*ORIGINAL PAPER*

# UMBRELLA MOTIONS IN 3-DIMENSIONAL LORENTZ SPACE

MERT ÇARBOĞA[1], YUSUF YAYLI[1]

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***Abstract.*** *In this study, umbrella matrices in 3D Lorentz space are obtained for the first time, and the effect of metric change on the matrix group is investigated. In addition, the relationship of these matrices with dual numbers was examined, and important results are obtained regarding their geometric and kinematic properties. The findings reveal that umbrella matrices can play an important role in Lorentz space.*
***Keywords:*** *umbrella matrix; Lorentzian geometry; kinematics; dual numbers.*

## 1. INTRODUCTION

### *1.1. BACKGROUND*

Umbrella matrices are significant orthogonal matrices with broad potential applications in mathematics, including geometry, kinematics, and statistics. These matrices have a determinant of +1 and leave the axis $\mathbf{1} = [1 \quad 1 \quad 1]^T \in \mathbb{R}^3_1$ invariant. They were first defined by Alisbah O.H. and later studied in the context of kinematics in [1]. Subsequently, Ozdamar E. studied the Lie group and Lie algebra of umbrella matrices in his doctoral thesis and the studies with these matrices were collected in the study [2], and Esin E. addressed their differential geometry using the Cayley transformation [3]. Additionally, in [4], this matrix group was generalized, and classifications were made regarding matrices $A$ or $A^T$ that leaves the axis $\mathbf{1}$ invariant for $GL(n, \mathbb{R})$. Recently, the relationship of this matrix group with pairwise comparison (PC) matrices was explored in [5], and the geometry of $3-$dimensional dual umbrella matrices were discussed in [6]. Moreover, in [7], the relationship between Euclidean and Lorentz umbrella matrices was defined. Extending these works in [8], generalized the Euclidean umbrella matrices, where considering the Euclidean metric, an orthogonal matrix group was obtained from skew-symmetric matrices with row and column sums equal to zero using the Cayley formula, resulting in matrices with row and column sums of 1 (i.e., $A\mathbf{1} = \mathbf{1}$ and $A^T\mathbf{1} = \mathbf{1}$ ).

In 1873, dual numbers were first introduced by William Clifford. Beyond finding applications in a variety of fields—*such as super spaces and algebraic geometry*—they also hold a particularly significant place in mechanics for analyzing the spatial motion of rigid bodies. In Eduard Study's 1891 work on quaternions, dual numbers were further studied, playing a crucial role in the development of dual quaternions. This mathematical structure continues to appear in many contemporary investigations and remains an active subject of research today (see [9-11]).

[1]Ankara University, Department of Mathematics, 06100 Ankara, Turkey. E-mail: mcarboga@ankara.edu.tr; yayli@science.ankara.edu.tr.

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### *1.2. OBJECTIVES AND CONTRIBUTIONS*

The primary aim of this study is to obtain and examine these matrices in $3-$dimensional Lorentz space for the first time. Özkaldı and Gündoğan obtained orthogonal matrices with a determinant of +1 in Lorentz space using the Cayley formula; however, they did not obtain matrices that leave the axis $\mathbf{1}$ invariant (see [12]).

By choosing the metric as (+, +, −), Lorentz umbrella matrices with row sums of 1 were obtained for the first time using the Euler-Rodrigues formula instead of the Cayley formula. These correspond to semi-skew symmetric matrices with row sums equal to zero in $3-$dimensional Lorentz space. Additionally, an intriguing aspect of this study is that if the Lorentz metric is chosen as (−, +, +), the resulting Lorentz umbrella matrices have column sums of 1. This condition sheds light on the formation of matrix groups represented in [4]. This study demonstrates that the choice of metric plays a significant role in determining Lorentz umbrella groups.

### *1.3. STRUCTURE OF THE PAPER*

This section provides information about the contents and structure of the paper. In section 2, some necessary basic information is provided. In section 3, focuses on how Lorentz umbrella matrices are obtained and demonstrates in Theorem 3.2 that this matrix group represents one-parameter motions. Subsequently, as the main result, forming a different umbrella matrix group with a change in metric is examined, and similar results are found. Moreover, examples of surfaces generated by umbrella matrices obtained with both metrics for some trajectory curves are provided in Example 3.1. Additionally, the Darboux matrix of the Umbrella motion is calculated in Theorem 3.3. In section 4, investigates Lorentz dual umbrella matrices using dual numbers and demonstrates that they correspond to a $\mathbf{1}-$axis screw motion. Furthermore, a ruled surface corresponding to this motion is obtained in Example 4.1. Finally, the relationship of infinitesimal motions in this space for the umbrella matrices $A(\theta)$ and $A(\hat{\theta})$ is discussed as a special case.

