ORIGINAL PAPER

ACCELERATING A CONVERGENCE TOWARD EULER-MASCHERONI CONSTANT

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Abstract. *We give in this paper a new sequence convergent to the Euler-Mascheroni constant.*

Keywords: *convergent sequence; Euler-Mascheroni constant; speed of convergence.*

1. INTRODUCTION

The Euler-Mascheroni constant, denoted as γ, is a fundamental mathematical constant that arises in various branches of mathematics, including number theory and analysis. First introduced by the Swiss mathematician Leonhard Euler in 1734, this constant is defined as the limiting difference between the harmonic series and the natural logarithm:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right). \tag{1}$$

Despite its simple definition, the Euler-Mascheroni constant has intrigued mathematicians for centuries due to its mysterious properties. This constant appears in numerous contexts, such as the analysis of the Gamma function, the Riemann zeta function, and in various integrals and series expansions.

It is not known whether γ is rational or irrational, and its exact nature remains one of the unsolved problems in mathematics. It seems that the difficulties in proving the rationality (or irrationality) of γ is since there are not known sufficient fast convergences to γ, as in case of the constant e, for example. This is the reason for which, in the recent past, many researchers were concentrated to construct increasingly convergence to γ.

It is the aim of this paper to construct a new, fast convergence to γ. We hope also that the ideas presented here inspire other mathematicians to obtain better results in the same direction.

Relation (1) can be rewritten in terms of numeric series as follows:

$$\gamma = \sum_{n=1}^{\infty}\left(\frac{1}{n} + \ln\frac{n}{n+1}\right). \tag{2}$$

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The associated sequence of partial sums is

$$\sigma_n = \sum_{k=1}^{n} \left(\frac{1}{k} + \ln \frac{k}{k+1} \right) = \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1)\,,$$

which converges to γ, too. We propose in this paper the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1/2} + \ln \frac{n}{n+1} \right), \tag{3}$$

that converges faster than series (2).

2. THE RESULTS

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that the series

$$\sum_{n=1}^{\infty} a_n = s$$

is convergent, with the sum equal to $s \in \mathbb{R}$. According to the definition,

$$\lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = s,$$

meaning that for large values of n, the expression $(a_1 + a_2 + \cdots + a_n)$ approximates the sum s of the series increasingly better. This approximation is better as the speed of convergence of the sequence $(a_1 + a_2 + \cdots + a_n)_{n\geq 1}$ is greater.

An useful tool for measuring the speed of convergence is the following lemma first given by Mortici [1].

Lemma 1. Let $(\mathrm{w_n})_{\mathrm{n}\geq 1}$ be a sequence converging to zero, such that

$$\lim_{n\to\infty} n^k (w_n - w_{n-1}) = l \in \mathbb{R},$$

for some $\mathrm{k} > 1$. Then

$$\lim_{n\to\infty} n^{k-1}\, w_n = \frac{l}{k-1}\,.$$

This lemma is useful in asymptotic analysis and other problems involving the Euler-gamma function and related functions. See, e.g., [2-9]. In case $w_n = a_1 + a_2 + \cdots + a_n - s$, we deduce that if

$$\lim_{n\to\infty} n^k a_n = l,$$

then

$$\lim_{n\to\infty} n^{k-1} (a_1 + a_2 + \cdots + a_n - s) = l.$$

In consequence, the sequence $(a_1 + a_2 + \cdots + a_n)_{n\geq 1}$ converges faster to s, as the sequence $(a_n)_{n\geq 1}$ faster converges to zero. We will say that a series $\sum_{n=1}^{\infty} a_n$ is *superior* to a series $\sum_{n=1}^{\infty} b_n$ if the sequence $(a_n)_{n\geq 1}$ converges faster to zero, than $(b_n)_{n\geq 1}$ does, that is:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

For the general term of series (2), we have:

$$\lim_{n\to\infty} n^2 \left(\frac{1}{n} + \ln\frac{n}{n+1}\right) = \frac{1}{2}.$$

In terms of Landau symbols, we have:

$$\frac{1}{n} + \ln\frac{n}{n+1} \sim O\left(\frac{1}{n^2}\right), \text{ as } n \to \infty.$$

The results in this paper are related to the following idea: for the sequence

$$g_n(a) = \frac{1}{n+a} + \ln\frac{n}{n+1}, \quad n \geq 1, \tag{4}$$

depending on a real parameter a, it holds:
(i) if $a \neq 1/2$, then

$$g_n(a) \sim O\left(\frac{1}{n^2}\right), \text{ as } n \to \infty;$$

(ii) if $a = 1/2$, then

$$g_n\left(\frac{1}{2}\right) \sim O\left(\frac{1}{n^3}\right), \text{ as } n \to \infty.$$

