ORIGINAL PAPER

STRONGLY NANO $(1,2)^{\star}$-g*-CLOSED SETS IN NANO BITOPOLOGICAL SPACES

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Abstract. *In this paper, we introduce the concept of strongly nano $(1,2)^{\star}$-$g^{\star}$-closed sets in nano bitopological space and we investigate the group of the structure of the set of all strongly nano $(1,2)^{\star}$-$g^{\star}$-closed sets.*

Keywords: *nano $(1,2)^{\star}$-closed; nano $(1,2)^{\star}$-$g^{\star}$-closed set; strongly nano $(1,2)^{\star}$-$g^{\star}$-closed set.*

1. INTRODUCTION AND PRELIMINARIES

Lellis Thivagar et al. [1] overtook all these theories by bringing out a new topologyto study the imperfect data in mathematics. The new topology is named Nano Topology because of its size. Whatever the universe's size, it is reduced to at most five open sets by defining the lower and upper approximations and boundary regions of subsets of a universe. The elements of the nano topology are called the nano open sets. Lellis Thivagar et al. [2] further established the weak forms of nano topology in Cech rough closure space. Some new notions in the concept of nano bitopological spaces were introduced by Bhuvaneswari et al. [3]. The main aim of this paper is to introduce the concept of strong nano $(1,2)^{\star}$-g*-closed sets in nano bitopological space and we investigate the group of the structure of the set of all strongly nano $(1,2)^{\star}$-g*-closed sets.Throughout this paper, U denote nano bitopological spaces $(\mathrm{U}, \tau_{\mathrm{R}_{1,2}}(\mathrm{X}))$We present here some basic definitions and results that will be used in the subsequent development of this work.

Definition 1.1. [4] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let X ⊆ U.

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $\mathrm{L_R(X)}$. That is, $\mathrm{L_R(X)} = \bigcup_{\mathrm{x} \in \mathrm{U}} \{\mathrm{R(x)}: \mathrm{R(x)} \subseteq \mathrm{X}\}$, where R(x) denotes the equivalence class determined by x.
2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $\mathrm{U_R(X)}$. That is, $\mathrm{U_R(X)} = \bigcup_{\mathrm{x} \in \mathrm{U}} \{\mathrm{R(x)}: \mathrm{R(x)} \cap \mathrm{X} \neq \phi\}$.
3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $\mathrm{B_R(X)}$. That is, $\mathrm{B_R(X)} = \mathrm{U_R(X)} - \mathrm{L_R(X)}$.

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Definition 1.2. [1] Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:
1. U and $\phi \in \tau_R(X)$,
2. The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$,
3. The intersection of the elements of any finite sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

Definition 1.3. [5] Let U be the universe, R be an equivalence relation on U and $\tau_{R_{1,2}}(X) = \bigcup_{,} \{\tau_{R_1}(X), \tau_{R_2}(X)\}$ where $X \subseteq U$. Then it satisfies the following axioms:
1. U and $\phi \in \tau_{R_{1,2}}(X)$.
2. The union of the elements of any sub-collection of $\tau_{R_{1,2}}(X)$ is in $\tau_{R_{1,2}}(X)$.
3. The intersection of the elements of any finite sub-collection of $\tau_{R_{1,2}}(X)$ is in $\tau_{R_{1,2}}(X)$.

Then $\tau_{R_{1,2}}(X)$ is a topology on U called the Nano bitopology on U with respect to X. $(U, \tau_{R_{1,2}}(X))$ is called the Nano bitopological space. Elements of the Nano bitopology are known as Nano $\tau_{1,2}$-open sets in U. Elements of $(\tau_{R_{1,2}}(X))^c$ are called Nano $\tau_{1,2}$-closed sets in $\tau_{R_{1,2}}(X)$.