## 2. PRELIMINARIES

In this section, we will provide some necessary information. First, we focus on defining a semi skew-symmetric matrix in the Lorentz space:

Let $S$ be a matrix of type n×n. If

$$S^T = -\varepsilon\, S\, \varepsilon,$$

matrix $S$ is called a *semi-skew symmetric matrix*, where

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is the sign matrix defined (See[13]).

Next, we examine the definition of the Euler-Rodrigues formula: Let $S$ be an anti-symmetric matrix. The equation is given in the form

$$A = I_3 + sin\theta S + (1 - cos\theta)S^2,$$

is called the *Euler -Rodrigues formula*, where $A$ is an orthogonal matrix (See[14]).

Now, let us define the umbrella matrix for $3-$dimensional space. Let $A$ be an orthogonal matrix. If

$$A\mathbf{1} = \mathbf{1},$$

then $A$ is called an *umbrella matrix*, where $\mathbf{1} = [1 \quad 1 \quad 1]^T \in \mathbb{R}_1^3$ $(See[2])$. To give the definition of the Darboux matrix for an orthogonal matrix $A$, let $y$ and $x$ be the position vectors, represented by column matrices, of a point $P$ in the fixed space $\Sigma^n$ and the moving space $\Sigma^n$, respectively. $A$ continuous series of displacements, given by

$$y = Ax + b$$

where the orthogonal matrix $A$ and the translation vector $b$ are functions of $a$ parameter $t$ which may be identified with time and is called motion. Now, we consider the rotational motion, given by

$$y(t) = A(t)x.$$

Taking the derivative,

$$\dot{y} = \dot{A}x$$

and

$$x = A^{-1}y,$$

we obtain $\dot{y} = \dot{A}A^{-1}y$. The matrix

$$W = \dot{A}A^{-1}$$

is called the *angular velocity* matrix [15] and is defined as an infinitesimal motion. An *infinitesimal linear transformation* is defined as a transformation whose matrix is

$$A = I_n + \varepsilon[b_{ij}]$$

where $[b_{ij}]$ is a skew-symmetric matrix and $\varepsilon$ denotes an infinitesimal quantity of the first order [16]. *Taylor series expansion of a dual function*: For $x + \varepsilon x^*$, the Taylor series expansion of the dual-function $f(x + \varepsilon x^*)$ is defined as

$$f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x).$$

.

Hence, the Taylor series expansions of the $\cos(x + \varepsilon x^*)$ and $\sin(x + \varepsilon x^*)$ functions are obtained as

$$\cos(x + \varepsilon x^*) = \cos(x) - \varepsilon x^* \sin x$$

and

$$\sin(x + \varepsilon x^*) = \sin(x) - \varepsilon x^* \cos x,$$

respectively [17].

## 3. OBTAINING UMBRELLA MATRICES IN LORENTZIAN SPACE

In 3D Lorentz space, we can consider the Euler–Rodrigues formula defined in this space to obtain umbrella matrices. By denoting the semi-skew symmetric matrix of Lorentz space as

$$S = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix},$$

we find that

$$S^2 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Then, by using the formula $A(\theta) = I_3 + sinh\theta S + (cosh\theta - 1)S^2$, the following matrix is obtained.

$$A(\theta) = \begin{bmatrix} 1 & 1 - sinh\theta - cosh\theta & sinh\theta + cosh\theta - 1 \\ sinh\theta - cosh\theta + 1 & 1 & cosh\theta - sinh\theta - 1 \\ sinh\theta - cosh\theta + 1 & 1 - sinh\theta - cosh\theta & 2cosh\theta - 1 \end{bmatrix}$$

This matrix, with the metric $(+,+,-)$, leaves the space-like axis $\mathbf{1} = [1 \quad 1 \quad 1]^T \in \mathbb{R}_1^3$ invariant and performs a boost motion of the angle. Moreover, ensuring that the row sums are 1 makes this matrix an umbrella matrix in 3-dimensional Lorentz space.

