ORIGINAL PAPER

CONSTRUCTING SOME NEW FRAMES AND ASSOCIATED CURVES USING THE DARBOUX VECTOR FIELDS OF THE FLC-FRAME

BAHAR UYAR DÜLDÜL[1]

Manuscript received: 02.05.2024; Accepted paper: 03.12.2024;
Published online: 30.03.2025.

Abstract. *In this paper, we define some new frames called the osculating Flc-frame, the normal Flc-frame, and the rectifying Flc-frame along a polynomial space curve using the Darboux vector of the Flc-frame and obtain the derivative equations according to these frames. Also, some new integral curves called the* d_{Flc}*-direction curve, the* $\overline{D}_o$*-direction curve, the* $\overline{D}_n$*-direction curve, and the* $\overline{D}_r$*-direction curve of a polynomial space curve are defined. Later, we give some theorems and results related to these curves.*

Keywords: *osculating Flc-frame; normal Flc-frame; rectifying Flc-frame.*

1. INTRODUCTION

In differential geometry, one of the best-known and most used frame fields for characterizing curves is the Frenet frame. The Frenet apparatus characterizes a curve and plays a very important role in determining the shape and the size of the curve. However, there are other frames used to characterize curves and surfaces, such as the Bishop frame (rotation-minimizing frame) and the Darboux frame, etc. The normal curvature, the geodesic curvature, and the geodesic torsion of the surfaces are found using the Darboux frame fields formed by the velocity vector of the curve lying on a surface and the normal vector of the surface. Thus, with the help of the Darboux frame, information is obtained about the character and the shape of the surfaces.

The Bishop frame [1], which can be placed at points where even curvatures vanish, is very suitable for engineering applications [2]. However, in differential geometry, this frame may not be very suitable when it cannot be calculated analytically [3]. Dede [4] recently described a new moving frame called the Flc (Frenet-like curve) frame, which has much easier calculations than the Frenet frame and has a more analytical form that the Bishop frame does not have.

Hananoi et al. [5] considered the Darboux frame $\{T, V, U\}$ along a regular curve γ lying on a smooth surface M where T is the unit tangent vector field of γ, U is the unit normal vector field of M restricted to γ and $V = T \times U$. They defined three vector fields $D_n(s)$, $D_r(s)$ and $D_o(s)$ which are called the normal Darboux vector field, the rectifying Darboux vector field, and the osculating Darboux vector field along γ, respectively:

$$D_n(s) = -\kappa_n(s)V(s) + \kappa_g(s)U(s),$$
$$D_r(s) = \tau_g(s)T(s) + \kappa_g(s)U(s),$$
$$D_o(s) = \tau_g(s)T(s) - \kappa_n(s)V(s),$$

[1] Yildiz Technical University, Faculty of Education, Department of Mathematics and Science Education, 34220 Istanbul, Turkey. E-mail: buduldul@yildiz.edu.tr.

https://doi.org/10.46939/J.Sci.Arts-25.1-a08

where s is the arc-length parameter of γ and κ_n, κ_g, τ_g are the normal curvature, the geodesic curvature, and the geodesic torsion of γ, respectively. Considering the vector fields $D_n(s)$, $D_r(s)$ and $D_o(s)$, Önder [6] defined three special curves on the surface as D_i-Darboux slant helices, where $i \epsilon \{o, n, r\}$.

Many researchers have examined associated curves and tried to reveal the relationships between the main curve and associated curves [7-12]. Among these associated curves, the most studied ones are spherical indicators, involute-evolute curve couple, Bertrand curve couple, Mannheim curve couple, and integral curves. Integral curves are one of the interesting curves among these curves since they are tangent to the vector field at every point.

Recently, Alkan et al. [13] defined three new moving orthogonal frames obtained based on the Darboux frame which are called osculator Darboux frame, normal Darboux frame, and rectifying Darboux frame, respectively. Also, they calculated osculator Darboux frame components and curvatures for a cylinder. Uyar Düldül [14] has studied some new frames using the Darboux vector field of the q-frame and defined some new integral curves of a space curve.

In this study, firstly, some new frames called osculating the Flc-frame, the normal Flc-frame, and the rectifying Flc-frame along a polynomial space curve using the Darboux vector field of the Flc-frame are defined. Then the derivative equations according to these frames are obtained. Also, we define some new integral curves called the d_{Flc}-direction curve, the $\overline{D}_o$-direction curve, the $\overline{D}_n$-direction curve, and the $\overline{D}_r$-direction curve of a polynomial space curve using the Darboux vector field of the Flc-frame and the osculating, normal, and rectifying Flc-frame vector fields. Thereafter, we give some theorems and results related to these curves.

2. PRELIMINARIES

Let $\alpha = \alpha(s)$ be a regular space curve with non-degenerate condition $\alpha'(s) \times \alpha''(s) \neq 0$. Then the Frenet frame vector fields are defined as

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \qquad N(s) = B(s) \times T(s), \quad B(s) = \frac{\alpha'(s) \times \alpha''(s)}{\|\alpha'(s) \times \alpha''(s))\|},$$

where T, N and B denote the tangent, the principal normal and the binormal vector fields of the curve α and the prime denotes the derivative with respect to s. Then the Frenet formulas are given by

$$\begin{cases} T'(s) = \kappa(s)v(s)N(s), \\ N'(s) = -\kappa(s)v(s)T(s) + \tau(s)v(s)B(s), \\ B'(s) = -\tau(s)v(s)N(s), \end{cases}$$

where $v(s) = ||\alpha'(s)||$ and the curvature and the torsion functions are

$$\kappa(s) = \frac{\|\alpha'(s) \times \alpha''(s)\|}{\|\alpha'(s)\|^3}, \qquad \tau(s) = \frac{\langle \alpha'(s) \times \alpha''(s), \alpha'''(s) \rangle}{\|\alpha'(s) \times \alpha''(s)\|^2},$$

respectively.