Definition 1.4. [3] If $(U, \tau_{R_{1,2}}(X))$ is a Nano bitopological space with respect to X where $X \subseteq U$ and if $A \subseteq U$, then
1. The Nano $(1,2)^\star$ closure of A is defined as the intersection of all Nano $(1,2)^\star$ closed sets containing A and it is denoted by $N_{\tau_{1,2}}$-cl(A).
2. The Nano $(1,2)^\star$ interior of A is defined as the union of all Nano $(1,2)^\star$ open subsets of A contained in A and it is denoted by $N_{\tau_{1,2}}$-int(A).

Definition 1.5. A subset G of a nano bitopological space $(U, \tau_{R_{1,2}}(X))$ is called
1. a nano $(1,2)^\star$-semi-open set [3] if $G \subseteq N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G))$.
2. a nano $(1,2)^\star$-α-open set [6] if $G \subseteq N_{\tau_{1,2}}\text{-int}(N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G)))$.
3. a nano $(1,2)^\star$-pre-open set [7] if $G \subseteq N_{\tau_{1,2}}\text{-int}(N_{\tau_{1,2}}\text{-cl}(G))$.
4. a nano $(1,2)^\star$-regular open set [5] if $G = N_{\tau_{1,2}}\text{-int}(N_{\tau_{1,2}}\text{-cl}(G))$.

The complements of the mentioned sets are called their respective closed sets.

Definition 1.6. A subset G of a nano bitopological space $(U, \tau_{R_{1,2}}(X))$ is called
1. a nano $(1,2)^\star$-g-closed set [8] if $N_{\tau_{1,2}}\text{-cl}(G) \subseteq H$ whenever $G \subseteq H$ and His nano $\tau_{1,2}$-open.
2. a nano $(1,2)^\star$-gs-closed set [3] if $N_{(1,2)^\star}\text{-scl}(G) \subseteq H$ whenever $G \subseteq H$ and H is nano $\tau_{1,2}$-open.
3. a nano $(1,2)^\star$-sg-closed set [3] if $N_{(1,2)^\star}\text{-scl}(G) \subseteq H$ whenever $G \subseteq H$ and H is nano $(1,2)^\star$-semi-open.
4. a nano $(1,2)^\star$-αg-closed set [6] if $N_{(1,2)^\star}\text{-}\alpha\text{cl}(G) \subseteq H$ whenever $G \subseteq H$ and G is nano $\tau_{1,2}$-open.
5. a nano $(1,2)^\star$-rg-closed set [5] if $N_{(1,2)^\star}\text{-rcl}(G) \subseteq H$ whenever $G \subseteq H$ and H is nano $(1,2)^\star$-regular open.
6. a nano $(1,2)^\star$-gsp-closed set [9] if $N_{(1,2)^\star}\text{-}\beta\text{cl}(G) \subseteq H$ whenever $G \subseteq H$ and H is nano $\tau_{1,2}$-open.

The complements of the mentioned sets are called their respective open sets.

Definition 1.7 [10] Let G be a subset of a nano bitopological space U. Then G is called a nano $(1,2)^\star$-$g^\star$-closed set if $N_{\tau_{1,2}}$-$cl(G) \subseteq H$ whenever $G \subseteq H$ and H is nano $(1,2)^\star$-g-open in U. The complement of nano $(1,2)^\star$-$g^\star$-closed set is called nano $(1,2)^\star$-$g^\star$-open.

2. STRONGLY NANO $(1,2)^\star$-g⋆-CLOSED SETS

Definition 2.1. Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space and G be its subset, then G is strongly nano $(1,2)^\star$-$g^\star$-closed set if $N_{\tau_{1,2}}$-$cl(N_{\tau_{1,2}}$-$int(G)) \subseteq H$ whenever $G \subseteq H$ and H is nano $(1,2)^\star$-g-open. The complement of strongly nano $(1,2)^\star$-$g^\star$-closed set is called strongly nano $(1,2)^\star$-$g^\star$-open.

Proposition 2.2. In a nano bitopological space $(U, \tau_{R_{1,2}}(X))$, every nano $\tau_{1,2}$-closed set is strongly nano $(1,2)^\star$-$g^\star$-closed.

