, such as atmospheric science, analytical chemistry, biochemistry, geochemistry, cyclotron technology, proton therapy, cancer treatment, and velocity selection (for details, see Refs. [5-6]).

In three-dimensional Riemannian manifolds, magnetic trajectories have been studied in [5-8]. The motion of a charged particle in the hyperbolic plane was examined by Comtet [5], while magnetic trajectories in complex projective spaces were analyzed by Adachi [9-21]. Cabrerizo et al. investigated the magnetic flow of the contact magnetic field [22]. In studies [2-12, 14], new characterizations of magnetic trajectories in 3-dimensional spaces E3E^3E3 and E13E_1^3E13 were presented. Additionally, research on alternative moving frames, electromagnetic theory, and electromagnetic curves has been conducted [4-15].

A helix, whose tangent vector makes a constant angle with a fixed direction, was introduced in [6], while Izumiya and Takeuchi defined the slant helix [16]. Kula and Yaylı provided special characterizations of the slant helix and showed that a curve of constant precession is a slant helix [18-19].

In this study, we construct magnetic curves associated with an alternative modified frame that can be employed in cases where the Frenet and $\{d_2$,C,W$\}$-frame cannot be defined at certain points (see [25]). This frame is referred to as the modified orthogonal $\{d_2$,C,W$\}$-frame. It is well-known that the trajectories of a charged particle in a magnetic field are called magnetic curves. When examining these trajectories, the adapted frame vectors defined along the curve are of critical importance. In this context, the new alternative orthogonal $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$-The frame described in this article is highly advantageous, as it is defined at every point on the magnetic curve.

Thus, we obtain the characterization of magnetic curves and electromagnetic curves associated with this frame. Additionally, we provide examples and visualize those using Mathematica and Maple programs.

2. PRELIMINARIES

In Euclidean 3-space, for all ς=(ς_1, ς_2, ς_3,) and υ=(υ_1, υ_2, υ_3,)∈X(M), the inner product is given by:

$$(\varsigma.\upsilon) = -\varsigma_1\,\upsilon_1 + \varsigma_2\,\upsilon_2 + \varsigma_3\upsilon_3. \tag{3}$$

The definition of the cross-product is:

$$\varsigma \wedge \upsilon = (-\varsigma_2\upsilon_3 + \varsigma_3\upsilon_2, \varsigma_3\upsilon_1 - \varsigma_1\upsilon_3, \text{-}\varsigma_2\upsilon_1 + \varsigma_1\upsilon_2). \tag{4}$$

The mixed product for all $\varsigma, \upsilon, \upsilon \in X(M)$ is given by

$$(\varsigma \wedge \upsilon.\upsilon) = dv(\varsigma, \upsilon, \upsilon) \tag{5}$$

where dv indicates the volume form on M. The definition of the semi-Riemannian curvature tensor is:

$$\Re(\varsigma, \upsilon)\upsilon = \nabla_\varsigma \nabla_\upsilon \upsilon - \nabla_\upsilon \nabla_\varsigma \upsilon - \nabla_{[\varsigma,\upsilon]}\upsilon. \tag{6}$$

The sectional curvature of the non-degenerate plane produced by $\{\varsigma, \upsilon\}$ is given by:

$$K(\varsigma, \upsilon) = \frac{(\Re(\varsigma, \upsilon)\varsigma.\upsilon)}{(\varsigma.\varsigma)(\upsilon.\upsilon) - (\varsigma.\upsilon)^2} \tag{7}$$

A semi-Riemannian space form is a manifold with constant sectional curvature CCC, and the curvature tensor is expressed as:

$$\Re(\varsigma, \upsilon)\upsilon = C\{(\upsilon.\varsigma)\upsilon - (\upsilon.\upsilon)\varsigma\} \tag{8}$$

Lemma 2.1. Let ρ be a curve with the Frenet apparatus $\{d_1, d_2, d_3, \kappa, \tau\}$. Then, the variations at t=0 of the curvatures and speed function of ρ are computed as follows:

$$\left(\frac{\partial v}{\partial t}(s,t)\right)\bigg|_{t=0} = v\left(\nabla_{d_1}\boldsymbol{B}.d_1\right) = 0, \tag{9}$$

$$\left(\frac{\partial \kappa}{\partial t}(s,t)\right)\bigg|_{t=0} = (R(\boldsymbol{B}, d_1)d_1.d_2) + \left(\nabla^2_{d_1}\boldsymbol{B}.d_2\right) - 2\kappa\left(\nabla_{d_1}\boldsymbol{B}.d_1\right) = 0, \tag{10}$$

$$\left(\frac{\partial \tau}{\partial t}(s,t)\right)\bigg|_{t=0} = [(R(\boldsymbol{B}, d_1)d_1.d_3) + (\nabla^2_{d_1}\boldsymbol{B}.d_3)]_s + (R(\boldsymbol{B}, d_1)d_2.d_3) + \tau(\nabla_{d_1}\boldsymbol{B}.d_1) + \kappa\left(\nabla_{d_1}\boldsymbol{B}.d_3\right) = 0, [5]. \tag{11}$$