Let us consider a curve $\alpha:[a,b]\to\mathbb{E}^n$, $\alpha(s)=(\alpha_i(s)), 1\leq i\leq n$. If each coordinate function α_i is polynomial then α is defined to be an n-dimensional polynomial space curve [15]. The degree of $\alpha(s)$ is identified by

$$\deg\alpha(s)=\max\{\deg(\alpha_1(s)),\deg(\alpha_2(s)),\cdots,\deg(\alpha_n(s))\}, [16].$$

The definition of the Frenet-like frame (Flc-frame) of a polynomial space curve of degree n is given by Dede [4] as following

$$T(s)=\frac{\alpha'(s)}{||\alpha'(s)||},\ \ D_1(s)=\frac{\alpha'(s)\times\alpha^{(n)}(s)}{||\alpha'(s)\times\alpha^{(n)}(s)||},\ \ D_2(s)=D_1(s)\times T(s),$$

where the prime $'$ indicates the differentiation with respect to s and $^{(n)}$ stands for the n^{th} derivative. The vectors D_1 and D_2 are called the binormal-like vector and the normal-like vector, respectively. The curvatures of the Flc-frame d_1, d_2 and d_3 are given by

$$d_1=\frac{\langle T',D_2\rangle}{v},\quad d_2=\frac{\langle T',D_1\rangle}{v},\quad d_3=\frac{\langle D_2',D_1\rangle}{v},$$

where $v=||\alpha'||$. The connection between the Frenet frame and the Flc-frame is given by

$$\begin{pmatrix}T\\D_2\\D_1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&\cos\theta&\sin\theta\\0&-\sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}T\\N\\B\end{pmatrix}$$

and the relations between the curvatures of these frames are

$$d_1=\kappa\cos\theta,\quad d_2=-\kappa\sin\theta,\quad d_3=\frac{d\theta}{v}+\tau,$$

where κ and τ are the curvatures of α and θ is the angle between the vectors N and D_2. The local rate of change for the Flc-frame called as the Frenet-like formulas can be expressed as

$$\begin{pmatrix}T'\\D_2'\\D_1'\end{pmatrix}=v\begin{pmatrix}0&d_1&d_2\\-d_1&0&d_3\\-d_2&-d_3&0\end{pmatrix}\begin{pmatrix}T\\D_2\\D_1\end{pmatrix}. \tag{1}$$

The Darboux vector d_{Flc} of the Flc-frame $\{T,D_2,D_1\}$ can be obtained in the following form

$$d_{Flc}=v(d_3T-d_2D_2+d_1D_1), [4].$$

Let us consider the vector fields D_o, D_n and D_r which are called the osculating Flc-vector field, the normal Flc-vector field, and the rectifying Flc-vector field along α, respectively:

$$D_o=v(d_3T-d_2D_2),\ \ D_n=v(-d_2D_2+d_1D_1),\qquad D_r=v(d_3T+d_1D_1).$$

Therefore, the normalized osculating Flc-vector field, the normalized normal Flc-vector field, and the normalized rectifying Flc-vector field along α are, respectively:

$$\overline{D}_o = \frac{d_3}{\sqrt{d_2^2 + d_3^2}} T - \frac{d_2}{\sqrt{d_2^2 + d_3^2}} D_2, \tag{2}$$

$$\overline{D}_n = -\frac{d_2}{\sqrt{d_1^2 + d_2^2}} D_2 + \frac{d_1}{\sqrt{d_1^2 + d_2^2}} D_1, \tag{3}$$

$$\overline{D}_r = \frac{d_3}{\sqrt{d_1^2 + d_3^2}} T + \frac{d_1}{\sqrt{d_1^2 + d_3^2}} D_1. \tag{4}$$

3. SOME NEW FRAMES ALONG A POLYNOMIAL SPACE CURVE

Now, let us define some new frames along a polynomial space curve using the Darboux vector fields of the Flc-frame.

3.1. THE OSCULATING FLC-FRAME

Let $\overline{D}_o$ be the normalized osculating Flc-vector field along the polynomial space curve α. So, from Eq.(2) we have $\overline{D}_o \in Sp\{T, D_2\}$ and $\overline{D}_o \perp D_1$. If we define $E_o = \overline{D}_o \times D_1$, then $\{\overline{D}_o, D_1, E_o\}$ is an orthonormal frame along α. Let us call this frame the osculating Flc-frame and obtain the derivative equations according to this frame.

Since $\overline{D}_o' \in Sp\{\overline{D}_o, D_1, E_o\}$,

$$\overline{D}_o' = a_1 \overline{D}_o + a_2 D_1 + a_3 E_o \tag{5}$$

can be written.

If we take the inner product of both sides of Eq.(5) with $\overline{D}_o$, we get $\langle \overline{D}_o', \overline{D}_o \rangle = a_1$. Since $||\overline{D}_o|| = 1$, we have $a_1 = 0$. So, we get

$$\overline{D}_o' = a_2 D_1 + a_3 E_o. \tag{6}$$

Taking the inner product of both sides of Eq.(6) with D_1 and E_o yield $\langle \overline{D}_o', D_1 \rangle = a_2$ and $\langle \overline{D}_o', E_o \rangle = a_3$. If we take

$$\sin\phi = \frac{d_3}{\sqrt{d_2^2+d_3^2}}, \quad \cos\phi = \frac{d_2}{\sqrt{d_2^2+d_3^2}}, \tag{7}$$

in Eq.(2), then we write

$$\overline{D}_o = \sin\phi T - \cos\phi D_2. \tag{8}$$

Differentiating this equation yields

$$\overline{D}_o' = \phi'\cos\phi T + \sin\phi T' + \phi'\sin\phi D_2 - \cos\phi D_2'.$$

From Eq.(1), we have

$$\overline{D}_o' = (\phi' + vd_1)\cos\phi T + (\phi' + vd_1)\sin\phi D_2 + (vd_2\sin\phi - vd_3\cos\phi)D_1.$$

So, using Eq.(7), we get $a_2 = 0$ and

$$\overline{D}_o' = (\phi' + vd_1)[\cos\phi T + \sin\phi D_2]. \tag{9}$$

Also from Eq.(8), we find

$$E_o = -\cos\phi T - \sin\phi D_2. \tag{10}$$

Additionally, using Eqs.(9) and (10) gives $a_3 = -(\phi' + vd_1)$. By doing some calculations, we have

$$a_3 = -\left[\left(\frac{d_3}{d_2}\right)'\left(\frac{d_2^2}{d_2^2 + d_3^2}\right) + vd_1\right]$$

and from Eq.(6),

$$\overline{D}_o' = -\left[\left(\frac{d_3}{d_2}\right)'\left(\frac{d_2^2}{d_2^2 + d_3^2}\right) + vd_1\right]E_o. \tag{11}$$

Similarly, since $D_1' \in Sp\{\overline{D}_o, D_1, E_o\}$,

$$D_1' = b_1\overline{D}_o + b_2D_1 + b_3E_o \tag{12}$$

can be written.