Proof: The proof is immediate from the definition of nano $\tau_{1,2}$-closed set.

Remark 2.3. The converse of Proposition 2.2 need not be true as shown in the following Example.

Example 2.4. Let

$$U = \{a, b, c, d\} \text{ with } U/R_1 = \{\{d\}, \{a, b, c\}\}$$

and

$$X = \{a, b, c\} \text{ then } \tau_{R_1}(X) = \{\phi, \{a, b, c\}, U\}$$

and let

$$U/R_2 = \{\{a\}, \{b, c, d\}\} \text{ and } X = \{a\}$$

then

$$\tau_{R_2}(X) = \{\phi, \{a\}, U\}.$$

Then the sets in {ϕ, {a}, {a, b, c}, U} are called $N_{\tau_{1,2}}$-open and the sets in {ϕ, {d}, {b, c, d}, U} are called $N_{\tau_{1,2}}$-closed. In the nano bitopological space $(U, \tau_{R_{1,2}}(X))$, then the subset {b} is strongly nano $(1,2)^\star$-$g^\star$-closed set but nano $\tau_{1,2}$-closed.

Proposition 2.5. If a subset G of a nano bitopological space $(U, \tau_{R_{1,2}}(X))$ is nano $(1,2)^\star$-$g^\star$-closed then it is strongly nano $(1,2)^\star$-$g^\star$-closed in U.

Proof: Suppose G is nano $(1,2)^\star$-$g^\star$-closed in U. Let H be a nano $\tau_{1,2}$-open set containing G in U. Then H contains $N_{\tau_{1,2}}$-$cl(A, B)$. Now $H \supseteq N_{\tau_{1,2}}$-$cl(G) \supseteq N_{\tau_{1,2}}$-$cl(N_{\tau_{1,2}}$-$int(G))$. Thus G is strongly nano $(1,2)^\star$-$g^\star$-closed in U.

Remark 2.6. The converse Proposition 2.5 is need not be true as shown in the following Example.

Example 2.7. In Example 2.4, then the subset {b} is a strongly nano $(1,2)^\star$-$g^\star$-closed set but not nano $(1,2)^\star$-$g^\star$-closed.

Theorem 2.8. If G is a subset of a nano bitopological space U is nano $\tau_{1,2}$-open and strongly nano $(1,2)^\star$-$g^\star$-closed then it is nano $\tau_{1,2}$-closed.

Proof: Suppose a subset G of U is both nano $\tau_{1,2}$-open and strongly nano $(1,2)^\star$-$g^\star$-closed. Now $G \supseteq N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $\supseteq N_{\tau_{1,2}}$-cl(G). Therefore $G \supseteq N_{\tau_{1,2}}$-cl(G). Since $N_{\tau_{1,2}}$-cl(G) $\supseteq$ G. We have $G \supseteq N_{\tau_{1,2}}$-cl(G). Thus G is nano $\tau_{1,2}$-closed in U.

Corollary 2.9. If G is both nano $\tau_{1,2}$-open and strongly nano $(1,2)^\star$-$g^\star$-closed in U then it is both nano $(1,2)^\star$-regular open and nano $(1,2)^\star$-regular closed-inU.

Proof: As G is nano $\tau_{1,2}$-open $G = N_{\tau_{1,2}}$-int(G) $= N_{\tau_{1,2}}$-int($N_{\tau_{1,2}}$-cl(G)), since G is nano $\tau_{1,2}$-closed. Thus G is nano $(1,2)^\star$-regular open. Again G is nano $\tau_{1,2}$-open in U, $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $= N_{\tau_{1,2}}$-cl(G). As G is nano $\tau_{1,2}$-closed $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $=$ G. Thus G is nano $(1,2)^\star$-regular closed.

Corollary 2.10. If G is both nano $\tau_{1,2}$-open and strongly nano $(1,2)^\star$-$g^\star$-closed then it is nano $(1,2)^\star$-rg-closed.