Assume that ρ is a curve in Euclidean space in three dimensions and that its curvature κ is non-zero everywhere. Then, we can define the Serret-Frenet frame $\{d_1, d_2, d_3\}$ as follows:

$$d_1' = \kappa d_2, d_2' = -\kappa d_1 + \tau d_3, d_3' = -\tau d_2. \tag{12}$$

If the curvature function κ is not identically zero but has discrete zero points, the following orthogonal frame can be defined [20],

$$d_1' = d_2,\ d_2' = -\kappa^2 d_1 + \frac{\kappa'}{\kappa} d_2 + \tau d_3, B' = -\tau d_2 + \frac{\kappa'}{\kappa} d_3. \tag{13}$$

In the following section, we introduce an alternatively orthogonal frame, denoted as $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$, which provides several important advantages for curve theory in differential geometry. Assume that ρ be a unit speed curve then along the curve ρ , the $\{d_2, C, W\}$ frame is given by

$$d_2' = fC,\ C' = -fd_2 + gW, W' = -gC,$$

where

$$f=\sqrt{\kappa^2+\tau^2}=\| d_2 \|, g=\sigma f, \sigma=\frac{(\tau/\kappa)'\kappa^2}{(\kappa^2+\tau^2)^{(3/2)}}, C=\frac{d_2'}{d_2}.$$

From the definition of the alternatively moving $\{d_2, C, W\}$ frame, C can not be defined when the curvature f(s)=0 for some points $s=s_0$. In the case where the curvature function f(s) is not identically zero but has discrete zero points, we can define the following alternatively orthogonal frame:

$$\mathbb{N}'=\frac{\kappa'}{\kappa}\mathbb{N}+\kappa\mathbb{C}, \mathbb{C}'=-\frac{f^2}{\kappa}\mathbb{N}+\frac{f'}{f}\mathbb{C}+g\mathbb{W}', \mathbb{W}'=-g\mathbb{C}+\frac{f'}{f}\mathbb{W}', \tag{14}$$

where

$$\mathbb{N}=\kappa d_2 , \mathbb{C}=f\mathbb{C} , \mathbb{W}=f\mathbb{W}.$$

Thus, we have the following relationships:

$$\mathbb{W}'=-g\mathbb{C}+\frac{f'}{f}\mathbb{W}', \tag{15}$$

$$\mathbb{C}=fC=-\kappa d_1+\tau d_3, \tag{16}$$

$$\mathbb{W}=f\mathbb{W}=\tau d_1+\kappa d_3. \tag{17}$$

Next, the relationships between the orthogonal frame and the Frenet frame are calculated as follows,

$$d_1=-\frac{\kappa}{f^2}\mathbb{C}+\frac{\tau}{f^2}\mathbb{W}, \tag{18}$$

$$d_2=\frac{C}{\kappa}, \tag{19}$$

$$d_3=\frac{\tau}{f^2}\mathbb{C}+\frac{\kappa}{f^2}\mathbb{W}. \tag{20}$$

Thus, the orthogonal frame $\{\mathbb{N},\mathbb{C},\mathbb{W}\}$ is defined at all points where the curvature f, (resp. κ or τ) is not zero. Additionally, the cross-product of the alternatively orthogonal frame is calculated as:

$$\mathbb{N}\wedge\mathbb{C}=\kappa\mathbb{W}, C\wedge\mathbb{W}=\frac{f^2}{\kappa}\mathbb{N}, \mathbb{W}\wedge\mathbb{N}=\kappa\mathbb{C}. \tag{21}$$

3. MAGNETIC CURVES BASED ON ORTHOGONAL $\{\mathbb{N},\mathbb{C},\mathbb{W}\}$-FRAME

In three-dimensional spaces, a magnetic vector field is defined as a vector field that is free of divergence. The Lorentz force law is specified geometrically by the skew-symmetric operator Φ, which describes the action of a magnetic vector field **B** on a charged particle as follows,

$$\Phi(X)=\boldsymbol{B}\wedge X. \tag{22}$$

A charged particle experiences the Lorentz force when it enters a magnetic field. These forces combine to form a new trajectory called a magnetic curve. By combining the Lorentz force law with the tangent vector of the particle, the following equation is obtained,

$$\Phi(d_1) = \boldsymbol{B} \wedge d_1 = \nabla_{d_1} d_1. \tag{23}$$

Magnetic curves with respect to alternative $\{d_2, C, W\}$ frame are defined as,

$$\Phi(d_2) = \boldsymbol{B} \wedge d_2 = \nabla_{d_1} d_2. \tag{24}$$

When analyzing the trajectory of a charged particle, the adapted frame fields along the curve describing the particle's path provide critical insights. Therefore, defining a frame at each point of the curve is highly beneficial. To analyze magnetic trajectories, we utilize the alternative orthogonal $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ frame. The following equations describe the Lorentz force associated with the orthogonal $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ frame.