If we take the inner product of both sides of Eq.(12) with $\overline{D}_o$, we obtain $\langle D_1', \overline{D}_o\rangle = b_1$. Using Eqs. (1), (7) and (8) gives $b_1 = 0$. Since $||D_1|| = 1$, we have $b_2 = 0$. Then, we get

$$D_1' = b_3E_o. \tag{13}$$

Taking the inner product of both sides of Eq.(13) with E_o yields $\langle D_1', E_o\rangle = b_3$. If we use Eqs.(1), (7) and (10), we obtain $b_3 = v\sqrt{d_2^2 + d_3^2}$ and

$$D_1' = v\sqrt{d_2^2 + d_3^2}\,E_o. \tag{14}$$

Furthermore, since $E_o' \in Sp\{\overline{D}_o, D_1, E_o\}$,

$$E_o' = c_1\overline{D}_o + c_2D_1 + c_3E_o \tag{15}$$

can be written.

If we take the inner product of both hand sides of Eq.(15) with $\overline{D}_o$, we have $\langle E_o', \overline{D}_o\rangle = c_1$. From Eqs. (1) and (10), we obtain

$$E_o' = (\phi' + vd_1)\sin\phi T - (\phi' + vd_1)\cos\phi D_2 - (vd_2\cos\phi + vd_3\sin\phi)D_1. \tag{16}$$

Using Eqs.(8) and (16) gives

$$c_1 = \left(\frac{d_3}{d_2}\right)' \left(\frac{d_2^2}{d_2^2 + d_3^2}\right) + vd_1.$$

From Eqs. (15) and (16), we find $c_2 = -v\sqrt{d_2^2 + d_3^2}$. Also, since $||E_o|| = 1$, we have $c_3 = 0$. Then from Eq.(15), we obtain

$$E_o' = \left[\left(\frac{d_3}{d_2}\right)' \left(\frac{d_2^2}{d_2^2 + d_3^2}\right) + vd_1\right]\overline{D}_o - v\sqrt{d_2^2 + d_3^2}\, D_1. \tag{17}$$

Thus, from Eqs. (11), (14), and (17), if we denote

$$\rho_o = \left(\frac{d_3}{d_2}\right)' \left(\frac{d_2^2}{d_2^2+d_3^2}\right) + vd_1 \quad \text{and} \quad \eta_o = v\sqrt{d_2^2 + d_3^2}\,,$$

we obtain

$$\begin{pmatrix} \overline{D}_o' \\ D_1' \\ E_o' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\rho_o \\ 0 & 0 & \eta_o \\ \rho_o & -\eta_o & 0 \end{pmatrix} \begin{pmatrix} \overline{D}_o \\ D_1 \\ E_o \end{pmatrix}. \tag{18}$$

We called ρ_o and η_o as the curvatures of α according to the osculating Flc-frame.

Definition 3.1. Let $\{\overline{D}_o, D_1, E_o\}$ be the osculating Flc-frame along the polynomial space curve α. The curve α is called a D_1-slant helix relative to the osculating Flc-frame if the vector field D_1 makes a constant angle with a fixed direction, i.e. $\langle D_1, U\rangle = \cos\psi$, where U is a constant unit vector and ψ is a constant angle.

Theorem 3.2. Let $\{\overline{D}_o, D_1, E_o\}$ be the osculating Flc-frame along the polynomial space curve α. The curve α is a D_1-slant helix relative to the osculating Flc-frame if and only if the function $\frac{\eta_o}{\rho_o}$ is constant (for $\eta_o \neq 0$ and $\rho_o \neq 0$).

Proof: Let α be a D_1-slant helix relative to the osculating Flc-frame. Then $\langle D_1, U\rangle = \cos\psi = c \neq 0$, where U is a unit of constant direction. So, we can write

$$U = \lambda_1\overline{D}_o + cD_1 + \lambda_2 E_o, \qquad (\lambda_i \in \mathbb{R}, i: 1,2). \tag{19}$$

Differentiating Eq. (19) yields

$$U' = \lambda_1\overline{D}_o' + \lambda_1'\overline{D}_o + cD_1' + \lambda_2' E_o + \lambda_2 E_o'$$

and using Eq. (18) gives

$$\begin{cases} \lambda_1' + \lambda_2\rho_o = 0, \\ \lambda_2\eta_o = 0, \\ \lambda_2' - \lambda_1\rho_o + c\eta_o = 0. \end{cases}$$

Since $\eta_o \neq 0$ and $\rho_o \neq 0$, we get $\lambda_2 = 0$ and $\lambda_1 = \text{constant}$. Then, we have $\frac{\eta_o}{\rho_o} = \text{constant}$. Conversely, let $\frac{\eta_o}{\rho_o}$ be constant. Choosing $\frac{\eta_o}{\rho_o} = \frac{\cos\psi}{\sin\psi}$ and taking $U = \cos\psi\overline{D}_o + \sin\psi D_1$, we get $U' = 0$ with the help of Eq.(18). So, the vector U is constant. Also, by the inner product of both sides of the equality $U = \cos\psi\overline{D}_o + \sin\psi D_1$ with D_1, we get $\langle U, D_1\rangle = \sin\psi$. Then, the constant vector U and the vector D_1 make a constant angle, i.e. the curve α is a D_1-slant helix.

Corollary 3.3. α is a D_1-slant helix $\Leftrightarrow$ $\frac{(d_3'd_2 - d_2'd_3) + vd_1(d_2^2 + d_3^2)}{v(d_2^2 + d_3^2)^{3/2}} = \text{constant}.$

3.2. THE NORMAL FLC-FRAME

Let $\overline{D}_n$ be the normalized normal Flc-vector field along the polynomial space curve α. Then, from Eq.(3) we have $\overline{D}_n \in Sp\{D_2, D_1\}$ and $\overline{D}_n \perp T$. If we define $E_n = \overline{D}_n \times T$, then $\{\overline{D}_n, T, E_n\}$ is an orthonormal frame along the curve α. Let us call this frame the normal Flc-frame and obtain the derivative equations according to this frame.

