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ON FIBONACCI AND LUCAS ELLIPTIC QUATERNIONS

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Abstract. *This paper presents a novel class of quaternions derived from the concept of elliptic quaternions. Specifically, we establish the Fibonacci and Lucas elliptic quaternions, serving as a natural extension of the Fibonacci and Lucas quaternions. Our study delves into the fundamental properties of these sequences, including recurrence relations, generating functions, Binet-like formulas, and Vajda's identity. We additionally establish relations involving both Fibonacci and Lucas elliptic quaternions.*

Keywords: *Quaternions; elliptic quaternions; Fibonacci numbers; generating function.*

1. INTRODUCTION

Quaternions were discovered by Sir William R. Hamilton in 1843 [1], and their theory expanded to include applications such as rotations in the early 20th century. One of the most significant properties of quaternions is that each unit quaternion represents a rotation, playing a crucial role in the study of rotations within 3-dimensional vector spaces. Quaternions find extensive use in fields like computer vision, computer graphics, animation, and kinematics. Similar to how each unit quaternion represents a rotation in Euclidean 3-space, utilizing unit elliptic quaternions for a given ellipsoid enables the generation of elliptical rotations. Understanding motion on an ellipsoid holds particular importance, especially considering that planets typically possess ellipsoidal shapes and follow elliptical orbits.

The concept of elliptic quaternions was introduced by Özdemir [2] and used for the creation of an elliptical rotation matrix. For a given ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$, the elliptic quaternion algebra $\mathbb{H}_{a_1,a_2,a_3}$ is

$$\mathbb{H}_{a_1,a_2,a_3} = \{q_0 + q_1 i + q_2 j + q_3 k | q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

an associative, non-commutative division ring and the basis $\{1, i, j, k\}$ satisfy the following multiplication rules:

$$\begin{aligned} i^2 = -a_1, j^2 = -a_2, k^2 = -a_3,, ijk = -\Delta \\ ij = -ji = \frac{\Delta}{a_3}k, jk = -kj = \frac{\Delta}{a_1}i, ki = -ik = \frac{\Delta}{a_2}j, \end{aligned} \tag{1.1}$$

where $\Delta := \sqrt{a_1a_2a_3}$ and $a_1, a_2, a_3 \in \mathbb{R}^+$. It is clear that if we take $a_1 = a_2 = a_3 = 1$, we get the usual quaternion algebra.

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For any given two elliptic quaternions $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$, the addition, subtraction, and scalar multiplication are defined as follows:

$$p \pm q = (p_0 \pm q_0) + (p_1 \pm q_1)i + (p_2 \pm q_2)j + (p_3 \pm q_3)k,$$
$$cp = cp_0 + cp_1 i + cp_2 j + cp_3 k, \text{where } c \in \mathbb{R}.$$

The product of two elliptic quaternions p and q is defined as:

$$\begin{aligned} pq &= p_0q_0 - a_1p_1q_1 - a_2p_2q_2 - a_3p_3q_3 \\ &+ \left(p_0q_1 + p_1q_0 + \frac{\Delta}{a_1}p_2q_3 - \frac{\Delta}{a_1}p_3q_2\right)i \\ &+ \left(p_0q_2 + p_2q_0 + \frac{\Delta}{a_2}p_3q_1 - \frac{\Delta}{a_2}p_1q_3\right)j \\ &+ \left(p_0q_3 + p_3q_0 + \frac{\Delta}{a_3}p_1q_2 - \frac{\Delta}{a_3}p_2q_1\right)k. \end{aligned}$$

The elliptic quaternion product can also be seen in Table 1.

Table 1.Elliptic quaternion product table.

	1	***i***	***j***	***k***
1	1	i	j	k
i	i	$-a_1$	$\frac{\Delta}{a_3}k$	$-\frac{\Delta}{a_2}j$
j	j	$-\frac{\Delta}{a_3}k$	$-a_2$	$\frac{\Delta}{a_1}i$
k	k	$\frac{\Delta}{a_2}j$	$-\frac{\Delta}{a_1}i$	$-a_3$

The norm of an elliptic quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ is defined as

$$N(q) := \sqrt{q\overline{q}} = \sqrt{q_0^2 + a_1q_1^2 + a_2q_2^2 + a_3q_3^2}$$

where $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$ is the conjugate of q. For details on elliptic quaternions, we refer to [2].

There are several studies on different types of sequences over quaternion algebra. In particular, Horadam [3] defined the Fibonacci and Lucas quaternions on real quaternion algebra $\mathbb{H}$ as:

$$Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k,$$

$$K_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k,$$

where F_n is the n-th Fibonacci number defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ with the initial conditions $F_0 = 0$ and $F_1 = 1$, and L_n is the nth Lucas number which satisfies the same recurrence relation as Fibonacci numbers but begins with $L_0 = 2$ and $L_1 = 1$. The Binet formulas for the Fibonacci and Lucas sequences are $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic polynomial

$x^2 - x - 1$. Note that $\alpha\beta = -1, \alpha + \beta = 1, \alpha - \beta = \sqrt{5}$. For more on quaternion sequences, we refer to [4-10].

In this paper, we introduce the Fibonacci and Lucas elliptic quaternions, which provide a natural generalization of the classical Fibonacci and Lucas quaternions by using the concept of elliptic quaternions. We investigate some basic properties of these sequences, such as recurrence relations, generating functions, Binet-like formulas, Vajda's identity, and Honsberger's identity. We also derive some relations including both Fibonacci and Lucas elliptic quaternions.