P**roposition 3.1**. Let ρ be a magnetic curve in Euclidean 3-space and $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ be orthogonal alternative frame along the curve ρ. Then the Lorentz force Φ satisfies the following equations,

$$\Phi(\mathbb{N}) = \kappa\mathbb{C}, \Phi(\mathbb{C}) = -\frac{f^2}{\kappa}\mathbb{N} + \frac{\omega}{f^2}\mathbb{W}, \Phi(\mathbb{W}) = -\frac{\omega}{f^2}\mathbb{C}. \tag{25}$$

Proof: Let ρ be a curve in three-dimensional Euclidean space. If ρ is a magnetic curve, it satisfies,

$$\Phi(d_2) = \boldsymbol{B} \wedge d_2 = \nabla_{d_1} d_2. \tag{26}$$

Using the magnetic curve equation, we compute

$$\Phi(N) = \boldsymbol{B} \wedge N = \kappa\boldsymbol{B} \wedge d_2 = \kappa\nabla_{d_1} d_2. \tag{27}$$

This leads to

$$\Phi(\mathbb{N}) = \kappa\mathbb{C}. \tag{28}$$

Next, we write $\Phi(\mathbb{C})$ in the form

$$\Phi(\mathbb{C}) = \varsigma_1\mathbb{N} + \varsigma_2\mathbb{C} + \varsigma_3\mathbb{W}. \tag{29}$$

From this expression, we compute the coefficients ς_1, ς_2 and ς_3,

$$\varsigma_1 = \frac{1}{\kappa^2}(\Phi(\mathbb{C}).\mathbb{N}) = -\frac{1}{\kappa^2}(\Phi(\mathbb{N}).\mathbb{C}) = -\frac{f^2}{\kappa}, \varsigma_2 = 0, \varsigma_3 = \frac{\omega}{f^2}. \tag{30}$$

By similar calculations, we determine $\Phi(\mathbb{W})$ as follows,

$$\Phi(\mathbb{W}) = -\frac{\omega}{f^2}. \tag{31}$$

Theorem 3.1. In Euclidean 3-space, let $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ represent an orthogonally-alternative frame along the curve ρ, where ρ is a magnetic curve. Then ρ is a magnetic curve in the Killing magnetic field B if and only if B can be expressed as the linear combination of the orthogonal frame $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$. Specially,

$$\boldsymbol{B} = \frac{\omega}{\kappa f^2}\mathbb{N} + \mathbb{W}. \tag{32}$$

Proof: Let ρ be a magnetic curve associated with the orthogonal frame $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$. Then the magnetic field B, associated with the curve ρ, can be expressed as,

$$\mathbf{B} = E_1\,\mathbb{N} + E_2\,\mathbb{C} + E_3\,\mathbb{W}. \tag{33}$$

To determine the coefficients E_1, E_2 and E_3, we compute the cross-products $\mathbf{B} \wedge \mathbb{N}$, $\mathbf{B} \wedge \mathbb{C}$, and $\mathbf{B} \wedge \mathbb{W}$. By comparing these expressions with the Lorentz force equations, we find

$$\boldsymbol{B} = \frac{\omega}{\kappa f^2}\mathbb{N} + \mathbb{W}. \tag{34}$$

Conversely, assume $\boldsymbol{B} = \frac{\omega}{\kappa f^2}\mathbb{N} + \mathbb{W}$. By computing $\Phi(d_1)$=$\mathbf{B}\wedge d_1$, we verify that ρ satisfies the magnetic curve condition. Therefore, ρ is a magnetic trajectory of the Killing magnetic field **B**.

Theorem 3.2. Consider a curve ρ in three-dimensional Euclidean space. The following differential equation is satisfied if ρ is a magnetic curve.

$$\frac{1}{\kappa}[((\frac{1}{\tau})(\frac{\omega}{f^2})'')' + \tau(\frac{\omega}{f^2})')]' + \frac{\kappa}{\tau}(\frac{\omega}{f^2})'') = 0, \omega = \frac{(\tau)' f^2}{\kappa}. \tag{35}$$

Proof: Let ρ represent a magnetic curve in Euclidean space. The magnetic vector field associated with the orthogonal frame $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ is given by,

$$\boldsymbol{B} = \frac{\omega}{\kappa f^2}\mathbb{N} + \mathbb{W}. \tag{36}$$

Using equation (9) and the derivatives of $\boldsymbol{B}$, we can derive

$$\omega\kappa - \tau' f^2 = 0. \tag{37}$$

Using the second derivative of **B** and substituting into equation (10), we compute,

$$-(\frac{\omega}{f^2})'' + \tau(\kappa' + \tau\frac{\omega}{f^2}) = 0. \tag{38}$$

Next, using equation (10), we compute,

$$\frac{1}{\kappa}[(\kappa' + \tau\frac{\omega}{f^2})' + \left(\tau\frac{\omega}{f^2})'\right]' + \kappa(\kappa' + \tau\frac{\omega}{f^2}) = 0, \tag{39}$$

Consequently, if we consider the above three equations, we obtain

$$\frac{1}{\kappa}[((\left(\frac{1}{\tau}\right)\left(\frac{\omega}{f^2}\right)'')' + \tau(\frac{\omega}{f^2})')]' + (\frac{\kappa}{\tau}\left(\frac{\omega}{f^2}\right)'') = 0. \tag{40}$$

Corollary 3.2. Let ρ be a curve in three-dimensional Euclidean space and let a point particle follow a magnetic trajectory along ρ under the magnetic vector field $\mathbf{B} = \frac{\omega}{\kappa f^2}\mathbb{N} + \mathbb{W}$. Denote the angle between **B** and ℕ by α . The following characterizations hold

i. If the angle α= π/2 between **B** and ℕ, the particle follows a circular helical trajectory.

ii. If the angle α (α≠π/2) between **B** and ℕ is constant, the particle follows a constant precession slant helical trajectory where the curvature f is constant. Specially

$$\kappa(s) = c\cos(\psi s), \tau(s) = -c\sin(\psi s)$$

where c is a constant.

i. If the angle $\alpha = 0$ between **B** and ℕ, the particle is not influenced by the magnetic field.