This follows from the asymptotic expansion:

$$g_n(a) = \left(\frac{1}{2} - a\right)\frac{1}{n^2} + \left(a^2 - \frac{1}{3}\right)\frac{1}{n^3} + \left(\frac{1}{4} - a^3\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right).$$

In case $a = 1/2$, we get:

$$g_n\left(\frac{1}{2}\right) = -\frac{1}{12n^3} + \frac{1}{8n^4} + O\left(\frac{1}{n^5}\right).$$

We give the following

Theorem 1.
a) The standard defining series of γ given by (2) is of order n^{-2}, since:

$$\lim_{n\to\infty} \left(\frac{1}{n} + \ln\frac{n}{n+1}\right) n^2 = \frac{1}{2} \neq 0.$$

b) The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1/2}+\ln\frac{n}{n+1}\right)$$

is of order n^{-3}, since:

$$\lim_{n\to\infty}\left(\frac{1}{n+1/2}+\ln\frac{n}{n+1}\right)n^3=-\frac{1}{12}\neq 0.$$

The complete asymptotic series of the sequence $g_n(a)$ defined in (4) can be constructed by using the following classical expansions:

$$\frac{1}{1+x}=1+\sum_{k=1}^{\infty}(-1)^k x^k,\quad x\in\mathbb{R}$$

and

$$\ln(1+x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^k,\quad x\in(-1,1].$$

Thus we have:

$$\frac{1}{n+a}=\frac{1}{n}\cdot\frac{1}{1+\frac{a}{n}}=\frac{1}{n}\left(1+\sum_{k=1}^{\infty}(-1)^k\frac{a^k}{n^k}\right)$$

and

$$\ln\left(1+\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n.$$

3. NUMERICAL COMPARISON

The difference between series (2) and (3) is the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1/2}\right)=\sum_{n=1}^{\infty}\frac{1}{n(2n+1)}=2-2\ln 2.$$

In conclusion, in order to prove the superiority of our new series over the classical series (2), we will present a numerical comparison of the partial sums

$$u_n=\sum_{k=1}^{n}\left(\frac{1}{k}+\ln\frac{k}{k+1}\right)-\gamma\to 0,$$

$$v_n=\sum_{k=1}^{n}\left(\frac{1}{k+1/2}+\ln\frac{k}{k+1}\right)-(\gamma-2+2\ln 2)\to 0.$$

n	u_n	v_n
1	-0.27036	-1.9900
10	-4.6143×10^{-2}	3.4386×10^{-4}
50	-9.8360×10^{-3}	1.6018×10^{-5}
100	-4.9587×10^{-3}	4.0845×10^{-6}
1000	-4.9958×10^{-4}	4.1583×10^{-8}
12500	-3.9997×10^{-5}	2.6662×10^{-10}

4. FURTHER STUDY

By replacing n by $n-1$ in series (3), we obtain the following series in a simple form:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}+\ln\frac{n-1/2}{n+1/2}\right).$$

In order to accelerate the general term of this series, we consider the family of sequences

$$q_n(a)=\frac{1}{n+\frac{a}{n}}+\ln\frac{n-1/2}{n+1/2}\ ,\quad n\geq 1.$$

As we have the following expansion:

$$q_n(a)=-\left(a+\frac{1}{12}\right)\frac{1}{n^3}+\left(a^2-\frac{1}{80}\right)\frac{1}{n^5}-\left(a^3+\frac{1}{448}\right)\frac{1}{n^7}+O\left(\frac{1}{n^9}\right),$$

we can state the following:

Theorem 2.
a) If $\mathrm{a}\neq -1/12$, then the sequence $(\mathrm{q_n(a)})_{\mathrm{n}\geq 1}$ converges to zero as n^{-3}.
b) If $\mathrm{a}=-1/12$, then the sequence $(\mathrm{q_n}(-1/12)_{\mathrm{n}\geq 1}$ converges to zero as n^{-5}. In this case,

$$q_n\left(-\frac{1}{12}\right)=-\frac{1}{180n^5}-\frac{5}{3024n^7}-\frac{1}{2592n^9}+O\left(\frac{1}{n^{11}}\right).$$

Remark 1. The numerical calculations presented in this paper were performed using the Maple software for symbolic computation.

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