2. MATERIALS AND METHODS

Throughout this section, we use the notation $E := E(a_1, a_2, a_3)$ for the ellipsoid $a_1x^2 + a_2y^2 + a_3z^2 = 1$.

Definition 1. The nth Fibonacci and Lucas elliptic quaternions on the ellipsoid E are defined respectively by

$$Q_{E,n} = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k,$$

$$K_{E,n} = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k,$$

where F_n is the nth Fibonacci number, L_n is the nth Lucas number, and the basis $\{1, i, j, k\}$ satisfies the multiplication rule in (1.1).

The norm of a Fibonacci elliptic quaternion $Q_{E,n}$is

$$N(Q_{E,n}) := \sqrt{Q_{E,n}\overline{Q_{E,n}}} = \sqrt{F_n^2 + a_1F_{n+1}^2 + a_2F_{n+2}^2 + a_3F_{n+3}^2}$$

where $\overline{Q_{E,n}} = F_n - F_{n+1}i - F_{n+2}j - F_{n+3}k$.

It is clear to see that the Fibonacci elliptic quaternions satisfy the following recurrence relation

$$Q_{E,n} = Q_{E,n-1} + Q_{E,n-2}, n \geq 2 \tag{2.1}$$

with initial values $Q_{E,0} = i + j + 2k$ and $Q_{E,1} = 1 + i + 2j + 3k$. The Lucas elliptic quaternions also satisfy the same recurrence relation but begin with initial values $K_{E,0} = 2 + i + 3j + 4k$ and $K_{E,1} = 1 + 3i + 4j + 7k$.

Now we give the generating functions and the Binet formulas of the Fibonacci and Lucas elliptic quaternions.

Theorem 1. The generating functions of the Fibonacci and Lucas elliptic quaternions are

$$G(x) = \frac{i + j + 2k + (1 + j + k)x}{1 - x - x^2}$$

and

$$H(x)=\frac{2+i+3j+4k+(-1+2i+j+3k)x}{1-x-x^2},$$

respectively.

Proof: Let $G(x)=\sum_{n=0}^{\infty}Q_{E,n}x^n$ be the generating function of the Fibonacci elliptic quaternions. From (2.1), we have

$$\begin{aligned}(1-x-x^2)\sum_{n=0}^{\infty}Q_{E,n}x^n &= \sum_{n=0}^{\infty}Q_{E,n}x^n-\sum_{n=0}^{\infty}Q_{E,n}x^{n+1}-\sum_{n=0}^{\infty}Q_{E,n}x^{n+2}\\ &=\sum_{n=0}^{\infty}Q_{E,n}x^n-\sum_{n=1}^{\infty}Q_{E,n-1}x^n-\sum_{n=2}^{\infty}Q_{E,n-2}x^n\\ &=Q_{E,0}+Q_{E,1}x+Q_{E,2}x^2+\cdots+Q_{E,n}x^n+\cdots\\ &\quad-\left(Q_{E,0}x+Q_{E,1}x^2+\cdots+Q_{E,n-1}x^n+\cdots\right)\\ &\quad-\left(Q_{E,0}x^2+Q_{E,1}x^3+\cdots+Q_{E,n-2}x^n+\cdots\right)\\ &=Q_{E,0}+\left(Q_{E,1}-Q_{E,0}\right)x+\sum_{n=2}^{\infty}Q_{E,n-2}x^{n+2}\\ &=Q_{E,0}+\left(Q_{E,1}-Q_{E,0}\right)x.\end{aligned}$$

By using the initial values, we get the desired result.
The generating function of Lucas elliptic quaternions can be derived similarly.

Theorem 2. The Binet-like formulas of the Fibonacci elliptic quaternions and Lucas elliptic quaternions are given respectively by

$$Q_{E,n}=\frac{\alpha^*\alpha^n-\beta^*\beta^n}{\alpha-\beta}$$

and

$$K_{E,n}=\alpha^*\alpha^n+\beta^*\beta^n,$$

where $\alpha^*:=1+\alpha i+\alpha^2j+\alpha^3k$ and $\beta^*:=1+\beta i+\beta^2j+\beta^3k$.

Proof: From the definition of the Fibonacci elliptic quaternion and the Binet formula of the Fibonacci sequence, we get

$$\begin{aligned}Q_{E,n}&=F_n+F_{n+1}i+F_{n+2}j+F_{n+3}k\\ &=\frac{\alpha^n-\beta^n}{\alpha-\beta}+\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)i+\left(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\right)j+\left(\frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\right)k\\ &=\frac{\alpha^n(1+\alpha i+\alpha^2j+\alpha^3k)}{\alpha-\beta}-\frac{\beta^n(1+\beta i+\beta^2j+\beta^3k)}{\alpha-\beta}\\ &=\frac{\alpha^*\alpha^n-\beta^*\beta^n}{\alpha-\beta}.\end{aligned}$$

The Binet formula of Lucas elliptic quaternions can be obtained in a similar manner.

Theorem 3. The exponential generating functions of the Fibonacci and Lucas elliptic quaternions are

$$\sum_{n=0}^{\infty} Q_{E,n}\frac{x^n}{n!}=\frac{\alpha^* e^{\alpha x}-\beta^* e^{\beta x}}{\alpha-\beta}$$

and

$$\sum_{n=0}^{\infty} K_{E,n}\frac{x^n}{n!}=\alpha^* e^{\alpha x}+\beta^* e^{\beta x},$$

respectively.