Theorem 2.11. If a subset G of a nano bitopological space U is both strongly nano $(1,2)^\star$-$g^\star$-closed and nano $(1,2)^\star$-semi-open open then it is nano $(1,2)^\star$-$g^\star$-closed.

Proof: Suppose G is both strongly nano $(1,2)^\star$-$g^\star$-closed and nano $(1,2)^\star$-semi-open in U. Let H be a nano $\tau_{1,2}$-open set containing G. As G is strongly nano $(1,2)^\star$-$g^\star$-closed, $H \supseteq N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)). Now $H \supseteq N_{\tau_{1,2}}$-cl(G). Since G is nano $(1,2)^\star$-semi-open.Thus G is nano $(1,2)^\star$-$g^\star$-closed in U.

Corollary 2.12. If a subset G of a nano bitopological space U is both strongly nano $(1,2)^\star$-$g^\star$-closed and nano $\tau_{1,2}$-open then it is nano $(1,2)^\star$-$g^\star$-closed.

Proof: As every nano $\tau_{1,2}$-open set is nano $(1,2)^\star$-semi-open by Theorem 2.11 the proof follows.

Theorem 2.13. A set G is strongly nano $(1,2)^\star$-$g^\star$-closed iff $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $-$ G contains no non empty nano $\tau_{1,2}$-closed.

Proof:

<u>Necessary part:</u>

Suppose that A is non empty nano $\tau_{1,2}$-closed subset of $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)). Now $A \subseteq N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $-$ G implies $A \subseteq N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $\cap G^c$, since $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $-$ G $= N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $\cap G^c$. Thus $A \subseteq N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)). Now $A \subseteq G^c$ implies $G \subseteq G^c$. Here A^c is nano $(1,2)^\star$-g-open and G is strongly nano $(1,2)^\star$- $g^\star$-closed, we have $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $\subseteq A^c$. Thus $A \subseteq (N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)))^c. Hence $A \subseteq (N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G))) $\cap$ $(N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)))$^c = \phi$. Therefore $A = \phi \Rightarrow N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $-$ G contains no non empty nano $\tau_{1,2}$-closed.

Sufficient part:

Let $G \subseteq H$, H is nano $(1,2)^\star$-g-open. Suppose that $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) is not contained in H then $(N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)))^c is a non empty nano $\tau_{1,2}$-closed set $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $-$ G which is a contradiction. Therefore $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $\subseteq$ H and hence G is strongly nano $(1,2)^\star$-g*-closed.

Corollary 2.14. A strongly nano $(1,2)^\star$-g*-closed set G is nano $(1,2)^\star$-regular closed iff $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $\supseteq$ G.

Proof: Assume that G is nano $(1,2)^\star$-regular closed. Since $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $=$ G, $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $-$ G $= \phi$ is nano $(1,2)^\star$-regular closed and hence nano $\tau_{1,2}$-closed.

Conversely, assume that $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $-$ G is nano $\tau_{1,2}$-closed by Theorem 2.13. $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $-$ G contains no nonempty nano $\tau_{1,2}$-closed set. Therefore $N_{\tau_{1,2}}$-cl$(N_{\tau_{1,2}}$-int(G)) $-$ G $= \phi$. Thus G is nano $(1,2)^\star$-regular closed.

Theorem 2.15. Suppose that $K \subseteq G \subseteq U$, K is strongly nano $(1,2)^\star$-g*-closed set relative to G and that both nano $\tau_{1,2}$-open and strongly nano $(1,2)^\star$-g*-closed subset of U then K is strongly nano $(1,2)^\star$-g*-closed set relative to U.