Since $\overline{D}_n' \in Sp\{\overline{D}_n, T, E_n\}$, we can write

$$\overline{D}_n' = a_1\overline{D}_n + a_2T + a_3E_n. \tag{20}$$

Taking the inner product of both sides of Eq.(20) with $\overline{D}_n$ gives $\langle \overline{D}_n', \overline{D}_n\rangle = a_1$. Since $||\overline{D}_n|| = 1$, we get $a_1 = 0$. Thus, we can write

$$\overline{D}_n' = a_2T + a_3E_n. \tag{21}$$

If we take the inner product of both sides of Eq.(21) with T and E_n, we obtain $\langle \overline{D}_n', T\rangle = a_2$ and $\langle \overline{D}_n', E_n\rangle = a_3$. Taking

$$\sin\Omega = \frac{d_2}{\sqrt{d_1^2 + d_2^2}}, \quad \cos\Omega = \frac{d_1}{\sqrt{d_1^2 + d_2^2}} \tag{22}$$

in Eq.(3) gives

$$\overline{D}_n = -\sin\Omega D_2 + \cos\Omega D_1. \tag{23}$$

If we differentiate this equation, we get

$$\overline{D}_n' = -\Omega'\cos\Omega D_2 - \sin\Omega D_2' - \Omega'\sin\Omega D_1 + \cos\Omega D_1'.$$

From Eq.(1), we have

$$\overline{D}'_n = (vd_1\sin\Omega - vd_2\cos\Omega)T - (\Omega' + vd_3)\cos\Omega D_2 - (\Omega' + vd_3)\sin\Omega D_1.$$

Thus, using Eq.(22), we find $a_2 = 0$ and we get

$$\overline{D}'_n = -(\Omega' + vd_3)[\cos\Omega D_2 + \sin\Omega D_1]. \quad (24)$$

Moreover using Eq.(23) gives

$$E_n = \cos\Omega D_2 + \sin\Omega D_1. \quad (25)$$

Further, from Eqs.(24) and (25), we get $a_3 = -(\Omega' + vd_3)$. By doing some calculations, we obtain

$$a_3 = -\left[\left(\frac{d_2}{d_1}\right)'\left(\frac{d_1^2}{d_1^2 + d_2^2}\right) + vd_3\right]$$

and from Eq.(21), we find

$$\overline{D}'_n = -\left[\left(\frac{d_2}{d_1}\right)'\left(\frac{d_1^2}{d_1^2 + d_2^2}\right) + vd_3\right]E_n. \quad (26)$$

Similarly, since $T' \in Sp\{\overline{D}_n, T, E_n\}$, we can write

$$T' = b_1\overline{D}_n + b_2T + b_3E_n. \quad (27)$$

Taking the inner product of both sides of Eq.(27) with $\overline{D}_n$ gives $\langle T', \overline{D}_n\rangle = b_1$. From Eqs.(1), (22) and (23), we obtain $b_1 = 0$. Since $||T|| = 1$, we get $b_2 = 0$. Thus, we have

$$T' = b_3E_n. \quad (28)$$

If we take the inner product of both sides of Eq.(28) with E_n and use Eqs. (1), (22) and (25), we obtain $b_3 = v\sqrt{d_1^2 + d_2^2}$. So, we get

$$T' = v\sqrt{d_1^2 + d_2^2}\, E_n. \quad (29)$$

Besides, since $E'_n \in Sp\{\overline{D}_n, T, E_n\}$,

$$E'_n = c_1\overline{D}_n + c_2T + c_3E_n \quad (30)$$

can be written.

Taking the inner product of both sides of Eq.(30) with $\overline{D}_n$ gives $\langle E'_n, \overline{D}_n\rangle = c_1$. Using Eqs. (1) and (25) gives

$$E'_n = -(vd_2\sin\Omega + vd_1\cos\Omega)T - (\Omega' + vd_3)\sin\Omega D_2 + (\Omega' + vd_3)\cos\Omega D_1. \quad (31)$$

From Eqs. (23) and (31), we obtain

$$c_1 = \left(\frac{d_2}{d_1}\right)' \left(\frac{d_1^2}{d_1^2 + d_2^2}\right) + v d_3.$$

Using Eqs. (22), (30), and (31) yields $c_2 = -v\sqrt{d_1^2 + d_2^2}$. Also, since $||E_n|| = 1$, we get $c_3 = 0$. Thus using Eq. (30) gives

$$E_n' = \left[\left(\frac{d_2}{d_1}\right)' \left(\frac{d_1^2}{d_1^2 + d_2^2}\right) + v d_3\right]\overline{D}_n - v\sqrt{d_1^2 + d_2^2}\, T. \tag{32}$$

Denoting

$$\rho_n = \left(\frac{d_2}{d_1}\right)' \left(\frac{d_1^2}{d_1^2+d_2^2}\right) + v d_3, \qquad \eta_n = v\sqrt{d_1^2 + d_2^2}$$

and using Eqs. (26), (29) and (32) yields

$$\begin{pmatrix} \overline{D}_n' \\ T' \\ E_n' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\rho_n \\ 0 & 0 & \eta_n \\ \rho_n & -\eta_n & 0 \end{pmatrix} \begin{pmatrix} \overline{D}_n \\ T \\ E_n \end{pmatrix}. \tag{33}$$

Here, ρ_n and η_n are called the curvatures of α according to the normal Flc-frame.

Theorem 3.4. Let $\{\overline{D}_n, T, E_n\}$ be the normal Flc-frame along the polynomial space curve α. The curve α is a helix relative to the normal Flc-frame if and only if the function $\frac{\eta_n}{\rho_n}$ is constant, for $\eta_n \neq 0$ and $\rho_n \neq 0$.

Proof: Let α be a helix relative to the normal Flc-frame. Then $\langle T, U\rangle = \cos\psi = c \neq 0$, where U is a unit constant direction. So,

$$U = \lambda_1 \overline{D}_n + cT + \lambda_2 E_n, \qquad (\lambda_i \in \mathbb{R}, i: 1,2) \tag{34}$$

can be written.

If we differentiate Eq. (34), we get

$$U' = \lambda_1 \overline{D}_n' + \lambda_1' \overline{D}_n + cT' + \lambda_2' E_n + \lambda_2 E_n'$$

and we use Eq. (33), we find

$$\begin{cases} \lambda_1' + \lambda_2 \rho_n = 0, \\ \lambda_2 \eta_n = 0, \\ \lambda_2' - \lambda_1 \rho_n + c\eta_n = 0. \end{cases}$$

Since $\eta_n \neq 0$ and $\rho_n \neq 0$, we have $\lambda_2 = 0$ and $\lambda_1 = \text{constant}$. So, we get $\frac{\eta_n}{\rho_n} = \text{constant}$. Conversely, let $\frac{\eta_n}{\rho_n}$ be constant. If we choose $\frac{\eta_n}{\rho_n} = \frac{\cos\psi}{\sin\psi}$ and take $U = \cos\psi \overline{D}_n + \sin\psi T$, we find $U' = 0$ by using Eq.(33). Then, the vector U is constant. In addition, having the inner product of both sides of the equality $U = \cos\psi \overline{D}_n + \sin\psi T$ with T, we get $\langle U, T\rangle =$

$\sin\psi$. Consequently, the constant vector U and the vector T make a constant angle, i.e. the curve α is a helix.