Proof: From the Binet formula of Fibonacci elliptic quaternions, we have

$$\begin{aligned}\sum_{n=0}^{\infty} Q_{E,n}\frac{x^n}{n!}&=\sum_{n=0}^{\infty}\left(\frac{\alpha^*\alpha^n-\beta^*\beta^n}{\alpha-\beta}\right)\frac{x^n}{n!}\\ &=\frac{\alpha^*}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!}-\frac{\beta^*}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!}\\ &=\frac{\alpha^* e^{\alpha x}-\beta^* e^{\beta x}}{\alpha-\beta}.\end{aligned}$$

Similarly, by using the Binet formula of the Lucas elliptic quaternions we get the desired result. Now we give the following lemma which gives a simplification to obtain the Vajda's like identity for the Fibonacci elliptic quaternions.

Lemma 1. Let $\omega:=\frac{1}{a_1}i+\frac{1}{a_2}j-\frac{1}{a_3}k$, $t:=1-a_1+a_2-a_3$, $t_1:=a_1F_1+a_2F_3+a_3F_5$, $t_2:=\frac{1}{2}(a_1F_2+a_2F_4+a_3F_6)$. For α^* and β^*, we have the following results:

(i) $\alpha^*\beta^*=K_{E,0}-t-\Delta\sqrt{5}\omega$,

(ii) $\beta^*\alpha^*=K_{E,0}-t+\Delta\sqrt{5}\omega$,

(iii) $(\alpha^*)^2=K_{E,0}-(1+t_1+t_2)+\sqrt{5}\left(Q_{E,0}-t_2\right)$,

(iv) $(\beta^*)^2=K_{E,0}-(1+t_1+t_2)-\sqrt{5}\left(Q_{E,0}-t_2\right)$.

Proof: We only give the proofs of the identities (i) and (iii). The remaining cases can be proven similarly.

(i) From the definition of α^* and β^*, we have

$$\begin{aligned}\alpha^*\beta^*&=(1+\alpha i+\alpha^2 j+\alpha^3 k)(1+\beta i+\beta^2 j+\beta^3 k)\\ &=1-a_1(\alpha\beta)-a_2(\alpha\beta)^2-a_3(\alpha\beta)^3\\ &\quad+\left(\beta+\alpha+\frac{\Delta}{a_1}(\alpha^2\beta^3-\alpha^3\beta^2)\right)i\end{aligned}$$

$$\begin{aligned}
&+\left(\beta^2+\alpha^2+\frac{\Delta}{a_2}(\alpha^3\beta-\alpha\beta^3)\right)j\\
&+\left(\beta^3+\alpha^3+\frac{\Delta}{a_3}(\alpha\beta^2-\alpha^2\beta)\right)k\\
&=1+a_1-a_2+a_3+\left(1-\frac{\Delta}{a_1}\sqrt{5}\right)i\\
&+\left(3-\frac{\Delta}{a_2}\sqrt{5}\right)j+\left(4+\frac{\Delta}{a_3}\sqrt{5}\right)k\\
&=1+a_1-a_2+a_3+K_{E,0}-2-\Delta\sqrt{5}\omega\\
&=K_{E,0}-t-\Delta\sqrt{5}\omega.
\end{aligned}$$

(iii) From the definition of α^*, we have

$$\begin{aligned}
(\alpha^*)^2 &= (1+\alpha i+\alpha^2 j+\alpha^3 k)(1+\alpha i+\alpha^2 j+\alpha^3 k)\\
&=1-a_1\alpha^2-a_2\alpha^4-a_3\alpha^6\\
&+(2\alpha)i+(2\alpha^2)j+(2\alpha^3)k\\
&=1-a_1\alpha^2-a_2\alpha^4-a_3\alpha^6+2\left(\frac{1+\sqrt{5}}{2}\right)i\\
&+2\left(\frac{3+\sqrt{5}}{2}\right)j+2\big(2+\sqrt{5}\big)k\\
&=1-a_1\alpha^2-a_2\alpha^4-a_3\alpha^6+K_{E,0}-2+\sqrt{5}Q_{E,0}\\
&=K_{E,0}-1-t_2-t_1+\sqrt{5}(Q_{E,0}-t_2)\\
&=K_{E,0}-(1+t_2+t_1)+\sqrt{5}\big(Q_{E,0}-t_2\big).
\end{aligned}$$

It is clear that we have the following results from the Lemma 1:

$$\begin{aligned}
\alpha^*\beta^*+\beta^*\alpha^* &= 2\big(K_{E,0}-t\big),\\
\alpha^*\beta^*-\beta^*\alpha^* &= -2\Delta\sqrt{5}\omega,\\
(\alpha^*)^2+(\beta^*)^2 &= 2\left(K_{E,0}-(1+t_1+t_2)\right),\\
(\alpha^*)^2-(\beta^*)^2 &= 2\sqrt{5}\big(Q_{E,0}-t_2\big).
\end{aligned}$$

By using the Binet formula of the Fibonacci elliptic quaternions and the above Lemma 1, we obtain the following identities.