Proof: Let $K \subseteq H$ and H be a nano $\tau_{1,2}$-open set in U. But given that $K \subseteq G \subseteq U$, therefore $K \subseteq G$ and $K \subseteq H$. This implies $K \subseteq G \cap H$. Since K is strongly nano $(1,2)^\star$-g*-closed relative to G,

$$N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)) \subseteq G \cap H.\text{(ie) } G \cap N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)) \subseteq G \cap H.$$

This implies

$$G \cap N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)) \subseteq H.$$

Thus

$$(G \cap (N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)))) \cup (N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)))^c \subseteq H \cup (N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)))^c$$

implies

$$G \cup (N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)))^c \subseteq H \cup (N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}((K))))^c.$$

Since G is strongly nano $(1,2)^\star$-g*-closed in U, we have

$$(N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G))) \subseteq H \cup (N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)))^c.$$

Also

$$K \subseteq G \Rightarrow N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)) \subseteq N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G)).$$

Thus

$N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(K)) ⊆ $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(K)) ⊆ H ∪ ($N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(K)))^c.

Therefore K is strongly nano $(1,2)^\star$-g*-closed set relative to U.

Corollary 2.16. Let G be strongly nano $(1,2)^\star$-g*-closed and suppose that A is nano $\tau_{1,2}$-closed then G ∩ A is strongly nano $(1,2)^\star$- g*-closed.

Proof: To show that G ∩ A is strongly nano $(1,2)^\star$-g*-closed, we have to show

$$N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G) \cap A) \subseteq H$$

whenever

$$G \cap A \subseteq H \text{ and } H$$

is nano $(1,2)^\star$-g-open. G ∩ A is nano $\tau_{1,2}$-closed in G and so strongly nano $(1,2)^\star$-g*-closed in K by Theorem 2.15. G ∩ A is strongly nano $(1,2)^\star$-g*-closed in U. Since G ∩ A ⊆ G ⊆ U.

Theorem 2.17. If G is strongly nano $(1,2)^\star$-g*-closed and G ⊆ K ⊆ $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) then K is strongly nano $(1,2)^\star$-g*-closed.

Proof: Given that

$$K \subseteq N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G))$$

then

$$N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)) \subseteq N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G)),$$

$$N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)) - K \subseteq N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G)) - G.$$

Since G ⊆ K. As G is strongly nano $(1,2)^\star$-g*-closed by the above theorem

$$N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(G)) - G$$

contains no non empty nano $\tau_{1,2}$-closed set,

$$N_{\tau_{1,2}}\text{-cl}(N_{\tau_{1,2}}\text{-int}(K)) - K$$

contains no empty nano $\tau_{1,2}$-closed set. Again by Theorem 2.15, K is strongly nano $(1,2)^\star$-g*-closed.

Theorem 2.18. Let G ⊆ A ⊆ U and suppose that G is strongly nano $(1,2)^\star$-g*-closed in U then G is strongly nano $(1,2)^\star$- g*-closed relative to A.

Proof: Given that G ⊆ A ⊆ U and G is strongly nano $(1,2)^\star$-g*-closed in U. To show that G is strongly nano $(1,2)^\star$- g*-closed relative to A, let G ⊆ A ∩ H, where H is nano $(1,2)^\star$-g-open in U. Since G is strongly nano $(1,2)^{\star}$-$g^{\star}$-closed in U, $G \subseteq H$ implies $N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $\subseteq$ H. (ie) $A \cap N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $\subseteq A \cap H$, where $A \cap N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) is nano $\tau_{1,2}$-closure of nano $\tau_{1,2}$-interior of G in A. Thus G is strongly nano $(1,2)^{\star}$-$g^{\star}$-closed relative to A.

Theorem 2.19. If a subset G of a nano bitopological space U is nano $(1,2)^{\star}$-gsp-closed then it is strongly nano $(1,2)^{\star}$-$g^{\star}$-closed.

Proof: Suppose that G is nano $(1,2)^{\star}$-gsp-closed in U, let H be nano $\tau_{1,2}$-open set containing G. Then $H \subseteq$ bsp-cl(G), $G \cup H \supseteq G \cup (N_{\tau_{1,2}}$-int($N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)))) which implies $H \supseteq N_{\tau_{1,2}}$-int($N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G))) as H is nano $\tau_{1,2}$-open. (ie) $H \supseteq N_{\tau_{1,2}}$-cl($N_{\tau_{1,2}}$-int(G)) $-$ G is strongly nano $(1,2)^{\star}$-$g^{\star}$-closed set in U.

