ORIGINAL PAPER

ON HOLOMORPHIC AND QUASI-KÄHLER-NORDEN METRICS IN THE TANGENT BUNDLE OF 4-WALKER MANIFOLDS

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Abstract. *Our goal is firstly to investigate holomorphic conditions of the Norden (Kähler-Norden) metrics g^C and resarch Quasi-Kähler-Norden metrics g^C on the tangent bundle $(T(M_4), \varphi^C, g^C)$ of 4-Walker manifolds M_4.*

Keywords: *Norden metrics; Walker 4-manifolds; holomorphic metrics; Quasi-Kähler-Norden metric; complete lifts; symbolic computation.*

1. INTRODUCTION

Work on the lifts of geometric objects on the manifold was started in the 1960s. Vertical and complete lifts of tensor areas were added to the tangent bundle by Yano and Kobayashi [1, 2]. Studies on the Walker manifold have been significantly expanded since 2004. Matsushita [3] has developed almost complex structures suitable for Walker 4-manifolds. Chaichi et al. [4] have explored the curvature properties of 4-dimensional Walker metrics. Davidov et al. investigated almost Kahler-Walker 4-manifolds [5] as well as Hermitian-Walker 4-manifolds [6]. Salimov [7] delved into certain properties of Norden-Walker metrics. These inquiries and findings have predominantly focused on 4-dimensional Walker manifolds.

In this study the structures and metrics on the Norden-Walker 4-manifolds will be transported to the tangent bundle with the help of the complete lifts and more general results will be obtained with the aid of computer programs than the results obtained on. The set $\mathfrak{I}_q^p(M_{2n})$ denotes all tensor fields of type (p,q) on M_{2n}. It is implied that manifolds, tensor fields, and connections are always assumed to be differentiable and belong to the class C^∞.

1.1. NORDEN (ANTI-HERMITIAN) AND HOLOMORHIC NORDEN (KÄHLER-NORDEN) METRICS

A metric g is classified as a Norden metric [8] if it satisfies the condition:

$$g(U,V) = -g(JU,JV)$$

or, equivalently,

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$$g(U,JV)=g(JU,V) \tag{1.1}$$

for any $U,V\in\Im_0^1(M_{2n})$.

A Norden metric g is called a holomorphic if

$$(\varnothing_J g)(U,V,W)=0 \tag{1.2}$$

for any $U,V,W\in\Im_0^1(M_{2n})$.

By setting $U=\partial_k, V=\partial_i$, $W=\partial_j$ in the equation (1.2), we see that the components $(\varnothing_J g)_{kij}$ of $\varnothing_J g$ with respect to the local coordinate system $x^1,\ldots,x^n$ may be expressed as follows:

$$(\varnothing_J g)_{kij}=J_k^m\partial_m g_{ij}-J_i^m\partial_k g_{mj}+g_{mj}(\partial_i J_k^m-\partial_k J_i^m)+g_{im}\partial_j J_k^m.$$

If (M_{2n},J,g) is a Norden manifold with holomorphic Norden metric g, we say that (M_{2n},J,g) is a holomorphic Norden manifold.

1.2. HOLOMORPHIC (ALMOST HOLOMORPHIC) TENSOR FIELDS

The pure tensor fields have been investigated by numerous authors (see, e.g., [2,9, 10, 11, 12, 13]). Specifically, for a $(0,q)$ −tensor field ω, the purity implies that for any $U_1,\ldots,U_q\in\Im_0^1(M_{2n})$, the following conditions must be satisfied:

$$\omega(JU_1,U_2,\ldots,U_q)=\omega(U_1,JU_2,\ldots,U_q)=\ldots=\omega(U_1,U_2,\ldots,JU_q).$$

An operator [2]

$$\varnothing_J:\Im_q^0(M_{2n})\longrightarrow\Im_{q+1}^0(M_{2n})$$

is defined for the pure tensor field ω as follows:

$$\begin{aligned}(\varnothing_J\omega)(U,V_1,V_2,\ldots,V_q)&=(JU)(\omega(V_1,V_2,\ldots,V_q))-U(\omega(JV_1,V_2,\ldots,V_q))\\&+\omega((L_{V_1}J)U,V_2,\ldots,V_q)+\ldots+\omega(V_1,V_2,\ldots,(L_{V_q}J)U),\end{aligned}$$

where L_V the denotes the Lie differentiation with respect to V.