Theorem 4. (Vajda's like identity) For nonnegative integers r, s and n, we have

$$Q_{E,n+r}Q_{E,n+s}-Q_{E,n}Q_{E,n+r+s}=(-1)^n F_r\big[\big(K_{E,0}-t\big)F_s+\Delta\omega L_s\big].$$

Proof: By using the Binet formula of Fibonacci elliptic quaternions, we get

$$\begin{aligned}
&5\big(Q_{E,n+r}Q_{E,n+s}-Q_{E,n}Q_{E,n+r+s}\big)\\
&=(\alpha^{n+r}\alpha^*-\beta^{n+r}\beta^*)(\alpha^{n+s}\alpha^*-\beta^{n+s}\beta^*)\\
&-(\alpha^n\alpha^*-\beta^n\beta^*)(\alpha^{n+r+s}\alpha^*-\beta^{n+r+s}\beta^*)\\
&=\alpha^n\beta^{n+r+s}(\alpha^*\beta^*)+\beta^n\alpha^{n+r+s}(\beta^*\alpha^*)\\
&-\alpha^{n+r}\beta^{n+s}(\alpha^*\beta^*)-\beta^{n+r}\alpha^{n+s}(\beta^*\alpha^*)
\end{aligned}$$

$$\begin{gathered}
= (\alpha^n\beta^{n+r+s} - \alpha^{n+r}\beta^{n+s})\alpha^*\beta^* + (\beta^n\alpha^{n+r+s} - \beta^{n+r}\alpha^{n+s})\beta^*\alpha^* \\
= (\alpha\beta)^n\big(\beta^{r+s}(\alpha^*\beta^*) + \alpha^{r+s}(\beta^*\alpha^*)\big) - (\alpha\beta)^n\big(\alpha^r\beta^s(\alpha^*\beta^*) + \beta^r\alpha^s(\beta^*\alpha^*)\big) \\
= (\alpha\beta)^n\big((\beta^{r+s} - \alpha^r\beta^s)\alpha^*\beta^* + (\alpha^{r+s} - \beta^r\alpha^s)\beta^*\alpha^*\big) \\
= (\alpha\beta)^n(\beta^r - \alpha^r)\beta^s(\alpha^*\beta^*) + (\alpha^r - \beta^r)\alpha^s(\beta^*\alpha^*) \\
= (\alpha\beta)^n(\alpha^r - \beta^r)\big(\alpha^s(\beta^*\alpha^*) - \beta^s(\alpha^*\beta^*)\big)
\end{gathered}$$

and by using Lemma 1 $(i) - (ii)$, we get

$$\begin{gathered}
5(Q_{E,n+r}Q_{E,n+s} - Q_{E,n}Q_{E,n+r+s}) \\
= (\alpha\beta)^n(\alpha^r - \beta^r)\left(\alpha^s\big(K_{E,0} - t + \Delta\sqrt{5}\omega\big) - \beta^s\big(K_{E,0} - t - \Delta\sqrt{5}\omega\big)\right) \\
= (\alpha\beta)^n\sqrt{5}F_r\left(\big(K_{E,0} - t\big)(\alpha^s - \beta^s) + \Delta\sqrt{5}\omega(\alpha^s + \beta^s)\right) \\
= (\alpha\beta)^n\sqrt{5}F_r\left(\big(K_{E,0} - t\big)\sqrt{5}F_s + \Delta\sqrt{5}\omega L_s\right) \\
= (\alpha\beta)^n 5F_r(\big(K_{E,0} - t\big)F_s + \Delta\omega L_s) \\
= (-1)^n 5F_r((K_{E,0} - t)F_s + \Delta\omega L_s).
\end{gathered}$$

Thus, we get the desired result.

It is clear that if we take $r, s \to m$ and $n \to n - m$ in the above theorem, we obtain the following result.

Corollary 1. (Catalan's like identity) For nonnegative integers n and m, such that n ≥ m, we have

$$Q_{E,n}^2 - Q_{E,n-m}Q_{E,n+m} = (-1)^{n-m}F_m\big[\big(K_{E,0} - t\big)F_m + \Delta\omega L_m\big].$$

It is clear that if we take $m = 1$ in the *Catalan's like identity*, we obtain the following result.

Corollary 2. (Cassini's like identity) For positive integer n, we have

$$Q_{E,n}^2 - Q_{E,n-1}Q_{E,n+1} = (-1)^{n-1}\left(\big(K_{E,0} - t\big) + \Delta\omega\right).$$

It is clear that if we take $s = m - n$ and $r = 1$ in Theorem 4 we obtain the following result.

Corollary 3. (d'Ocagne's like identity) For nonnegative integers n and m, such that m ≥ n, we have

$$Q_{E,n+1}Q_{E,m} - Q_{E,n}Q_{E,m+1} = (-1)^n\big[\big(K_{E,0} - t\big)F_{m-n} + \Delta\omega L_{m-n}\big].$$

Theorem 5. (Honsberger's like identity) For positive integers n and m, we have

$$Q_{E,m-1}Q_{E,n} + Q_{E,m}Q_{E,n+1} = F_{m+n}(K_{E,0} - (1 + t_1 + t_2)) + L_{m+n}(Q_{E,0} - t).$$