Corollary 3.5. α is a helix $\Leftrightarrow$ $\frac{(d_2'd_1-d_1'd_2)+vd_3(d_1^2+d_2^2)}{v(d_1^2+d_2^2)^{3/2}} = \text{constant}.$

3.3. THE RECTIFYING FLC-FRAME

Let $\overline{D}_r$ be the normalized rectifying Flc-vector field along the polynomial space curve α. From Eq.(4), $\overline{D}_r \in \mathrm{Sp}\{T, D_1\}$ and $\overline{D}_r \perp D_2$. If we define $E_r = \overline{D}_r \times D_2$, then $\{\overline{D}_r, D_2, E_r\}$ is an orthonormal frame along the curve α. Let us call this frame the rectifying Flc-frame and obtain the derivative equations according to this frame.

Since $\overline{D}_r' \in \mathrm{Sp}\{\overline{D}_r, D_2, E_r\}$,

$$\overline{D}_r' = a_1\overline{D}_r + a_2D_2 + a_3E_r \tag{35}$$

can be written. If we take the inner product of both sides of Eq. (35) with $\overline{D}_r$, we get $\langle\overline{D}_r', \overline{D}_r\rangle = a_1$. Since $||\overline{D}_r|| = 1$, we have $a_1 = 0$. Therefore,

$$\overline{D}_r' = a_2D_2 + a_3E_r. \tag{36}$$

Taking the inner product of both sides of Eq. (36) with D_2 and E_r give $\langle\overline{D}_r', D_2\rangle = a_2$ and $\langle\overline{D}_r', E_r\rangle = a_3$. If we denote

$$\sin\Theta = \frac{d_3}{\sqrt{d_1^2 + d_3^2}}, \quad \cos\Theta = \frac{d_1}{\sqrt{d_1^2 + d_3^2}}, \tag{37}$$

in Eq. (4), then we have

$$\overline{D}_r = \sin\Theta T + \cos\Theta D_1. \tag{38}$$

Differentiating this equation yields

$$\overline{D}_r' = \Theta'\cos\Theta T + \sin\Theta T' - \Theta'\sin\Theta D_1 + \cos\Theta D_1'.$$

From Eq. (1), we get

$$\overline{D}_r' = (\Theta' - vd_2)\cos\Theta T + (vd_1\sin\Theta - vd_3\cos\Theta)D_2 - (\Theta' - vd_2)\sin\Theta D_1.$$

Thus, using Eq. (37), we have $a_2 = 0$. So, we get

$$\overline{D}_r' = (\Theta' - vd_2)[\cos\Theta T - \sin\Theta D_1]. \tag{39}$$

Besides from Eq. (38), we find

$$E_r = -\cos\Theta T + \sin\Theta D_1. \tag{40}$$

Additionally, using Eqs. (39) and (40) gives $a_3 = -(\Theta' - vd_2)$. By doing some calculations, we obtain

$$a_3 = -\left[\left(\frac{d_3}{d_1}\right)'\left(\frac{d_1^2}{d_1^2 + d_3^2}\right) - vd_2\right]$$

and from Eq. (36), we have

$$\overline{D}_r' = -\left[\left(\frac{d_3}{d_1}\right)'\left(\frac{d_1^2}{d_1^2 + d_3^2}\right) - vd_2\right]E_r. \tag{41}$$

Similarly, since $D_2' \in Sp\{\overline{D}_r, D_2, E_r\}$,

$$D_2' = b_1\overline{D}_r + b_2 D_2 + b_3 E_r \tag{42}$$

can be written. If we take the inner product of both sides of Eq.(42) with $\overline{D}_r$, we obtain $\langle D_2', \overline{D}_r\rangle = b_1$. Using Eqs. (1), (37) and (38) gives $b_1 = 0$. Since $||D_2|| = 1$, we obtain $b_2 = 0$. Then, we get

$$D_2' = b_3 E_r. \tag{43}$$

Taking the inner product of both sides of Eq.(43) with E_r yields $\langle D_2', E_r\rangle = b_3$. If we use Eqs. (1), (37), and (40), we obtain $b_3 = v\sqrt{d_1^2 + d_3^2}$ and

$$D_2' = v\sqrt{d_1^2 + d_3^2}\,E_r. \tag{44}$$

Furthermore, since $E_r' \in Sp\{\overline{D}_r, D_2, E_r\}$,

$$E_r' = c_1\overline{D}_r + c_2 D_2 + c_3 E_r \tag{45}$$

can be written. If we take the inner product of both hand sides of Eq.(45) with $\overline{D}_r$, we have $\langle E_r', \overline{D}_r\rangle = c_1$. From Eqs. (1) and (40), we get

$$E_r' = (\Theta' - vd_2)\sin\Theta T - (vd_3\sin\Theta + vd_1\cos\Theta)D_2 + (\Theta' - vd_2)\cos\Theta D_1. \tag{46}$$

Using Eqs.(38) and (46) gives

$$c_1 = \left(\frac{d_3}{d_1}\right)'\left(\frac{d_1^2}{d_1^2 + d_3^2}\right) - vd_2.$$

From Eqs. (37), (45), and (46), we find $c_2 = -v\sqrt{d_1^2 + d_3^2}$. Also, since $||E_r|| = 1$, we have $c_3 = 0$. Then From Eq.(45), we obtain

$$E_r' = \left[\left(\frac{d_3}{d_1}\right)'\left(\frac{d_1^2}{d_1^2 + d_3^2}\right) - vd_2\right]\overline{D}_r - v\sqrt{d_1^2 + d_3^2}\,D_2. \tag{47}$$

Thus from Eqs.(41), (44) and (47), if we denote

$$\rho_r = \left(\frac{d_3}{d_1}\right)' \left(\frac{d_1^2}{d_1^2+d_3^2}\right) - vd_2 \quad \text{and} \quad \eta_r = v\sqrt{d_1^2 + d_3^2},$$

we get

$$\begin{pmatrix} \overline{D}_r' \\ D_2' \\ E_r' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\rho_r \\ 0 & 0 & \eta_r \\ \rho_r & -\eta_r & 0 \end{pmatrix} \begin{pmatrix} \overline{D}_r \\ D_2 \\ E_r \end{pmatrix}.$$

Here, ρ_r and η_r are the curvatures of α according to the rectifying Flc-frame.

Definition 3.6. Let $\{\overline{D}_r, D_2, E_r\}$ be the rectifying Flc-frame along the polynomial space curve α. The curve α is called a D_2-slant helix relative to the rectifying Flc-frame if the vector field D_2 makes a constant angle with a fixed direction, i.e. $\langle D_2, U\rangle = \cos\psi$, where U is a constant unit vector and ψ is a constant angle.

Theorem 3.7. Let $\{\overline{D}_r, D_2, E_r\}$ be the rectifying Flc-frame along the polynomial space curve α. The curve α is D_2-slant helix relative to the rectifying Flc-frame if and only if the function $\frac{\eta_r}{\rho_r}$ is constant (for $\eta_r \neq 0$ and $\rho_r \neq 0$).