When J represents a complex structure on M_{2n} and the tensor field $\varnothing_J\omega$ vanishes, the complex tensor field $\overset{*}{\omega}$ on $X_n(\mathbf{C})$ is referred to as holomorphic [2, 9, 12]. Consequently, a holomorphic tensor field $\overset{*}{\omega}$ on $X_n(\mathbf{C})$ is manifested on M_{2n} as a pure tensor field ω, satisfying the condition:

$$(\varnothing_J\omega)(U,V_1,V_2,\ldots,V_q)=0$$

for any $U,V_1,\ldots,V_q\in\Im_0^1(M_{2n})$. Hence, such a tensor field ω on M_{2n} is also referred to as a holomorphic tensor field. When J is an almost complex structure on M_{2n}, a tensor field ω satisfying $\varnothing_J\omega=0$ is termed almost holomorphic.

2. MAIN RESULTS

2.1. THE COMPLETE LIFTS OF THE NORDEN-WALKER METRICS ON $T(M_4)$

The expression for the Walker metric is given by:

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \alpha & \theta \\ 0 & 1 & \theta & \beta \end{pmatrix} \tag{2.1}$$

where α, β and θ are functions depending on the coordinate values (x, y, z, t). g^C on $T(M_4)$ with lokal components g_{ij} defined by

$$g^C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \partial\alpha & \partial\theta & 1 & 0 & \alpha & \theta \\ 0 & 0 & \partial\theta & \partial\beta & 0 & 1 & \theta & \beta \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha & \theta & 0 & 0 & 0 & 0 \\ 0 & 1 & \theta & \beta & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

In addition, we get the inverse of the metric tensor g^C defined by (2.2) as follows

$$(g^C)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\alpha & -\theta & 1 & 0 \\ 0 & 0 & 0 & 0 & -\theta & -\beta & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\alpha & -\theta & 1 & 0 & -\partial\alpha & -\partial\theta & 0 & 0 \\ -\theta & -\beta & 0 & 1 & -\partial\theta & -\partial\beta & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.3}$$

where α,β,θ are smooth functions and the complete lifts $\alpha^C, \beta^C, \theta^C$ of α,β,θ have the lokal expressions

$$\alpha^C = y^i \partial_i \alpha = \partial\alpha,$$
$$\beta^C = y^i \partial_i \beta = \partial\beta,$$
$$\theta^C = y^i \partial_i \theta = \partial\theta$$

of the coordinate values (x, y, z, t, k, l, m, n).

2.2. THE COMPLETE LIFTS OF THE PROPER ALMOST COMPLEX STRUCTURE J

Let an almost complex structure on M_4 be Φ, which satisfied:
(i) $\Phi^2 = -I$,
(ii) $\Phi\partial_x = \partial_y$, $\Phi\partial_y = -\partial_x$,($\Phi$ induces a positive $\frac{\pi}{2}$ −rotation on D)

(iii) $g(\Phi U, V) = g(U, \Phi V)$ (Nordenian property),
$(i), (ii), (iii)$ properties define Φ non-uniquely, i.e.

$$\left\{\begin{array}{l} \Phi\partial_y = -\partial_x, \Phi\partial_x = \partial_y, \\ \Phi\partial_z = \omega\partial_x + \frac{1}{2}(\beta+\alpha)\partial_y - \partial_t, \\ \Phi\partial_t = -\frac{1}{2}(\beta+\alpha)\partial_x + \omega\partial_y + \partial_z. \end{array}\right\}$$

In addition, Φ has the local components

$$\Phi = (\Phi_j^i) = \begin{pmatrix} 0 & -1 & \omega & -\frac{1}{2}(\beta+\alpha) \\ 1 & 0 & \frac{1}{2}(\beta+\alpha) & \omega \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

according to the natural frame $\{\partial_x, \partial_y, \partial_z, \partial_t\}$, where an arbitrary function is $\omega = \omega(x, y, z, t)$.

Consequently, we set ω equal to θ. Subsequently, g establishes a singular almost complex structure.

$$J = (J_j^i) = \begin{pmatrix} 0 & -1 & \theta & -\frac{1}{2}(\beta+\alpha) \\ 1 & 0 & \frac{1}{2}(\beta+\alpha) & \theta \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.4}$$

(M_4, J, g) is named almost Norden-Walker manifold.