**Theorem 3.1.** The matrix $A(\theta)$ leaves the plane $x_1 + x_2 - x_3 = 0$ invariant.

*Proof:* Since $A(\theta)$ is an umbrella matrix, we can write the equation

$$A(\theta)\mathbf{1} = \mathbf{1}$$

Now, assuming $y = [y_1 \quad y_2 \quad y_3]^T$, we have $y = A(\theta)x$. From here, we find

$$x = A(\theta)^{-1} y \tag{1}$$

and we can write $x_1 + x_2 - x_3 = 0$ as

$$[1 \quad 1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \\ -x_3 \end{bmatrix} = 0. \tag{2}$$

Combining Eqs. (1) and (2), we obtain

$$\begin{aligned}[1 \quad 1 \quad 1]A(\theta)^{-1}y &= [1 \quad 1 \quad 1](IA(\theta)^T I)y \\ &= [1 \quad 1 \quad -1]A(\theta)^T Iy \\ &= [1 \quad 1 \quad 1]Iy \\ &= [1 \quad 1 \quad -1]y \\ &= y_1 + y_2 - y_3.\end{aligned}$$

Thus, the matrix A leaves the plane $x_1 + x_2 - x_3 = 0$ invariant.

**Theorem 3.2.** The matrix $A(\theta)$ represents a one-parameter motion.

*Proof:* To show that $A(\theta)$ represents a one-parameter motion, we need to obtain the equation

$$A(\theta_1 + \theta_2) = A(\theta_1)A(\theta_2). \tag{3}$$

From here, if we consider the Euler-Rodrigues formula for matrices $A(\theta_1)$ and $A(\theta_2)$, we can write the right-hand side of Eq.(3) as

$$A(\theta_1)A(\theta_2) = (I_3 + sinh\theta_1 S + (cosh\theta_1 - 1)S^2)(I_3 + sinh\theta_2 S + (cosh\theta_2 - 1)S^2). \tag{4}$$

Rearranging Eq. (4) with the equalities $S^3 = S$ and $S^4 = S^2$ , we get

$$\begin{aligned}A(\theta_1)A(\theta_2) = I_3 &+ (sinh\theta_1 cosh\theta_2 + cosh\theta_1 sinh\theta_2)S \\ &+(cosh\theta_1 cosh\theta_2 + sinh\theta_1 sinh\theta_2 - 1)S^2.\end{aligned} \tag{5}$$

Finally, consider the equations

$$sinh(\theta_1 + \theta_2) = sinh\theta_1 cosh\theta_2 + cosh\theta_1 sinh\theta_2,$$

$$cosh(\theta_1 + \theta_2) = cosh\theta_1 cosh\theta_2 + sinh\theta_1 sinh\theta_2$$

along with Eq.(5), we obtain

$$A(\theta_1)A(\theta_2) = I_3 + sinh(\theta_1 + \theta_2)S + (cosh(\theta_1 + \theta_2) - 1)S^2$$

and we find

$$A(\theta_1 + \theta_2) = A(\theta_1)A(\theta_2).$$

This shows that the matrix $A(\theta)$ corresponds to a one-parameter motion. The proof is complete.

Let us now consider the metric as $(-,+,+)$, we obtain the semi skew-symmetric matrix $\bar{S}$ as

$$\bar{S} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Hence, $\bar{S}^2$ is obtained as

$$\bar{S}^2 = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix},$$

and from the equation $\bar{A}(\theta) = I_3 + sinh\theta\bar{S} + (cosh\theta - 1)\bar{S}^2$, we find the matrix

$$\bar{A}(\theta) = \begin{bmatrix} 2cosh\theta - 1 & -sinh\theta + cosh\theta - 1 & sinh\theta + cosh\theta - 1 \\ -sinh\theta - cosh\theta + 1 & 1 & -sinh\theta - cosh\theta + 1 \\ sinh\theta - cosh\theta + 1 & sinh\theta - cosh\theta + 1 & 1 \end{bmatrix}.$$

The transposition of this matrix $\bar{A}(\theta)$, leaves the axis $\mathbf{1} = [1 \quad 1 \quad 1]^T \in \mathbb{R}_1^3$ fixed, and similar situations to the previous section can be obtained below.