Proof: The proof of the theorem can be done in a similar way to the proof of Theorem 3.2. and Theorem 3.4.

Corollary 3.8. α is a D_2-slant helix $\Leftrightarrow$ $\frac{(d_3' d_1 - d_1' d_3) - vd_2(d_1^2+d_3^2)}{v(d_1^2+d_3^2)^{3/2}} = \text{constant}.$

4. SOME ASSOCIATED CURVES WITH THE DARBOUX FLC-VECTOR FIELDS

Now, let us define some new associated curves with a polynomial space curve using the Darboux vector field of the Flc-frame and the osculating, normal, and rectifying Flc-frame vector fields.

4.1. d_{Flc}-DIRECTION CURVE

Definition 4.1. Let α be a polynomial space curve in $\mathbb{E}^3$, $\{T, D_2, D_1\}$ be the Flc-frame along α and d_{Flc} be the Flc-Darboux vector field of α. The integral curve of the vector field d_{Flc} is called d_{Flc}-direction curve of α. Namely, if $\beta(s)$ is the d_{Flc}-direction curve of α, then $d_{Flc}(s) = \beta'(s)$, where $d_{Flc} = v(d_3T - d_2D_2 + d_1D_1)$.

Theorem 4.2. Let β be the d_{Flc}-direction curve of a polynomial space curve α. Then the Frenet vectors $\{T_\beta, N_\beta, B_\beta\}$ and the curvatures κ_β and τ_β of β can be obtained as

$$T_\beta = \frac{1}{\sqrt{d_1^2 + d_2^2 + d_3^2}}(d_3T - d_2D_2 + d_1D_1),$$

$$N_\beta = \frac{1}{\sqrt{d_1^2 + d_2^2 + d_3^2}\,\Gamma}\{[(d_1^2 + d_2^2)d_3' - d_3(d_1 d_1' + d_2 d_2')]T$$
$$-[(d_1^2 + d_3^2)d_2' - d_2(d_1 d_1' + d_3 d_3')]D_2$$
$$+[(d_2^2 + d_3^2)d_1' - d_1(d_2 d_2' + d_3 d_3')]D_1\},$$

$$B_\beta = \frac{1}{\Gamma}[(d_1 d_2' - d_2 d_1')T + (d_1 d_3' - d_3 d_1')D_2 + (d_2 d_3' - d_3 d_2')D_1],$$

$$\kappa_\beta = \frac{\Gamma}{v(d_1^2 + d_2^2 + d_3^2)^{3/2}},$$

$$\tau_\beta = \frac{\Delta}{v^2\Gamma^2},$$

where

$$\Gamma = \sqrt{(d_1 d_2' - d_2 d_1')^2 + (d_1 d_3' - d_3 d_1')^2 + (d_2 d_3' - d_3 d_2')^2},$$

and

$$\Delta = \sum_{i=1}^{3} [v'' d_k + 2v' d_k' + v d_k'' + v^2(d_i d_j' - d_j d_i')](d_i d_j' - d_j d_i'), \quad i, j, k = 1,2,3 \text{ (cyclic)}.$$

Proof: Since β is the d_{Flc}-direction curve of α, we can write $\beta' = d_{Flc}$ and we get the tangent vector T_β of β as

$$T_\beta = \frac{\beta'}{||\beta'||} = \frac{1}{\sqrt{d_1^2 + d_2^2 + d_3^2}}(d_3 T - d_2 D_2 + d_1 D_1).$$

Also, we have

$$\beta'' = (v d_3)' T - (v d_2)' D_2 + (v d_1)' D_1.$$

So, we find

$$\beta' \times \beta'' = v^2[(d_1 d_2' - d_2 d_1')T + (d_1 d_3' - d_3 d_1')D_2 + (d_2 d_3' - d_3 d_2')D_1]$$

and

$$||\beta' \times \beta''|| = v^2\Gamma,$$

where

$$\Gamma = \sqrt{(d_1 d_2' - d_2 d_1')^2 + (d_1 d_3' - d_3 d_1')^2 + (d_2 d_3' - d_3 d_2')^2}.$$

Then, we obtain the binormal vector B_β of β as

$$B_\beta = \frac{\beta' \times \beta''}{||\beta' \times \beta''||} = \frac{1}{\Gamma}[(d_1 d_2' - d_2 d_1')T + (d_1 d_3' - d_3 d_1')D_2 + (d_2 d_3' - d_3 d_2')D_1].$$

Moreover, we find the normal vector N_β of β as

$$\mathrm{N}_\beta = \mathrm{B}_\beta \times \mathrm{T}_\beta = \frac{1}{\sqrt{d_1^2+d_2^2+d_3^2}\,\Gamma}\{[(d_1^2+d_2^2)d_3' - d_3(d_1d_1'+d_2d_2')]\mathrm{T} \\ -[(d_1^2+d_3^2)d_2' - d_2(d_1d_1'+d_3d_3')]\mathrm{D}_2 \\ +[(d_2^2+d_3^2)d_1' - d_1(d_2d_2'+d_3d_3')]\mathrm{D}_1\}.$$

Also, the first curvature κ_β of β can be obtained as

$$\kappa_\beta = \frac{||\beta'\times\beta''||}{||\beta'||^3} = \frac{\Gamma}{v(d_1^2+d_2^2+d_3^2)^{3/2}}.$$

Besides, we get

$$\beta''' = [v''d_3 + 2v'd_3' + vd_3'' + v^2(d_1d_2' - d_2d_1')]\mathrm{T} \\ -[v''d_2 + 2v'd_2' + vd_2'' + v^2(d_3d_1' - d_1d_3')]\mathrm{D}_2 \\ +[v''d_1 + 2v'd_1' + vd_1'' + v^2(d_2d_3' - d_3d_2')]\mathrm{D}_1.$$

Moreover, we have

$$\langle\beta'\times\beta'',\beta'''\rangle = v^2\{[v''d_3 + 2v'd_3' + vd_3'' + v^2(d_1d_2' - d_2d_1')](d_1d_2' - d_2d_1') \\ +[v''d_2 + 2v'd_2' + vd_2'' + v^2(d_3d_1' - d_1d_3')](d_3d_1' - d_1d_3') \\ +[v''d_1 + 2v'd_1' + vd_1'' + v^2(d_2d_3' - d_3d_2')](d_2d_3' - d_3d_2')\}.$$