Theorem 2.20. In a nano bitopological space $(U, \tau_{R_{1,2}}(X))$, every strongly nano $(1,2)^{\star}$-$g^{\star}$-closed set is a nano $(1,2)^{\star}$-αg-closed set and hence nano $(1,2)^{\star}$-gs-closed.

Proof: Let G be a strongly nano $(1,2)^{\star}$-$g^{\star}$-closed set of $(U, \tau_{R_{1,2}}(X))$. By Theorem 2.19, G is nano $(1,2)^{\star}$-g-closed and nano $(1,2)^{\star}$-αg-closed. Then we know that every nano $(1,2)^{\star}$-g-closed set is nano $(1,2)^{\star}$-gs-closed. By Theorem 2.19, every strongly nano $(1,2)^{\star}$- $g^{\star}$-closed set is nano $(1,2)^{\star}$-gs-closed.

3. MORE ON STRONGLY NANO $(1,2)^{\star}$-$g^{\star}$-OPEN SETS

Definition 3.1. Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space and $x \in U$. A subset J of U is said to be strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of x if there exists a strongly nano $(1,2)^{\star}$-$g^{\star}$-open set H such that $x \in H \subset J$.

The collection of all strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhoods of $x \in J$ is called a strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood system at x and shall be denoted by $SNG^{\star}N(x)$.

Definition 3.2. Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space and G be a subset of U. G subset J of U is said to be strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of G if there exists a strongly nano $(1,2)^{\star}$-$g^{\star}$-open set H such that $G \in H \subseteq J$.

Definition 3.3. Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space and G be a subset of U. A point $x \in G$ is said to be a strongly nano $(1,2)^{\star}$-$g^{\star}$-interior point of G, if G is strongly nano $(1,2)^{\star}$-$g^{\star}N(x)$. The set of all strongly nano $(1,2)^{\star}$-$g^{\star}$-interior points of G is called a strongly nano $(1,2)^{\star}$-$g^{\star}$-interior points of G is called a strongly nano $(1,2)^{\star}$-$g^{\star}$-interior of G and is denoted by $N_{(1,2)^{\star}}$-$sg^{\star}int(G)$. $N_{(1,2)^{\star}}$-$sg^{\star}int(G) = \bigcup\{H$: H is strongly nano $(1,2)^{\star}$-$g^{\star}$-open, $H \subset G\}$.

Definition 3.4. Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space and G be a subset of U. A point $x \in G$ is said to be a strongly nano $(1,2)^{\star}$-$g^{\star}$-closure of G. Then $N_{(1,2)^{\star}}$-$sg^{\star}cl(G) = \bigcap\{E: G \subset E$ is strongly nano $(1,2)^{\star}$-$g^{\star}$-closed $N_{(1,2)^{\star}}G^{\star}C(U)\}$.

Theorem 3.5. A subset G of a nano bitopological space is strongly nano $(1,2)^{\star}$-$g^{\star}$-open if it is a strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of each points.

Proof: Let H be a subset of a nano bitopological space be strongly nano $(1,2)^{\star}$-$g^{\star}$-open. Then for every $x \in U$, $x \in H \subseteq H$, and therefore H is a strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of each of the points.

Theorem 3.6. Let $(U, \tau_{R_{1,2}}(X))$ be a nano bitopological space. If G is strongly nano $(1,2)^{\star}$-$g^{\star}$-closed subset of U and $x \in SN_{(1,2)^{\star}}G^{\star}CL(G)$ if and only if for any strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood J of x in U, $J \cap G \neq \phi$.

Proof: Let us assume that there is a strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood J of the point x in U such that $J \cap G = \phi$. There exists a strongly nano $(1,2)^{\star}$-$g^{\star}$-open set H of U such that $x \in H \subseteq J$. Therefore we have $H \cap G = \phi$ and so $x \in U - H$. Then $SN_{(1,2)^{\star}}G^{\star}CL(G) \in U - H$ and therefore $x \notin SN_{(1,2)^{\star}}G^{\star}CL(G)$, which contradicts the hypothesis that $x \in SN_{(1,2)^{\star}}G^{\star}CL(G)$. Therefore $J \cap G \neq \phi$.