J^C of the proper almost complex structure J on $4-$manifold M_4 with lokal components J_j^i defined by

$$J^c = \begin{pmatrix} 0 & -1 & \theta & -\frac{1}{2}(\beta+\alpha) & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2}(\beta+\alpha) & \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial\theta & -\frac{1}{2}(\partial\beta+\partial\alpha) & 0 & -1 & \theta & -\frac{1}{2}(\beta+\alpha) \\ 0 & 0 & \frac{1}{2}(\partial\beta+\partial\alpha) & \partial\theta & 1 & 0 & \frac{1}{2}(\beta+\alpha) & \theta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.5}$$

where α,β,θ are smooth functions and the complete lifts $\alpha^C, \beta^C, \theta^C$ of the α,β,θ have the local expressions

$$\alpha^C = y^i\partial_i\alpha = \partial\alpha,$$

$$\beta^C = y^i \partial_i \beta = \partial\beta,$$

$$\theta^C = y^i \partial_i \theta = \partial\theta$$

of the coordinate values (x, y, z, t, k, l, m, n).

Remark 1. From (2.5) we obtain that in the case of $\alpha = -\beta$ and $\theta = 0, J$ is integrable.

2.3. INTEGRABILITY CONDITIONS OF THE COMPLETE LIFTS J^C OF THE PROPER ALMOST COMPLEX STRUCTURE

Theorem 1. Let J be the proper almost complex structure on (M_4, J, g) and J^C be the complete lifts of the proper almost complex structure on $T(M_4)$. Then J^C is integrable if and only if the following PDEs hold:

$$0 = -\theta_t + \theta\theta_x + (\partial\theta)\theta_k + (\partial\theta)_n - \theta(\partial\theta)_k + \frac{1}{2}[(\alpha+\beta)\theta_y + ((\partial\alpha) \\ +(\partial\beta))\theta_l - (\alpha+\beta)(\partial\theta)_l + (\alpha_z + \beta_z) - ((\partial\alpha)_m + (\partial\beta)_m)]$$

$$0 = \theta_z + \theta\theta_y + (\partial\theta)\theta_l - (\partial\theta)_m - \theta(\partial\theta)_l - \frac{1}{2}[(\alpha+\beta)\theta_x - ((\partial\alpha) \\ +(\partial\beta))\theta_k + (\alpha+\beta)(\partial\theta)_k + (\alpha_t + \beta_t) - ((\partial\alpha)_n + (\partial\beta)_n)]$$

$$0 = -\theta_n + \theta(\theta_k) + \frac{1}{2}(\alpha_m + \beta_m + (\alpha+\beta)\theta_l)$$

$$0 = -\theta_m - \theta(\theta_l) - \frac{1}{2}(\alpha_n + \beta_n - (\alpha+\beta)\theta_k)$$

$$(\partial\alpha)_k + (\partial\beta)_k + 2(\partial\theta)_l = 0, (\partial\alpha)_l + (\partial\beta)_l - 2(\partial\theta)_k = 0$$

$$(\partial\alpha)_y + (\partial\beta)_y - 2(\partial\theta)_x = 0, (\partial\alpha)_x + (\partial\beta)_x + 2(\partial\theta)_y = 0 \tag{2.6}$$

$$\alpha_k + \beta_k + 2\theta_l = 0, \alpha_l + \beta_l - 2\theta_k = 0$$

$$\alpha_x + \beta_x + 2\theta_y = 0, \alpha_y + \beta_y - 2\theta_x = 0$$

Thus, we see that if $\alpha = -\beta$ and $\theta = 0$, then J^C is integrable.

Let $(T(M_4), J^C, g^C)$ be a Norden-Walker manifold ($N_{J^C} = 0$) and $\alpha = \beta$. Then the equation (2.6) reduces to

$$\begin{matrix} \alpha_x = -\theta_y \\ \alpha_y = \theta_x \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_k = -\theta_l \\ \alpha_l = \theta_k \end{matrix} \tag{2.7}$$

from which follows

$$\begin{matrix} \alpha_{xx} + \alpha_{yy} = 0 \\ \theta_{xx} + \theta_{yy} = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_{kk} + \alpha_{ll} = 0 \\ \theta_{ll} + \theta_{kk} = 0 \end{matrix}$$

e.g., the functions α and θ are harmonic according to the arguments x, y and k, l.

Theorem 2. If the triple $(T(M_4), J^c, g^c)$ is Norden manifold and $\alpha = \beta$, then α and θ all harmonic with respect to the arguments x, y and k, l.