So, the second curvature τ_β of β can be found as

$$\tau_\beta = \frac{\langle\beta'\times\beta'',\beta'''\rangle}{||\beta'\times\beta''||^2} = \frac{\Delta}{v^2\Gamma^2},$$

where

$$\Delta = \sum_{i=1}^{3}[v''d_k + 2v'd_k' + vd_k'' + v^2(d_id_j' - d_jd_i')](d_id_j' - d_jd_i'),\quad i,j,k = 1,2,3 \text{ (cyclic)}.$$

4.2. $\overline{D}_o$-DIRECTION CURVE

Definition 4.3. Let α be a polynomial space curve in $\mathbb{E}^3$, $\{\mathrm{T},\mathrm{D}_2,\mathrm{D}_1\}$ be the Flc-frame along α and $\overline{\mathrm{D}}_o$ be the unit osculating Flc-frame vector field of α. The integral curve of the vector field $\overline{\mathrm{D}}_o$ is called $\overline{\mathrm{D}}_o$-direction curve of α. Namely, if $\gamma(s)$ is the $\overline{\mathrm{D}}_o$-direction curve of α, then $\overline{\mathrm{D}}_o(s) = \gamma'(s)$.

Theorem 4.4. Let γ be the $\overline{\mathrm{D}}_o$-direction curve of a polynomial space curve α. Then the Frenet vectors $\{\mathrm{T}_\gamma,\mathrm{N}_\gamma,\mathrm{B}_\gamma\}$ and the curvatures κ_γ and τ_γ of γ can be found as

$$\mathrm{T}_\gamma = \overline{\mathrm{D}}_o,\qquad \mathrm{N}_\gamma = -\varepsilon\frac{\mathrm{D}_1'}{||\mathrm{D}_1'||},\qquad \mathrm{B}_\gamma = \varepsilon\mathrm{D}_1,\qquad \kappa_\gamma = \varepsilon\rho_o,\qquad \tau_\gamma = \eta_o,$$

where $\varepsilon = \pm 1$.

Proof: Since γ is the $\overline{\mathrm{D}}_\mathrm{o}$-direction curve of α, we can write $\gamma' = \overline{\mathrm{D}}_\mathrm{o}$ and we obtain the tangent vector T_γ of γ as

$$\mathrm{T}_\gamma = \overline{\mathrm{D}}_\mathrm{o} = \frac{1}{\sqrt{\mathrm{d}_2^2 + \mathrm{d}_3^2}}(\mathrm{d}_3\mathrm{T} - \mathrm{d}_2\mathrm{D}_2).$$

Differentiating this equation with respect to s and using Eq.(1), we find

$$\overline{\mathrm{D}}_\mathrm{o}' = \frac{(\mathrm{d}_3'\mathrm{d}_2 - \mathrm{d}_2'\mathrm{d}_3) + \mathrm{v}\mathrm{d}_1(\mathrm{d}_2^2 + \mathrm{d}_3^2)}{(\mathrm{d}_2^2 + \mathrm{d}_3^2)^{3/2}}(\mathrm{d}_2\mathrm{T} + \mathrm{d}_3\mathrm{D}_2). \tag{48}$$

Since

$$\rho_\mathrm{o} = \frac{(\mathrm{d}_3'\mathrm{d}_2 - \mathrm{d}_2'\mathrm{d}_3) + \mathrm{v}\mathrm{d}_1(\mathrm{d}_2^2 + \mathrm{d}_3^2)}{\mathrm{d}_2^2 + \mathrm{d}_3^2},$$

from Eq.(48), we obtain

$$\overline{\mathrm{D}}_\mathrm{o}' = -\rho_\mathrm{o}\frac{\mathrm{D}_1'}{||\mathrm{D}_1'||}.$$

So, we get $||\overline{\mathrm{D}}_\mathrm{o}'|| = \varepsilon\rho_\mathrm{o}$, where $\varepsilon = \pm 1$. Thus, since $\mathrm{N}_\gamma = \frac{\overline{\mathrm{D}}_\mathrm{o}'}{||\overline{\mathrm{D}}_\mathrm{o}'||}$, we have

$$\mathrm{N}_\gamma = -\varepsilon\frac{\mathrm{D}_1'}{||\mathrm{D}_1'||}.$$

Moreover, since $\mathrm{B}_\gamma = \mathrm{T}_\gamma \times \mathrm{N}_\gamma$, we obtain the binormal vector as $\mathrm{B}_\gamma = \varepsilon\mathrm{D}_1$. Besides, for the curvatures of γ, we get

$$\kappa_\gamma = ||\mathrm{T}_\gamma'|| = \varepsilon\rho_\mathrm{o}, \qquad \tau_\gamma = -\langle \mathrm{B}_\gamma', \mathrm{N}_\gamma\rangle = \eta_\mathrm{o}.$$

Corollary 4.5. γ is a general helix if and only if α is a D_1-slant helix.

Proof: The proof can be seen by using Corollary 3.3.

4.3. $\overline{D}_n$-DIRECTION CURVE

Definition 4.6. Let α be a polynomial space curve in $\mathbb{E}^3$, $\{\mathrm{T}, \mathrm{D}_2, \mathrm{D}_1\}$ be the Flc-frame along α and $\overline{\mathrm{D}}_\mathrm{n}$ be the unit normal Flc-frame vector field of α. The integral curve of the vector field $\overline{\mathrm{D}}_\mathrm{n}$ is called $\overline{\mathrm{D}}_\mathrm{n}$-direction curve of α. Namely, if $\zeta(\mathrm{s})$ is the $\overline{\mathrm{D}}_\mathrm{n}$-direction curve of α, then $\overline{\mathrm{D}}_\mathrm{n}(\mathrm{s}) = \zeta'(\mathrm{s})$.

Theorem 4.7. Let ζ be the $\overline{\mathrm{D}}_\mathrm{n}$-direction curve of a polynomial space curve α. Then the Frenet vectors $\{\mathrm{T}_\zeta, \mathrm{N}_\zeta, \mathrm{B}_\zeta\}$ and the curvatures κ_ζ and τ_ζ of ζ can be found as

$$T_\zeta = \overline{D}_n, \qquad N_\zeta = -\varepsilon \frac{T'}{||T'||}, \qquad B_\zeta = \varepsilon T, \qquad \kappa_\zeta = \varepsilon \rho_n, \qquad \tau_\zeta = \eta_n,$$

where $\varepsilon = \pm 1$.