Proof: By using the Binet formula for the Fibonacci elliptic quaternion we have

$$\begin{gathered}
5\left(Q_{E,m-1}Q_{E,n}+Q_{E,m}Q_{E,n+1}\right) \\
=(\alpha^{m-1}\alpha^*-\beta^{m-1}\beta^*)(\alpha^n\alpha^*-\beta^n\beta^*) \\
+(\alpha^m\alpha^*-\beta^m\beta^*)(\alpha^{n+1}\alpha^*-\beta^{n+1}\beta^*) \\
=(\alpha^{m+n-1}(\alpha^*)^2+\beta^{m+n-1}(\beta^*)^2+\alpha^{m+n+1}(\alpha^*)^2+\beta^{m+n+1}(\beta^*)^2 \\
-\alpha^{m-1}\beta^n\alpha^*\beta^*-\beta^{m-1}\alpha^n\beta^*\alpha^*-\alpha^m\beta^{n+1}\alpha^*\beta^*-\beta^m\alpha^{n+1}\beta^*\alpha^*) \\
=(\alpha^{m+n}(\alpha-\beta)(\alpha^*)^2-\beta^{m+n}(\alpha-\beta)(\beta^*)^2 \\
-\alpha^m(\alpha^{-1}\beta^n+\beta^{n+1})\alpha^*\beta^*-\beta^m(\beta^{-1}\alpha^n+\alpha^{n+1})\beta^*\alpha^*) \\
=(\alpha-\beta)(\alpha^{m+n}(\alpha^*)^2-\beta^{m+n}(\beta^*)^2)
\end{gathered}$$

and using the Lemma 1 $(iii)-(iv)$, we get

$$\begin{gathered}
5\left(Q_{E,m-1}Q_{E,n}+Q_{E,m}Q_{E,n+1}\right) \\
=\sqrt{5}\left(\alpha^{m+n}\left(K_{E,0}-(1+t_1+t_2)+\sqrt{5}\left(Q_{E,0}-t_2\right)\right)\right. \\
\left.-\beta^{m+n}\left(K_{E,0}-(1+t_1+t_2)-\sqrt{5}\left(Q_{E,0}-t_2\right)\right)\right) \\
=\sqrt{5}\left((\alpha^{m+n}-\beta^{m+n})\left(K_{E,0}-(1+t_1+t_2)\right)\right. \\
\left.+(\alpha^{m+n}+\beta^{m+n})\sqrt{5}\left(Q_{E,0}-t_2\right)\right) \\
=\sqrt{5}\left(F_{m+n}\sqrt{5}\left(K_{E,0}-(1+t_1+t_2)\right)+L_{m+n}\sqrt{5}\left(Q_{E,0}-t_2\right)\right) \\
=5F_{m+n}\left(K_{E,0}-(1+t_1+t_2)\right)+L_{m+n}\left(Q_{E,0}-t_2\right).
\end{gathered}$$

Thus, we get the desired result.

Next, we give some relations between Fibonacci and Lucas elliptic quaternions.

Theorem 6. For nonnegative integers $\mathrm{n,r,s}$ with $\mathrm{s}\geq\mathrm{r}$, we have

$$K_{E,n+r}Q_{E,n+s}-K_{E,n+s}Q_{E,n+r}=2(-1)^{n+r}F_{s-r}\left(K_{E,0}-t\right).$$

Proof: By using the Binet formulas for the Fibonacci and Lucas elliptic quaternions, we have

$$\begin{gathered}
\sqrt{5}\left(K_{E,n+r}Q_{E,n+s}-K_{E,n+s}Q_{E,n+r}\right) \\
=(\alpha^*\alpha^{n+r}+\beta^*\beta^{n+r})(\alpha^*\alpha^{n+s}-\beta^*\beta^{n+s}) \\
-(\alpha^*\alpha^{n+s}+\beta^*\beta^{n+s})(\alpha^*\alpha^{n+r}-\beta^*\beta^{n+r}) \\
=(\alpha^*)^2\alpha^{2n+r+s}-\alpha^*\beta^*\alpha^{n+r}\beta^{n+s}+\beta^*\alpha^*\alpha^{n+s}\beta^{n+r}-(\beta^*)^2\beta^{2n+r+s} \\
-(\alpha^*)^2\alpha^{2n+r+s}+\alpha^*\beta^*\alpha^{n+s}\beta^{n+r}-\beta^*\alpha^*\alpha^{n+r}\beta^{n+s}+(\beta^*)^2\beta^{2n+r+s} \\
=\alpha^*\beta^*(\alpha\beta)^n(\alpha^s\beta^r-\alpha^r\beta^s)+\beta^*\alpha^*(\alpha\beta)^n(\alpha^s\beta^r-\alpha^r\beta^s) \\
=(\alpha\beta)^n(\alpha^s\beta^r-\alpha^r\beta^s)(\alpha^*\beta^*+\beta^*\alpha^*) \\
=2(-1)^{n+r}(\alpha^{s-r}-\beta^{s-r})\left(K_{E,0}-t\right) \\
=2(-1)^{n+r}\sqrt{5}F_{s-r}\left(K_{E,0}-t\right).
\end{gathered}$$

Thus, we get the desired result.

Theorem 7. For nonnegative integer n, we have

$$(i)K_{E,n}^2+Q_{E,n}^2=\frac{6}{5}\left(\left(K_{E,0}-(1+t_1+t_2)\right)L_{2n}+5\left(Q_{E,0}-t_2\right)F_{2n}\right)+\frac{8}{5}\left(K_{E,0}-t\right)(-1)^n,$$

$$(ii) K_{E,n}^2 - Q_{E,n}^2 = \frac{4}{5}\left(\left(K_{E,0} - (1 + t_1 + t_2)\right) L_{2n} + 5\left(Q_{E,0} - t_2\right) F_{2n}\right) + \frac{12}{5}\left(K_{E,0} - t\right)(-1)^n.$$