3.1. Electromagnetic curves according to $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ *−frame*

Curves that satisfy the Lorentz force Φ, electric field E, and magnetic field **B** under the following conditions are defined as electromagnetic curves,

$$\Phi(E) = \boldsymbol{B} \wedge E = \nabla_{d_1} E. \tag{41}$$

Firstly, we calculate the direction of the electric field under the condition that *E* is perpendicular to ℕ. Hence, we write

$$\left(E. \mathbb{N}\ (s)\right) = 0. \tag{42}$$

We also assume the derivative of *E* has the form,

$$\nabla_{d_1} E = \upsilon_1 \mathbb{N}(s) + \upsilon_2 \mathbb{C}(s) + \upsilon_3 \mathbb{W}(s). \tag{43}$$

By differentiating equation (42) with respect to s and using the formula (6), we compute,

$$\left(\nabla_{d_1} E. \mathbb{N}(s)\right) = -\kappa\left(E. \mathbb{C}(s)\right). \tag{44}$$

From equations (42) and (43), we derive,

$$\upsilon_1 = -\frac{1}{\kappa}\left(E. \mathbb{C}(s)\right). \tag{45}$$

Assuming the magnitude of *E* is constant (*E.E=c*, *c=const.*), its derivative yields,

$$\left(\nabla_{d_1} E. E\right) = 0. \tag{46}$$

Using equation (43), we get

$$\upsilon_2\big(E.\mathbb{C}(s)\big) = -\upsilon_3\big(E.\mathbb{W}(s)\big). \tag{47}$$

If both $(E.\mathbb{W}(s)) \neq 0$ and $(E.\mathbb{C}(s)) \neq 0$, then υ_2 and υ_3 are proportional

$$\upsilon_2 = \upsilon\big(E.\mathbb{W}(s)\big) \text{ and } \upsilon_3 = -\upsilon\big(E.\mathbb{C}(s)\big). \tag{48}$$

From equations (43)-(48), we obtain

$$\nabla_{d_1}E = -\frac{1}{\kappa}(E.\mathbb{C})\mathbb{N}(s) + \Big(\upsilon\big(E.\mathbb{W}(s)\big)\Big)\mathbb{C}(s) + \Big(-\upsilon\big(E.\mathbb{C}(s)\big)\Big)\mathbb{W}(s). \tag{49}$$

We can rewrite this as,

$$\nabla_{d_1}E = -\frac{1}{\kappa}(E.\mathbb{C})\mathbb{N}(s) + \upsilon - \frac{1}{\kappa}\big(E \wedge \mathbb{N}(s)\big). \tag{50}$$

If we assume that E is parallel transported ($\upsilon = 0$), the equation simplifies to,

$$\nabla_{d_1}E = -\frac{1}{\kappa}(E.\mathbb{N})\mathbb{N}(s). \tag{51}$$

Assuming $f^2E = (E.\mathbb{C}(s))\mathbb{C}(s) + \big(E.\mathbb{W}(s)\big)\mathbb{W}(s)$, we can write,

$$E = \frac{1}{f^2}(E.\mathbb{C}(s))\mathbb{C}(s) + \frac{1}{f^2}(E.\mathbb{W}(s))\mathbb{W}(s)). \tag{52}$$

Differentiating and comparing with Equation (51), we derive,

$$\begin{bmatrix} (E.\mathbb{C}(s))' \\ (E.\mathbb{W}(s))' \end{bmatrix} = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \begin{bmatrix} \big(E.\mathbb{C}(s)\big) \\ \big(E.\mathbb{W}(s)\big) \end{bmatrix}. \tag{53}$$

Without loss of generality, assuming $E.E = c$, the electric field can be expressed as,

$$E = \sin\alpha\,\mathbb{C}(s) + \cos\alpha\,\mathbb{W}(s), f = \text{const.} \tag{54}$$

Differentiating equation (54), we obtain

$$\begin{aligned}\nabla_{d_1}E = \left(-\frac{f^2}{\kappa}\sin\alpha\right)\mathbb{N}(s) + \left(\frac{d\alpha}{ds}\cos\alpha - g\cos\alpha\right)\mathbb{C}(s) + \\ + \left(g\sin\alpha - \frac{d\alpha}{ds}\sin\alpha\right)\mathbb{W}(s).\end{aligned} \tag{55}$$

Simplifying further,

$$\nabla_{d_1}E = \left(-\frac{f^2}{\kappa}\sin\alpha\right)\mathbb{N}(s) + \frac{1}{\kappa}\left(\frac{d\alpha}{ds} - g\right)E \wedge \mathbb{N}. \tag{56}$$