Conversely, suppose that $x \notin SN_{(1,2)^{\star}}G^{\star}CL(G)$. Then there exists a strongly nano $(1,2)^{\star}$-$g^{\star}$-closed set H of U such that $G \subseteq H$ and $x \notin H$. Thus $x \in U - H$ and $U - H$ is strongly nano $(1,2)^{\star}$-$g^{\star}$-open in U and hence $U - H$ is strongly nano $(1,2)^{\star}$-$g^{\star}$-open in U and hence $U - H$ is a strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of x in U. But $G \cap (U - H) = \phi$ which is a contradiction. Hence $x \in SN_{(1,2)^{\star}}G^{\star}CL(G)$.

Theorem 3.7. Let (X, Y) be a nano bitopological space and $x \in U$. Let strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x)$ be a collection of all strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of x. Then

1. Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x) \neq \phi$ and x belongs to each member of Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x)$.
2. The intersection of the any two members of strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x)$ is again a member of strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x)$.
3. If $J \in$ Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x)$ and $A \subseteq J$, then $A \in$ Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x)$.
4. Each member $J \in$ Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x)$ is a superset of a member $H \in$ Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(x)$ where H is a strongly nano $(1,2)^{\star}$-$g^{\star}$-open set.

Proof:

1. Since U is strongly nano $(1,2)^{\star}$-$g^{\star}$-open set containing a, it is a strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of every $a \in U$. Hence there exists at least one strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood namely U for each $a \in U$ there is strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a) \neq \phi$. Let $J \in$ strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a)$, J is a strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of a, then there exists a strongly nano $(1,2)^{\star}$-$g^{\star}$-open set H such that $a \in H \subseteq J$, so $a \in J$. therefore a belongs to every number J strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a)$.
2. Let $J \in$ Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a)$ and $A \in$ strongly nano $(1,2)^{\star}$-$g^{\star}N(a)$. There exists strongly nano $(1,2)^{\star}$-$g^{\star}$-open set H and E such that $a \in H \subseteq J$ and $a \in E \subseteq A$. Hence $a \in H \cap E \subseteq A \cap J$. Note that $H \cap E$ is a strongly nano $(1,2)^{\star}$-$g^{\star}$-open set. Therefore it follows that $J \cap A$ is a strongly nano $(1,2)^{\star}$- $g^{\star}$-neighborhood of a. Hence $J \cap A \in$ strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a)$.
3. If $J \in$ strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a)$ then there is a strongly nano $(1,2)^{\star}$-$g^{\star}$-open set H such that $a \in H \subseteq J$. Since $A \subseteq J$, A is a strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of a. Hence $A \in$ Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a)$.
4. Let $J \in$ strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a)$ then there exists a strongly nano $(1,2)^{\star}$-$g^{\star}$-open set H, such that $a \in H \subseteq J$. Since H is strongly nano $(1,2)^{\star}$-$g^{\star}$-open set and $a \in H$, H is strongly nano $(1,2)^{\star}$-$g^{\star}$-neighborhood of a. Therefore $H \in$ Strongly $N_{(1,2)^{\star}}$-$g^{\star}N(a)$ and also $H \subseteq J$.

Theorem 3.8. Let U be a nano bitopological space. If G is strongly nano $(1,2)^\star$-g*-closed subset of U and $x \in SN_{(1,2)^\star}G^\star int(G)$ if and only if for any strongly nano $(1,2)^\star$-g*-neighborhood J of x in U, $J \cap G \neq \phi$.