2.4. EXAMPLE OF NORDEN-WALKER METRIC

Suppose α equals β and let $h(x, y, k, l)$ represent a function of variables x, y, k and l that is harmonic with respect to α, for instance.

$$h(x, y, k, l) = e^x \cos y + e^k \cos l.$$

We define

$$\begin{aligned} \alpha = \partial(x, y, z, t, k, l, m, n) &= h(x, y, k, l) + \omega(z, t, m, n) \\ &= e^x \cos y + e^k \cos l + \omega(z, t, m, n), \end{aligned}$$

where ω represents any smooth functions z, t, m and n.

In that case, α is also harmonic according to x, y, k and l. We obtain

$$\begin{matrix} \alpha_x = e^x \cos y \\ \alpha_y = -e^x \sin y \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_k = e^k \cos l \\ \alpha_l = -e^k \sin l. \end{matrix}$$

From equation (2.7), we derive the partial differential equation (PDE) for θ to fulfill as:

$$\begin{matrix} \theta_x = \alpha_y = -e^x \sin y \\ \theta_y = -\alpha_x = -e^x \cos y \end{matrix} \quad \text{and} \quad \begin{matrix} \theta_k = \alpha_l = -e^k \sin l \\ \theta_l = -\alpha_k = -e^k \cos l. \end{matrix}$$

For these PDE_S, we get a value of θ_1 by solutions of the equation $\theta_y = -\alpha_x = -e^x \cos y$ and a value of θ_2 by solutions of the equation $\theta_l = -\alpha_k = -e^k \cos l$. Thus, we obtain the results

$$\theta_1 = -e^x \sin y + \beta_1(z, t, k, l, m, n)$$

$$\theta_2 = -e^k \sin l + \beta_2(x, y, z, t, m, n),$$

where β_1 and β_2 represent arbitrary smooth functions of z, t, k, l, m, n as well as x, y, z, t, m, n, respectively. Consequently, the full lift of the Norden-Walker metric exhibits components in relation to θ_1 and θ_2, correspondingly.

2.5. HOLOMORPHIC CONDITIONS OF THE COMPLETE LIFTS g^C OF THE NORDEN-WALKER METRICS

If (M_4, J, g) represents an almost Norden-Walker manifold and

$$(\varnothing_J g)_{kij} = J_k^m \partial_m g_{ij} - J_i^m \partial_k g_{mj} + g_{mj}(\partial_i J_k^m - \partial_k J_i^m) + g_{im} \partial_j J_k^m = 0, \tag{2.8}$$

then, as demonstrated in Section 1, J is integrable and the triple (M_4, J, g) is denoted as a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold.

By substituting equations (2.2) and (2.5) into (2.8), we observe the non-vanishing components of $(\varnothing_{J^C} g^C)_{kij}$. Consequently, we establish the following theorem:

Theorem 3. If (M_4, J, g) denotes an almost Norden-Walker manifold, then the triple $(T(M_4), J^C, g^C)$ constitutes a holomorphic Norden (Kähler-Norden) manifold if and only if the following partial differential equations are satisfied:

$$\alpha_x = \alpha_y = \alpha_k = \alpha_l = \alpha_m = \alpha_n = 0$$

$$(\partial\alpha)_x = (\partial\alpha)_y = (\partial\alpha)_k = (\partial\alpha)_l = 0$$

$$\beta_x = \beta_y = \beta_k = \beta_l = \beta_m = \beta_n = 0$$

$$(\partial\beta)_x = (\partial\beta)_y = (\partial\beta)_z = (\partial\beta)_k = (\partial\beta)_l = 0$$

$$\theta_x = \theta_y = \theta_k = \theta_l = \theta_m = \theta_n = 0$$

$$(\partial\theta)_x = (\partial\theta)_y = (\partial\theta)_k = (\partial\theta)_l = 0$$

$$(\partial\alpha)_t - 2(\partial\theta)_z = 0, 2\theta_t + (\partial\beta)_m - 2(\partial\theta)_n = 0$$

$$\beta_z - \theta_t + (\partial\theta)_n = 0, \beta_z - \alpha_z + (\partial\alpha)_m + (\partial\beta)_m = 0$$

$$2\theta_z - (\partial\alpha)_n = 0, \alpha_t - \beta_t - 4\theta_z + (\partial\alpha)_n + (\partial\beta)_n = 0$$