If we take $\frac{d\alpha}{ds} = g$, the electric field E is parallel to $\mathbb{N}(s)$. Using the Fermi-Walker parallel transportation law

$$\frac{\mathrm{dX}^{\mathrm{FW}}}{\mathrm{ds}} = \frac{\mathrm{dX}}{\mathrm{ds}} \pm (\mathrm{X}.\mathrm{d}_2)\frac{\mathrm{d}(\mathrm{d}_2(\mathrm{s}))}{\mathrm{ds}} + \left(\mathrm{X}.\frac{\mathrm{d}(\mathrm{d}_2(\mathrm{s}))}{\mathrm{ds}}\right)\mathrm{d}_2(\mathrm{s}). \tag{57}$$

we compute

$$\nabla_{\mathrm{d}_1}\mathrm{E}^{\mathrm{FW}} = \nabla_{\mathrm{d}_1}\mathrm{E} + \frac{1}{\kappa^2}\left(\mathrm{E}.\frac{\mathrm{d}\mathbb{N}(\mathrm{s})}{\mathrm{ds}}\right)\mathbb{N}(\mathrm{s}). \tag{58}$$

Finally, the simplifies to,

$$\nabla_{\mathrm{d}_1}\mathrm{E} = -\frac{1}{\kappa}\big(\mathrm{E}.\mathbb{C}(\mathrm{s})\big)\mathbb{N}(\mathrm{s}). \tag{59}$$

The electric field along the curve ρ becomes

$$\mathrm{E} = \sin(\int \mathrm{gds})\mathbb{C}(\mathrm{s}) + \cos(\int \mathrm{gds})\mathbb{W}(\mathrm{s}). \tag{60}$$

To compute the electromagnetic curves, we use the relation $\Phi(\mathrm{X}) = \mathbf{B} \wedge \mathrm{X}$ and equation (56). This yields

$$\Phi(E).\mathbb{N}(s) = -\Phi(\mathbb{N}(s).E), \Phi(E).\mathbb{C}(s) = -\Phi(\mathbb{C}(s).E), \Phi(E).\mathbb{W}(s) = -\Phi(\mathbb{W}(s).E) \tag{61}$$

Using equations (54) and (61), we can derive the following matrix representation for the Lorentz force,

$$\begin{bmatrix}\Phi(\mathbb{N}(\mathrm{s}))\\ \Phi(\mathbb{C}(\mathrm{s}))\\ \Phi(\mathbb{W}(\mathrm{s}))\end{bmatrix} = \begin{bmatrix}0 & \kappa(\mathrm{s}) & 0\\ -\frac{\mathrm{f}^2(\mathrm{s})}{\kappa(\mathrm{s})} & 0 & \upsilon\mathrm{f}^2(\mathrm{s})\\ 0 & -\upsilon\mathrm{f}^2(\mathrm{s}) & 0\end{bmatrix}\begin{bmatrix}\mathbb{N}(\mathrm{s})\\ \mathbb{C}(\mathrm{s})\\ \mathbb{W}(\mathrm{s})\end{bmatrix}. \tag{62}$$

We assume that the magnetic vector field **B** can be expressed as,

$$\mathbf{B} = \mathrm{a}_1\mathbb{N}(\mathrm{s}) + \mathrm{a}_2\mathbb{C}(\mathrm{s}) + \mathrm{a}_3\mathbb{W}(\mathrm{s}). \tag{63}$$

Using the Lorentz force identities,

$$\Phi(\mathbb{N}) = \mathbf{B} \wedge \mathbb{N}, \Phi(\mathbb{C}) = \mathbf{B} \wedge \mathbb{C}, \Phi(\mathbb{W}) = \mathbf{B} \wedge \mathbb{W}, \tag{64}$$

and substituting into equations (62), the components of the magnetic vector field **B** can be computed as,

$$\mathbf{B} = -\upsilon\frac{\mathrm{f}^2}{\kappa}\mathbb{N}(\mathrm{s}) + \mathbb{W}(\mathrm{s}). \tag{65}$$

Taking the Fermi-Walker parallel transportation law into account, we can state the Lorentz force as,

$$\begin{bmatrix} \Phi(\mathbb{N}(s)) \\ \Phi(\mathbb{C}(s)) \\ \Phi(\mathbb{W}(s)) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\frac{f^2(s)}{\kappa(s)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{N}(s) \\ \mathbb{C}(s) \\ \mathbb{W}(s) \end{bmatrix}. \tag{66}$$

Thus, the magnetic vector field simplifies to,

$$\boldsymbol{B} = \mathbb{W}(s). \tag{67}$$

Corollary 3.3. Let ρ a unit speed curve with respect to the $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ frame. Then the electric field E along the curve ρ parallel is transported under the Fermi-Walker transportation law.