Proof: From the definition of $\overline{D}_n$-direction curve, we can write

$$T_\zeta = \overline{D}_n = \frac{1}{\sqrt{d_1^2 + d_2^2}}(-d_2 D_2 + d_1 D_1).$$

If we differentiate this equation with respect to s and applying Eq.(1), we obtain

$$\overline{D}_n' = \frac{(d_1' d_2 - d_2' d_1) - v d_3 (d_1^2 + d_2^2)}{(d_1^2 + d_2^2)^{3/2}}(d_1 D_2 + d_2 D_1). \tag{49}$$

Since

$$\rho_n = \frac{(d_2' d_1 - d_1' d_2) + v d_3 (d_1^2 + d_2^2)}{d_1^2 + d_2^2},$$

from Eq.(49), we get

$$\overline{D}_n' = -\rho_n \frac{T'}{||T'||}.$$

So, we have $||\overline{D}_n'|| = \varepsilon \rho_n$, where $\varepsilon = \pm 1$. Thus, since $N_\zeta = \frac{\overline{D}_n'}{||\overline{D}_n'||}$, we find

$$N_\zeta = -\varepsilon \frac{T'}{||T'||}.$$

Also, the definition of the binormal vector B_ζ, we find $B_\zeta = \varepsilon T$. On the other hand, the curvatures of the $\overline{D}_n$-direction curve ζ can be obtained

$$\kappa_\zeta = ||T_\zeta'|| = \varepsilon \rho_n, \qquad \tau_\zeta = -\langle B_\zeta', N_\zeta \rangle = \eta_n.$$

Corollary 4.8. ζ is a general helix if and only if α is a general helix.

Proof: The proof can be seen by using Corollary 3.5.

4.4. $\overline{D}_r$-DIRECTION CURVE

Definition 4.9. Let α be a polynomial space curve in $\mathbb{E}^3$, $\{T, D_2, D_1\}$ be the Flc-frame along α and $\overline{D}_r$ be the unit rectifying the Flc-frame vector field of α. The integral curve of the vector field $\overline{D}_r$ is called $\overline{D}_r$-direction curve of α. Namely, if $\varphi(s)$ is the $\overline{D}_r$-direction curve of α, then $\overline{D}_r(s) = \varphi'(s)$.

Theorem 4.10. Let φ be the $\overline{D}_r$-direction curve of a polynomial space curve α. Then the Frenet vectors $\{T_\varphi, N_\varphi, B_\varphi\}$ and the curvatures κ_φ and τ_φ of φ can be found as

$$T_\varphi = \overline{D}_r, \qquad N_\varphi = -\varepsilon \frac{D_2'}{||D_2'||}, \qquad B_\varphi = \varepsilon D_2, \qquad \kappa_\varphi = \varepsilon \rho_r, \qquad \tau_\varphi = \eta_r,$$

where $\varepsilon = \pm 1$.

Proof: Taking into account the definition of the $\overline{D}_r$-direction curve, we get

$$T_\varphi = \overline{D}_r = \frac{1}{\sqrt{d_1^2 + d_3^2}} (d_3 T + d_1 D_1).$$

Differentiating this equation with respect to s, using Eq.(1) and doing some calculations, we obtain

$$\overline{D}_r' = \frac{(d_3' d_1 - d_1' d_3) - v d_2 (d_1^2 + d_3^2)}{(d_1^2 + d_3^2)^{3/2}} (d_1 T - d_3 D_1). \tag{50}$$

Since

$$\rho_r = \frac{(d_3' d_1 - d_1' d_3) - v d_2 (d_1^2 + d_3^2)}{d_1^2 + d_3^2},$$

from Eq.(50), we find

$$\overline{D}_r' = -\rho_r \frac{D_2'}{||D_2'||}.$$

So, we get $||\overline{D}_r'|| = \varepsilon \rho_r$, where $\varepsilon = \pm 1$. Since $N_\varphi = \frac{\overline{D}_r'}{||\overline{D}_r'||}$, we have

$$N_\varphi = -\varepsilon \frac{D_2'}{||D_2'||}.$$

Thus, the binormal vector B_φ of the curve φ is obtained as $B_\varphi = \varepsilon D_2$. Also, the curvature and the torsion of the $\overline{D}_r$-direction curve φ can be found as

$$\kappa_\varphi = ||T_\varphi'|| = \varepsilon \rho_r, \qquad \tau_\varphi = -\langle B_\varphi', N_\varphi \rangle = \eta_r.$$

Corollary 4.11. φ is a general helix if and only if α is a D_2-slant helix.

Proof: The proof can be seen by using Corollary 3.8.

5. CONCLUSION

In this paper, using the Darboux vector of the Flc-frame, the osculating Flc-frame, the normal Flc-frame, and the rectifying Flc-frame are defined along a polynomial space curve. Also, the derivative equations according to these frames are obtained. Besides, some new integral curves called the d_{Flc}-direction curve, the $\overline{D}_o$-direction curve, the $\overline{D}_n$-direction curve, and the $\overline{D}_r$-direction curve of a polynomial space curve are defined and some results related to these curves are given. Similar studies can be done using other frames in the future.

REFERENCES

[1] Bishop, R. L., *American Mathematical Monthly*, **82**(3), 246, 1975.
[2] Jüttler, B., Mäurer, C., *Computer-Aided Design*, **31**(1), 73, 1999.
[3] Dede, M., *Mathematical Sciences and Applications E-Notes*, **10**(4), 190, 2022.
[4] Dede, M., *Houston Journal of Mathematics*, **45**(3), 707, 2019.
[5] Hananoi, S., Noriaki, I., Izumiya, S., *Beitrage zur Algebra und Geometrie*, **56**, 575, 2015.
[6] Önder, M., arXiv:2201.09684.
[7] Choi, J. H., Kim, Y. H., *Applied Mathematics and Computation*, **218**(18), 9116, 2012.
[8] Macit, N., Düldül, M., *Turkish Journal of Mathematics*, **38**, 1023, 2014.
[9] Körpınar, T., Sarıaydın, M. T., Turhan, E., *Advanced Modelling and Optimization*, **15**(3), 713, 2013.
[10] Yılmaz, S., *Kirklareli University Journal of Engineering and Science* **1**, 66, 2015.
[11] Güven, I. A., *Open Mathematics*, **18**, 1332, 2020.
[12] Echabbi, N., Chahdi, A. O., *International Journal of Mathematics and Mathematical Sciences*, Article ID 3814032, 2021.
[13] Alkan, A., Kocayiğit, H., Ağırman Aydın, T., *Sigma Journal of Engineering and Natural Sciences*, **42**(4), 1023, 2024.
[14] Uyar Düldül, B., *Aims Mathematics*, **9**(7), 17871, 2024.
[15] Farouki, R. T., Han, C. Y., Manni, C., Sestini, A., *Journal of Computational and Applied Mathematics*, **162**(2), 365, 2004.
[16] Larson, R., *Elementary Linear Algebra*, The Pennsylvania State University, Boston, 2012.