$$\alpha_z + \beta_z - (\partial\alpha)_m = 0, \alpha_t + \beta_t - 2(\partial\theta)_m - (\partial\beta)_n = 0$$

$$\theta_z - \alpha_t + (\partial\theta)_m = 0$$

2.6. QUASI-KÄHLER-NORDEN-WALKER MANIFOLD

A Norden-Walker manifold (M, J, g) meeting the condition $\varnothing_k g_{ij} + 2\nabla_k G_{ij}$ being zero is termed a quasi-Kähler manifold, where G is defined by $G_{ij} = J_i^m g_{mj}$ [10]. Furthermore, for the covariant derivative ∇G of the associated metric G set $(\nabla G)_{ijk} = \nabla_i G_{jk}$ on the Norden-Walker $4-$manifold.

Theorem 4. A triplet $(T(M_4), J^C, g^C)$ forms a quasi-Kähler-Norden manifold if and only if the following partial differential equations are satisfied:

$$\alpha_x = \alpha_y = \alpha_k = \alpha_l = \alpha_m = \alpha_n = 0$$

$$(\partial\alpha)_x = (\partial\alpha)_y = (\partial\alpha)_k = (\partial\alpha)_l = 0$$

$$\beta_x = \beta_y = \beta_k = \beta_l = \beta_m = \beta_n = 0$$

$$(\partial\beta)_x = (\partial\beta)_y = (\partial\beta)_z = (\partial\beta)_k = (\partial\beta)_l = 0$$

$$\theta_x = \theta_y = \theta_k = \theta_l = \theta_m = \theta_n = 0$$

$$(\partial\theta)_x = (\partial\theta)_y = (\partial\theta)_k = (\partial\theta)_l$$

$$3(\partial\theta)_z - 2(\partial\alpha)_t = 0{,}3\theta_z - \alpha_t - (\partial\alpha)_n + (\partial\theta)_m = 0$$

$$2(\partial\theta)_z - (\partial\alpha)_t = 0, (\partial\alpha)_m - \theta_t - \alpha_z + (\partial\theta)_n = 0$$

$$\beta_z - \theta_t + (\partial\theta)_n = 0, \beta_z + \theta_t - (\partial\theta)_n + (\partial\beta)_m = 0$$

$$\theta_z - \alpha_t + (\partial\theta)_m = 0, -\beta_t - \theta_z + (\partial\beta)_n + (\partial\theta)_m = 0$$

3. CONCLUSION

In this study it was investigated holomorphic conditions of the Norden (Kähler-Norden) metrics g^C. Later, studied the Quasi-Kähler-Norden metrics g^C on the tangent bundle $(T(M_4), \varphi^C, g^C)$ of 4-Walker manifolds M_4.

REFERENCES

[1] Kobayashi, S., Nomizu, K., *Foundations of differential geometry II*, Interscience Publishers a division of John Wiley and Sons, New York-London. Sydney, 1969.
[2] Yano, K., Ako, M., *Kodai Math. Sem. Rep.*, **20**(4), 414, 1968.
[3] Matsushita, Y., *Journal of Geometry and Physics*, **52**(1), 88, 2004.
[4] Chaichi, M., Garcia-Rio, E., Matsushita, Y., *Classical and Quantum Gravity,* **22**, 559, 2005.
[5] Davıdov, J., Diaz-Ramos, C., Garcia-Rio, E., Matsushita, Y., Muskarov, O., Vazquez-Lorenzo, R., Journal of Geometry and Physics, **57**(3), 1075-1088, 2007.
[6] Davıdov, J., Diaz-Ramos, C., Garcia-Rio, E., Matsushita, Y., Muskarov, O., Vazquez-Lorenzo, R., *Journal of Geometry and Physics*, **58**(3), 307, 2008.
[7] Salimov, A.A., Iscan, M., *Kodai Mathematical Journal*, **33**(2), 283, 2010.
[8] Norden, A.P., *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, **4**, 145, 1960.
[9] Kruchkovich, G.I., *Trudy Sem. Vektor Tenzor Anals*, *Moscow University*, **16**, 174, 1972.
[10] Salimov, A.A., *Tensor* (N.S), **55**, 142, 1994.
[11] Salimov, A.A., *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, **40**(10), 52, 1996.
[12] Tachibana, S., *Tohoku Mathematical Journal*, **12**(2), 208, 1960.
[13] Vishnevskii, V. V., *Journal of Mathematical Sciences*, **108**(2), 151, 2002.