Corollary 3.4. Let ρ a unit speed curve with respect to the $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ frame, where the electric field E perpendicular to N. Then ρ is an electromagnetic curve if and only if it is a constant precession curve with curvatures

$$\mathrm{K(s)} = -\mathrm{k}\sin(\varphi \mathrm{s}), \tau(\mathrm{s}) = \mathrm{k}\cos(\varphi \mathrm{s}),$$

where φ and k are constants. The associated electric and magnetic fields are

$$E = \sin(\int \mathrm{gds})\mathbb{C}(\mathrm{s}) + \cos(\int \mathrm{gds})\mathbb{W}(\mathrm{s}), \mathbf{B} = \mathbb{W}(\mathrm{s}).$$

Corollary 3.5. Let ρ a unit speed curve with respect to the $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ frame, where the electric field E perpendicular to $\mathbb{C}$. Then ρ is an electromagnetic curve if and only if it is a circular helix. The associated electric and magnetic fields are

$$E = \sin\alpha\, \mathbb{C}(\mathrm{s}) + \cos(\alpha \mathbb{W}(\mathrm{s}), \mathbf{B} = \frac{\mathrm{g}}{\kappa}\mathbb{N}(\mathrm{s}) + \mathbb{W}(\mathrm{s}), \tag{68}$$

where α is a constant.

Corollary 3.6. Let ρ a unit speed curve with respect to the $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ the frame where the electric field E perpendicular to W . Then ρ is an electromagnetic curve if and only if it is a circle with constant radius f. The associated electric and magnetic fields are

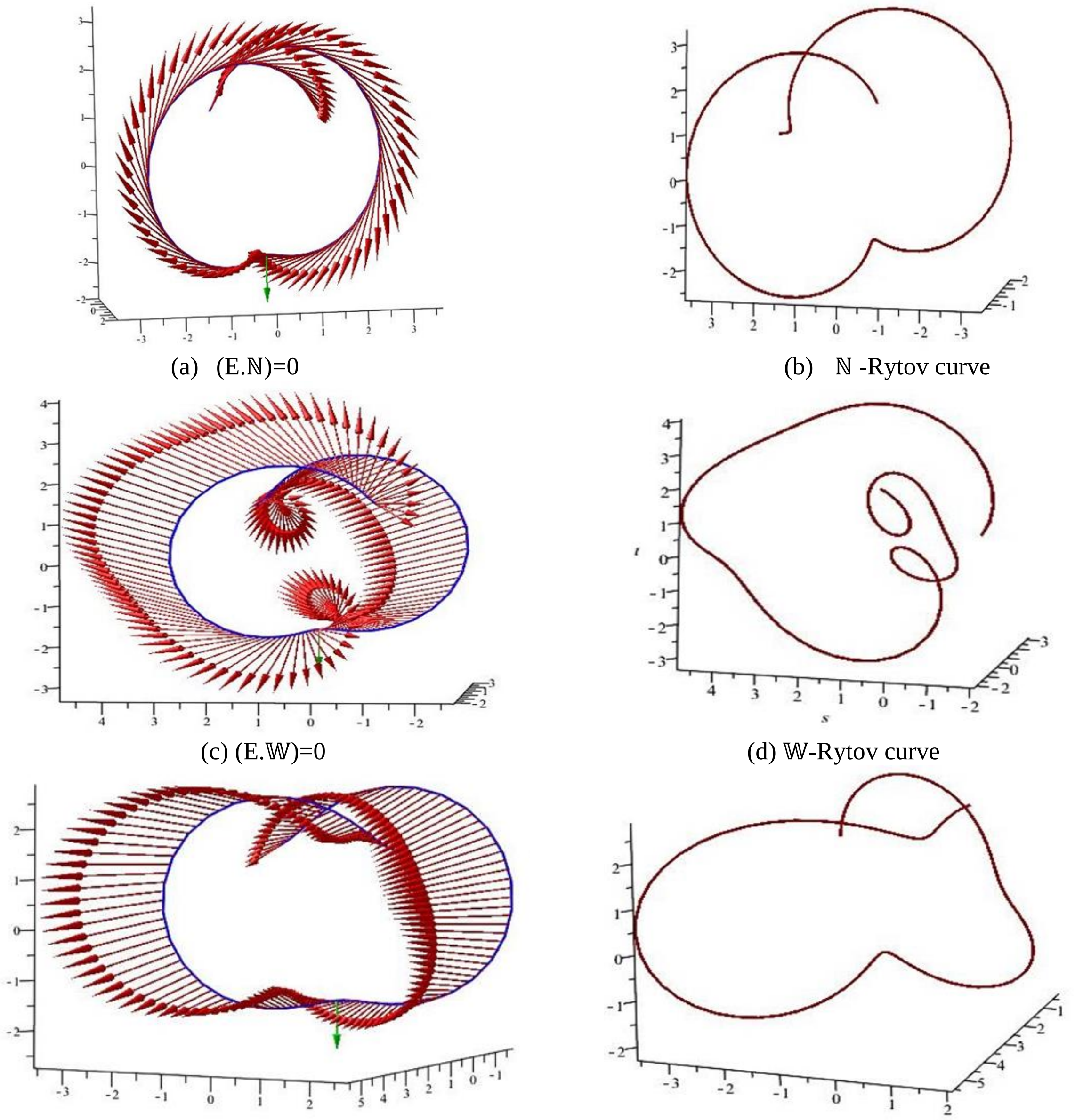
$$\mathrm{E} = \sin(\int \mathrm{fds})\mathbb{N}(\mathrm{s}) + \cos(\int \mathrm{fds})\mathbb{C}(\mathrm{s}), \mathrm{B} = \frac{\mathrm{g}}{\mathrm{f}^2\kappa}\mathbb{N}(\mathrm{s}). \tag{69}$$