Proof: Let us assume that there is strongly nano $(1,2)^\star$-g*-neighborhood J of the point x in U such that $J \cap G = \phi$. There exists a strongly nano $(1,2)^\star$-g*-open set H of U such that $x \in H \subseteq J$. Therefore we have $H \cap G = \phi$ and so $x \in U - H$. Then $SN_{(1,2)^\star}G^\star CL(G) \in U - H$ and therefore $x \notin SN_{(1,2)^\star}G^\star(G)$, which is a contradiction to the hypothesis that $x \in SN_{(1,2)^\star}G^\star CL(G)$. Therefore $J \cap G \neq \phi$.

Conversely, suppose that $x \notin SN_{(1,2)^\star}G^\star CL(G)$, then there exists a strongly nano $(1,2)^\star$-g*-closed set H of U such that $G \subseteq H$ and $x \notin H$. Thus $x \in U - H$ and $U - H$ is strongly nano $(1,2)^\star$-g*-open in U and hence $U - H$ is a strongly nano $(1,2)^\star$-g*-neighborhood of x in U. But $G \cap (U - H) = \phi$ which is a contradiction. Hence $x \in SN_{(1,2)^\star}G^\star CL(G)$.

Theorem 3.9. If G is a subset of U, then $SN_{(1,2)^\star}G^\star int(G) = \bigcup\{H: H$ is strongly nano $(1,2)^\star$-g*-open, $H \subset G\}$.

Proof: Let G be a subset of U, $x \in SN_{(1,2)^\star}G^\star int(G) \Leftrightarrow$ x is a strongly nano $(1,2)^\star$-g*-interior point of G, G is a strongly nano $(1,2)^\star$-g$^\star N(x)$ which implies that there exists a strongly nano $(1,2)^\star$-g*-open set H such that $x \in H \subset G$, $x \in \bigcup\{H: H$ is strongly nano $(1,2)^\star$-g*-open, $H \subset G\}$. Hence $SN_{(1,2)^\star}G^\star int(G) = \bigcup\{H: H$ is strongly nano $(1,2)^\star$-g*-open, $H \subset G\}$.

4. CONCLUSIONS

In this research, was introduced and studied the concept of strongly nano $(1,2)^\star$-g*-closed sets and their properties in nano bitopological space. Also, we have investigated the group of the structure of the set of all strongly nano $(1,2)^\star$-g*-closed sets in this paper. We investigate some extensions of nano bitopology, which are defined by loosening the constraints of nanobitopology, for a variety of uses, including selecting among various topological concepts and ideas on nano bitopology and obtaining suitable models to address some real-life problems in the medical field. To this end, we have recently defined some strong nano $(1,2)^\star$-g*-closed sets and functions namely strong nano $(1,2)^\star$-g*-closed maps and strong nano $(1,2)^\star$-g*-homeomorphism.

REFERENCES

[1] Thivagar, M. L., Richard, C., *International Journal of Mathematics and Statistics Invention*, **1**(1), 3, 2013.
[2] Thivagar, M. L., Kavitha, J., *Missouri Journal of Mathematical Sciences*, **29**(1), 80, 2017.
[3] Bhuvaneswari, K., Priyadharshini, J. S., *International Research Journal of Mathematics, Engineering and IT (IRJMEIT)*, **3**(3), 1, 2016.
[4] Pawlak, Z., *International Journal of Computer and Information Sciences*, **11**(5), 341, 1982.

[5] Bhuvaneswari, K., Sheela, K., *International Journal of Mathematical Archive*, **7**(10), 109, 2016.
[6] Bhuvaneswari, K., Rasya Banu, H., *International Journal of Scientific Progress and Research*, **126**(43), 68, 2018.
[7] Bhuvaneswari, K., Srividhya, R., *International Journal of Science and Research*, **5**(3), 2230, 2016.
[8] Bhuvaneswari, K., Karpagam, K., *International Journal of Mathematics and its Applications*, **4**(1B), 149, 2016.
[9] Biswas, R., Asokan, R., *International Journal of Mathematics and Computer Research*, **12**(12), 4652, 2024.
[10] Biswas, R., Asokan, R., Savarimuthu, S.J., Thiripuram, A., *Indian Journal Natural Sciences*, **15**(86), 82630, 2024.