Proof: (i) By using the Binet formulas for the Fibonacci and Lucas elliptic quaternions and by using Lemma 1, we have

$$\begin{aligned}
K_{E,n}^2 + Q_{E,n}^2 &= (\alpha^*\alpha^n + \beta^*\beta^n)^2 + \frac{1}{5}(\alpha^*\alpha^n - \beta^*\beta^n)^2 \\
&= (\alpha^*\alpha^n + \beta^*\beta^n)(\alpha^*\alpha^n + \beta^*\beta^n) \\
&\quad + \frac{1}{5}(\alpha^*\alpha^n - \beta^*\beta^n)(\alpha^*\alpha^n - \beta^*\beta^n) \\
&= (\alpha^*)^2\alpha^{2n} + \alpha^*\beta^*(\alpha\beta)^n + \beta^*\alpha^*(\alpha\beta)^n + (\beta^*)^2\beta^{2n} \\
&\quad + \frac{1}{5}((\alpha^*)^2\alpha^{2n} - \alpha^*\beta^*(\alpha\beta)^n - \beta^*\alpha^*(\alpha\beta)^n + (\beta^*)^2\beta^{2n}) \\
&= (\alpha^*)^2\alpha^{2n}\left(1 + \frac{1}{5}\right) + (\beta^*)^2\beta^{2n}\left(1 + \frac{1}{5}\right) \\
&\quad + (\alpha^*\beta^* + \beta^*\alpha^*)(\alpha\beta)^n\left(1 - \frac{1}{5}\right) \\
&= \left(1 + \frac{1}{5}\right)\left(K_{E,0} - (1 + t_1 + t_2) + \sqrt{5}\left(Q_{E,0} - t_2\right)\right)\alpha^{2n} \\
&\quad + \left(1 + \frac{1}{5}\right)\left(K_{E,0} - (1 + t_1 + t_2) - \sqrt{5}\left(Q_{E,0} - t_2\right)\right)\beta^{2n} \\
&\quad + 2\left(1 - \frac{1}{5}\right)\left(K_{E,0} - t\right)(-1)^n \\
&= \frac{6}{5}\left(\left(K_{E,0} - (1 + t_1 + t_2)\right) L_{2n} + 5\left(Q_{E,0} - t_2\right) F_{2n}\right) + \frac{8}{5}\left(K_{E,0} - t\right)(-1)^n.
\end{aligned}$$

(ii) By using the Binet formulas for the Fibonacci and Lucas elliptic quaternions and by using Lemma 1, we have

$$\begin{aligned}
K_{E,n}^2 - Q_{E,n}^2 &= (\alpha^*\alpha^n + \beta^*\beta^n)^2 - \frac{1}{5}(\alpha^*\alpha^n - \beta^*\beta^n)^2 \\
&= (\alpha^*\alpha^n + \beta^*\beta^n)(\alpha^*\alpha^n + \beta^*\beta^n) - \frac{1}{5}(\alpha^*\alpha^n - \beta^*\beta^n)(\alpha^*\alpha^n - \beta^*\beta^n) \\
&= (\alpha^*)^2\alpha^{2n} + \alpha^*\beta^*(\alpha\beta)^n + \beta^*\alpha^*(\alpha\beta)^n + (\beta^*)^2\beta^{2n} \\
&\quad - \frac{1}{5}((\alpha^*)^2\alpha^{2n} - \alpha^*\beta^*(\alpha\beta)^n - \beta^*\alpha^*(\alpha\beta)^n + (\beta^*)^2\beta^{2n}) \\
&= \frac{4}{5}\left(\alpha^{2n}(K_{E,0} - (1 + t_1 + t_2\right) + \sqrt{5}(Q_{E,0} - t_2)) \\
&\quad + \beta^{2n}(K_{E,0} - (1 + t_1 + t_2) - \sqrt{5}(Q_{E,0} - t_2)) \\
&\quad + \frac{6}{5}\left((\alpha\beta)^n 2\left(K_{E,0} - t\right)\right) \\
&= \frac{4}{5}((K_{E,0} - (1 + t_1 + t_2))(\alpha^{2n} + \beta^{2n}) + \sqrt{5}(Q_{E,0} - t_2)(\alpha^{2n} - \beta^{2n})) \\
&\quad + \frac{6}{5}(-1)^n 2\left(K_{E,0} - t\right) \\
&= \frac{4}{5}\left(K_{E,0} - (1 + t_1 + t_2)\right)L_{2n} + 5(Q_{E,0} - t_2)F_{2n}) + \frac{12}{5}(-1)^n\left(K_{E,0} - t\right).
\end{aligned}$$

Thus we get the desired result.

Theorem 8. For nonnegative integers n and m with m ≥ n we have

$$Q_{E,n}K_{E,m} - K_{E,m}Q_{E,n} = 2(-1)^{n+1}\Delta\omega L_{m-n}.$$

Proof: By using the Binet formulas for the Fibonacci and Lucas elliptic quaternions, we have

$$\begin{gathered}
\sqrt{5}\big(Q_{E,n}K_{E,m} - K_{E,m}Q_{E,n}\big) \\
= (\alpha^*\alpha^n - \beta^*\beta^n)(\alpha^*\alpha^m + \beta^*\beta^m) \\
-(\alpha^*\alpha^m + \beta^*\beta^m)(\alpha^*\alpha^n - \beta^*\beta^n) \\
= (\alpha^*)^2\alpha^{n+m} + \alpha^*\beta^*\alpha^n\beta^m - \beta^*\alpha^*\alpha^m\beta^n - (\beta^*)^2\beta^{n+m} \\
-(\alpha^*)^2\alpha^{n+m} + \alpha^*\beta^*\alpha^m\beta^n - \beta^*\alpha^*\alpha^n\beta^m + (\beta^*)^2\beta^{n+m} \\
= \alpha^*\beta^*\alpha^n\beta^m - \beta^*\alpha^*\alpha^m\beta^n + \alpha^*\beta^*\alpha^m\beta^n - \beta^*\alpha^*\alpha^n\beta^m \\
= (\alpha^*\beta^* - \beta^*\alpha^*)(\alpha^n\beta^m + \alpha^m\beta^n) \\
= 2(-1)^{n+1}\Delta\sqrt{5}\omega(\beta^{m-n} + \alpha^{m-n}) \\
= 2(-1)^{n+1}\Delta\sqrt{5}\omega L_{m-n}.
\end{gathered}$$

Thus, we get the desired result.