Examples 3.3. Let $\rho : I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve equipped with the alternately orthogonal $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$-frame. If we choose the curvatures as

$$\kappa(s) = sin(0.5s) \text{ and } \tau(s) = -cos(0.5s),$$

then the Fermi-Walker transportations of the electric field associated with the electromagnetic curve ρ are presented in Fig. 1((a),(c),(e)). Furthermore, when the electric field is transported along the electromagnetic curve according to the Fermi-Walker parallel transportation rule, the tip of the electric field traces out a curve called the Rytov curve, which is shown in Fig. 1((b),(d),(f)).

When $\langle \mathbb{N}, E \rangle = 0$, $\langle \mathbb{C}, E \rangle = 0$, $\langle \mathbb{W}, E \rangle = 0$, the resulting curve is called the $\mathbb{N}$ -Rytov curve, $\mathbb{C}$ -Rytov curve, $\mathbb{W}$ -Rytov curve, respectively.



(a) (E.$\mathbb{N}$)=0 (b) $\mathbb{N}$ -Rytov curve

(c) (E.$\mathbb{W}$)=0 (d) $\mathbb{W}$-Rytov curve

(e) (E$\mathbb{C}$)=0 (f) $\mathbb{C}$-Rytov curve

Figure 1. Fermi Walker parallel transportation of the electric field E and corresponding Rytov curves along the electromagnetic curve, shown with respect to alternatively orthogonal $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$-frame.

Examples 3.4. Let $\rho: I \subset R \to R^3$ be a magnetic curve with the alternately orthogonal $\{N, C, W\}$ frame. If the curvatures are chosen as

$$\kappa(s) = -c\sin(\psi s) \text{ and } \tau(s) = c\cos(\psi s)$$

and we solve the following differential equations using a symbolic computation tool like Mathematica,

$$d_1'[s] = \varsigma[s]d_2[s], d_2'[s] = -\varsigma[s]d_1[s] + \mu[s]d_3[s], d_3'[s] = -\mu[s][s], \tag{70}$$