Theorem 9. For Fibonacci and Lucas elliptic quaternions, we have the following sum formulas:

$$(i)\sum_{r=1}^{n} Q_{E,r} = Q_{E,n+2} - Q_{E,2},$$

$$(ii)\sum_{r=1}^{n} K_{E,r} = K_{E,n+2} - K_{E,2},$$

$$(iii)\sum_{i=0}^{n} \binom{n}{i} Q_{E,i+r} = Q_{E,2n+r},$$

$$(iv)\sum_{i=0}^{n} \binom{n}{i} K_{E,i+r} = K_{E,2n+r}.$$

Proof: (i) and (ii) can be proven by mathematical induction.

(iii) By using the Binet formula of Fibonacci elliptic quaternion and by using binomial expansion, we get

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} Q_{E,i+r} &= \sum_{i=0}^{n} \binom{n}{i}\left(\frac{\alpha^*\alpha^{i+r} - \beta^*\beta^{i+r}}{\alpha - \beta}\right) \\
&= \frac{\alpha^*\alpha^r}{\alpha - \beta}\sum_{i=0}^{n} \binom{n}{i}\alpha^i - \frac{\beta^*\beta^r}{\alpha - \beta}\sum_{i=0}^{n} \binom{n}{i}\beta^i \\
&= \frac{\alpha^*\alpha^r}{\alpha - \beta}(1+\alpha)^n - \frac{\beta^*\beta^r}{\alpha - \beta}(1+\beta)^n
\end{aligned}$$

$$= \frac{\alpha^* \alpha^{2n+r} - \beta^* \beta^{2n+r}}{\alpha - \beta} = Q_{E,2n+r}.$$

(iv) By using the Binet formula of Lucas elliptic quaternion and by using binomial expansion, we get

$$\begin{aligned}\sum_{i=0}^{n} \binom{n}{i} K_{E,i+r} &= \sum_{i=0}^{n} \binom{n}{i} \left(\alpha^* \alpha^{i+r} + \beta^* \beta^{i+r}\right) \\ &= \alpha^* \alpha^r \sum_{i=0}^{n} \binom{n}{i} \alpha^i + \beta^* \beta^r \sum_{i=0}^{n} \binom{n}{i} \beta^i \\ &= \alpha^* \alpha^r (1+\alpha)^n + \beta^* \beta^r (1+\beta)^n \\ &= \alpha^* \alpha^{2n+r} + \beta^* \beta^{2n+r} \\ &= K_{E,2n+r}.\end{aligned}$$

Thus, we get the desired result.

Theorem 10. For Fibonacci and Lucas elliptic quaternions, we have the followings:

$$(i) \sum_{i=0}^{n} \binom{n}{i} Q_{E,2i+r} = \begin{cases} 5^{\frac{n}{2}} Q_{E,n+r}, & \text{if } n \text{ is even,} \\ 5^{\frac{n-1}{2}} K_{E,n+r}, & \text{if } n \text{ is odd,} \end{cases}$$

$$(ii) \sum_{i=0}^{n} \binom{n}{i} K_{E,2i+r} = \begin{cases} 5^{\frac{n}{2}} K_{E,n+r}, & \text{if } n \text{ is even,} \\ 5^{\frac{n+1}{2}} Q_{E,n+r}, & \text{if } n \text{ is odd,} \end{cases}$$

$$(iii) \sum_{i=0}^{n} \binom{n}{i} (-1)^i Q_{E,2i+r} = (-1)^n Q_{E,n+r},$$

$$(iv) \sum_{i=0}^{n} \binom{n}{i} (-1)^i K_{E,2i+r} = (-1)^n K_{E,n+r}.$$

Proof: (i) By using the Binet formula of Fibonacci elliptic quaternion and by using binomial expansion, we get

$$\begin{aligned}\sum_{i=0}^{n} \binom{n}{i} Q_{E,2i+r} &= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{\alpha^* \alpha^{2i+r} - \beta^* \beta^{2i+r}}{\alpha - \beta}\right) \\ &= \frac{\alpha^* \alpha^r}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} \alpha^{2i} - \frac{\beta^* \beta^r}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} \beta^{2i} \\ &= \frac{\alpha^* \alpha^r}{\alpha - \beta} (1+\alpha^2)^n - \frac{\beta^* \beta^r}{\alpha - \beta} (1+\beta^2)^n \\ &= \frac{\alpha^* \alpha^r \left(\alpha\sqrt{5}\right)^n - \beta^* \beta^r \left(-\beta\sqrt{5}\right)^n}{\alpha - \beta}\end{aligned}$$

$$=\begin{cases}5^{\frac{n}{2}}Q_{E,n+r}, & \text{if } n \text{ is even,}\\ 5^{\frac{n-1}{2}}K_{E,n+r}, & \text{if } n \text{ is odd.}\end{cases}$$