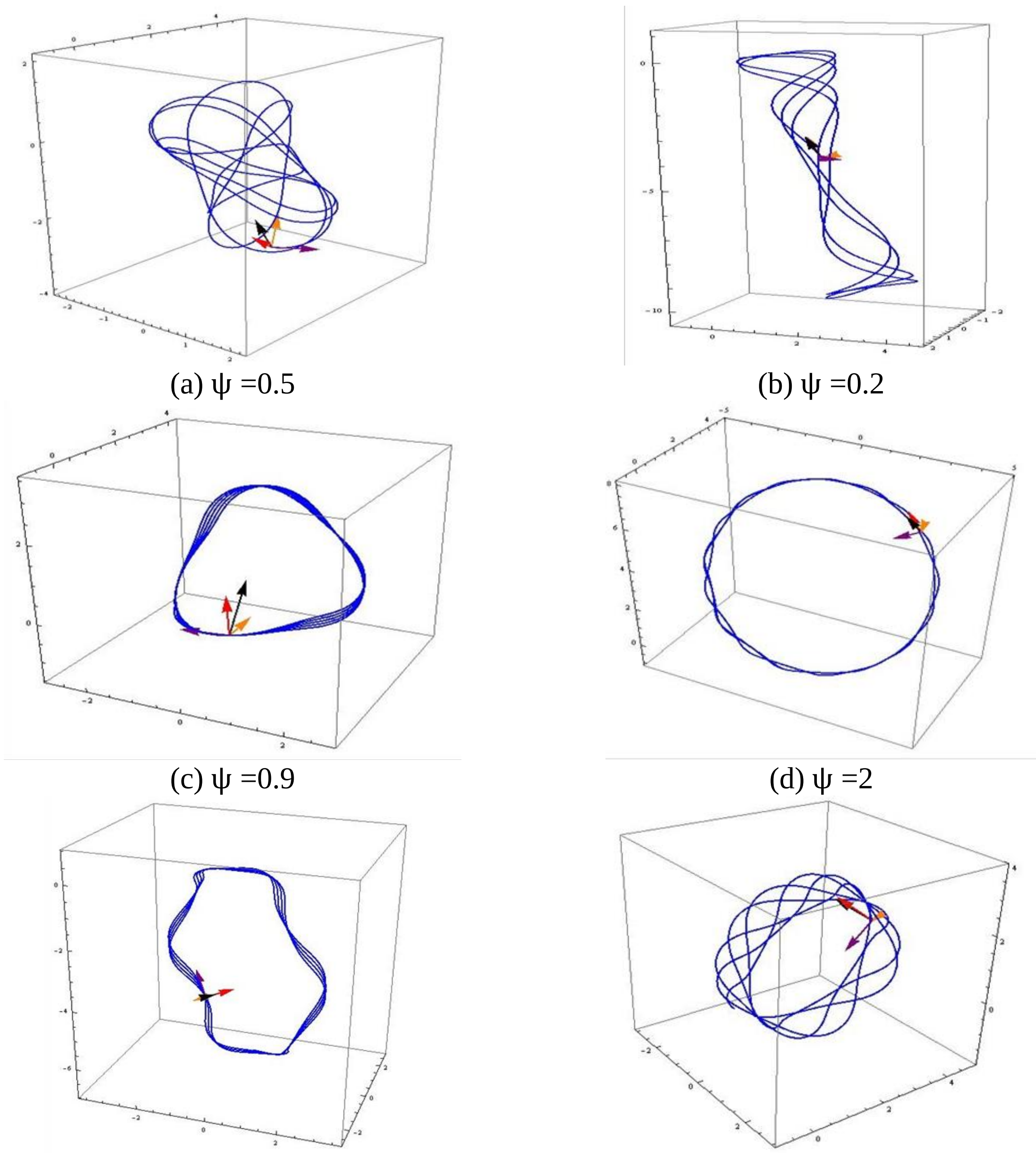
we define orthogonal components as,

- $\mathbb{N}[s] := \rho''[s]$, (orange)
- $\mathbb{C}[s] := \mu[s]d_3[s] - \varsigma[s]d_1[s]$, (purple)
- $\mathbb{W}[s] := \mu[s]d_1[s] + \varsigma[s]d_3[s]$, (blue)

Further, the velocity vector can be expressed as,

$$V[s] = \psi\kappa[s]\mathbb{N} + \mathbb{W}(\text{black}).$$

By solving these relationships and generating magnetic curves with the orthogonal $\{\mathbb{N}, \mathbb{C}, W\}$-frame for given values of ψ and $c = 1$, the resulting curves are visualized in Fig. 2.



(a) ψ =0.5 (b) ψ =0.2

(c) ψ =0.9 (d) ψ =2

(e) ψ =1.5 (f) ψ =1

Figure 2. Magnetic (resp. electromagnetic) and curves for some values of ψ (resp. ϕ)

4. CONCLUSIONS

Helical magnetic curves associated with the Frenet frame in Euclidean 3-space exhibit axes parallel to the magnetic vector field. When a charged particle moves parallel to the magnetic field, the Lorentz force is zero because it always acts perpendicular to the particle's velocity. Conversely, when the magnetic field is perpendicular to the velocity vector, the Lorentz force reaches its maximum, causing the particle to follow a circular trajectory. If the angle between the velocity vector and the magnetic field remains constant but is not perpendicular, the particle follows a helical trajectory (see [5-7]).

The most significant contribution of this study lies in the characterization of magnetic curves using the alternative $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$-frame. Specifically, this approach addresses the challenge of defining magnetic and electromagnetic curves at points where conventional frames, such as $\{d_2, C, W\}$ or $\{d_1, d_2, d_3\}$, fail to be well-defined. This situation arises particularly at singular points where certain parameters, such as, $f = 0$, $\tau = 0$, or $\kappa = 0$ (as discussed in [9]), cause these traditional frames to break down. By using the $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$-frame, the study provides a more robust method for analyzing magnetic and electromagnetic properties, especially in cases where conventional approaches fail to yield meaningful results.

Magnetic and electromagnetic curves are shown to be constant precession curves when the curvatures satisfy

$$\kappa(s) = c\, cos(\psi s) \text{ and } \tau(s) = -c\, sin(\psi s).$$

Although these curvatures vanish at certain points, the orthogonal $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$-frame remains well-defined at all points along the curve.

In the geometric interpretation of the motion of a charged particle in a magnetic field, the adapted frame elements and their curvatures along the particle's trajectory provide valuable insights. Therefore, having a well-defined adapted frame at every point on the magnetic trajectory curve offers a significant advantage. In this context, the new alternative orthogonal $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ frame, as described in [25], proves to be particularly useful, as it is defined at every point along the magnetic curve, even where the curvatures , κ or τ is vanish. This allows us to characterize the magnetic curves associated with this frame.

According to our approach, the following results are found:

– If the vector $\mathbb{N}$ is parallel to $\boldsymbol{B}$ as the charged particle enters the magnetic field, the Lorentz force becomes zero.

– If $(\mathbb{N}.\boldsymbol{B}) = 0$ as the charged particle enters the magnetic field, the particle follows a circular helical trajectory.

– If $(\boldsymbol{B}.\mathbb{N}) = cos\theta, \theta = const.$, the particle follows a trajectory of constant precession and slant helices under the influence of the Lorentz force.

– If $(E.\mathbb{N}) = 0$, the related electromagnetic curves are constant precession curves.

– If $(E.\mathbb{C}) = 0$, the related electromagnetic curves are circular helices.

– If $(E, \mathbb{W}) = 0$, the related electromagnetic curves are circles.

Additionally, we present several examples and visualize them using Mathematica and Maple programs. Different magnetic curves can be generated in various ways with these tools. Furthermore, other special curves can be characterized with respect to the modified orthogonal frame $\{\mathbb{N}, \mathbb{C}, \mathbb{W}\}$ in Euclidean 3-space.

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