(ii) By using the Binet formula of Lucas elliptic quaternion and by using binomial expansion, we get

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}K_{E,2i+r} &= \sum_{i=0}^{n}\binom{n}{i}\left(\alpha^*\alpha^{2i+r}+\beta^*\beta^{2i+r}\right)\\ &= \alpha^*\alpha^r\sum_{i=0}^{n}\binom{n}{i}\alpha^{2i}+\beta^*\beta^r\sum_{i=0}^{n}\binom{n}{i}\beta^{2i}\\ &= \alpha^*\alpha^r(1+\alpha^2)^n+\beta^*\beta^r(1+\beta^2)^n\\ &= \alpha^*\alpha^r\left(\alpha\sqrt{5}\right)^n+\beta^*\beta^r\left(-\beta\sqrt{5}\right)^n\\ &= \begin{cases}5^{\frac{n}{2}}K_{E,n+r}, & \text{if } n \text{ is even,}\\ 5^{\frac{n+1}{2}}Q_{E,n+r}, & \text{if } n \text{ is odd.}\end{cases}\end{aligned}$$

(iii) By using the Binet formula of Fibonacci elliptic quaternion and by using binomial expansion, we get

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^iQ_{E,2i+r} &= \sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(\frac{\alpha^{2i+r}\alpha^*-\beta^{2i+r}\beta^*}{\alpha-\beta}\right)\\ &= \frac{\alpha^*\alpha^r}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{2i}-\frac{\beta^*\beta^r}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{2i}\\ &= \frac{\alpha^*\alpha^r}{\alpha-\beta}(1-\alpha^2)^n-\frac{\beta^*\beta^r}{\alpha-\beta}(1-\beta^2)^n\\ &= \frac{\alpha^*\alpha^r}{\alpha-\beta}(-\alpha)^n-\frac{\beta^*\beta^r}{\alpha-\beta}(-\beta)^n\\ &= (-1)^nQ_{E,n+r}.\end{aligned}$$

(iv) By using the Binet formula of Lucas elliptic quaternion and by using binomial expansion, we get

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^iK_{E,2i+r} &= \sum_{i=0}^{n}\binom{n}{i}(-1)^i\left(\alpha^{2i+r}\alpha^*+\beta^{2i+r}\beta^*\right)\\ &= \alpha^*\alpha^r\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{2i}+\beta^*\beta^r\sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{2i}\\ &= \alpha^*\alpha^r(1-\alpha^2)^n+\beta^*\beta^r(1-\beta^2)^n\\ &= \alpha^*\alpha^r(-\alpha)^n+\beta^*\beta^r(-\beta)^n\\ &= (-1)^nK_{E,n+r}.\end{aligned}$$

Thus, we get the desired result.

Next, we give some binomial identities for Fibonacci and Lucas elliptic quaternions. Since the proof is similar to the previous theorem, we skip it.

Theorem 11. For Fibonacci and Lucas elliptic quaternions, we have the followings:

$$(i)\sum_{i=0}^{n}\binom{n}{2i}Q_{E,4i}=\begin{cases}\frac{1}{2}\left(5^{\frac{n}{2}}+1\right)Q_{E,n}, & \text{if } n \text{ is even,}\\ \frac{1}{2}\left(5^{\frac{n-1}{2}}K_{E,n}-Q_{E,n}\right), & \text{if } n \text{ is odd,}\end{cases}$$

$$(ii)\sum_{i=0}^{n}\binom{n}{2i}K_{E,4i}=\begin{cases}\frac{1}{2}\left(5^{\frac{n}{2}}+1\right)K_{E,n}, & \text{if } n \text{ is even,}\\ \frac{1}{2}\left(5^{\frac{n+1}{2}}Q_{E,n}-K_{E,n}\right), & \text{if } n \text{ is odd,}\end{cases}$$

$$(iii)\sum_{i=0}^{n}\binom{n}{2i+1}Q_{E,4i+1}=\begin{cases}\frac{1}{2}\left(5^{\frac{n}{2}}-1\right)Q_{E,n}, & \text{if } n \text{ is even,}\\ \frac{1}{2}\left(5^{\frac{n-1}{2}}K_{E,n}+Q_{E,n}\right), & \text{if } n \text{ is odd,}\end{cases}$$

$$(iv)\sum_{i=0}^{n}\binom{n}{2i+1}K_{E,4i+1}=\begin{cases}\frac{1}{2}\left(5^{\frac{n}{2}}-1\right)K_{E,n}, & \text{if } n \text{ is even,}\\ \frac{1}{2}\left(5^{\frac{n+1}{2}}Q_{E,n}+K_{E,n}\right), & \text{if } n \text{ is odd.}\end{cases}$$

4. CONCLUSIONS

In this paper, we introduce the Fibonacci and Lucas elliptic quaternions, extending the classical Fibonacci and Lucas quaternions through the incorporation of elliptic quaternion concepts. We explore the fundamental properties of these sequences, including recurrence relations, generating functions, and Binet-like formulas. By using the Binet formula of Fibonacci elliptic quaternions we derive Vajda’s identity and Honsberger’s identity. Additionally, we establish relations encompassing both Fibonacci and Lucas elliptic quaternions. An intriguing direction for further exploration is the study of a generalization of Fibonacci and Lucas elliptic quaternions. In particular, one could consider bi-periodic Fibonacci and Lucas sequences [11-13] and investigate the elliptic quaternion structure associated with them.

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