**Proposition 3.1.** The matrix $\bar{A}^T(\theta)$ leaves the plane $-x_1 + x_2 + x_3 = 0$ invariant.

**Proposition 3.2.** The matrix $\bar{A}(\theta)$ represents a one-parameter motion.

Let us now obtain surfaces generated by an umbrella matrix and a trajectory curve in Euclidean space. Then, we will examine surfaces that share the same trajectory curves with the matrices $A(\theta)$ and $\bar{A}^T(\theta)$.

**Example 3.1.** The umbrella matrix in the 3-dimensional Euclidean space obtained in the study of [3] is known as

$$\mathrm{A(t)} = \frac{1}{1+3\mathrm{t}^2}\begin{bmatrix} 1-\mathrm{t}^2 & 2(t^2+t) & 2(t^2-t) \\ 2(t^2-t) & 1-\mathrm{t}^2 & 2(t^2+t) \\ 2(t^2+t) & 2(t^2-t) & 1-\mathrm{t}^2 \end{bmatrix}.$$

Here, if we consider the matrix A together with the curves $\alpha(u) = (sinu, cosu, 0)$ *and* $\beta(u) = (\cosh u, \sinh u, \sin u)$, we obtain the surfaces $H_1(t,u) = \mathrm{A(t)}\,.\,\alpha(u)$ and $H_2(t,u) = \mathrm{A(t)}.\,\beta(u)$ respectively. Hence, the surfaces are drawn as follows (Fig. 1):

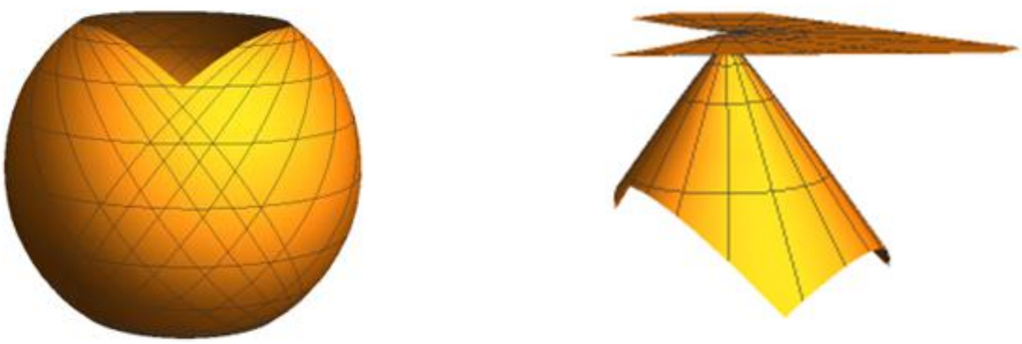

**Figure 1. $\mathrm{H_1}$ (Left) and $\mathrm{H_2}$ (Right) surfaces.**

Than, the surfaces formed by umbrella matrices $A(\theta)$ and $\bar{A}^T(\theta)$ with orbital curve $\alpha(u)$ are obtained as $F(u,\theta) = A(\theta)\,.\,\alpha(u)$ and $\bar{F}(u,\theta) = \bar{A}^T(\theta).\,\alpha(u)$ respectively. Hence, the surfaces are drawn as follows (Fig. 2):

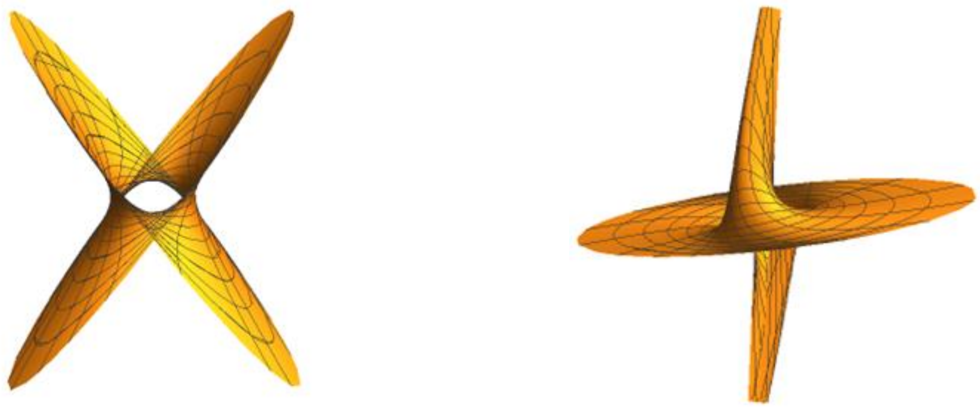

**Figure 2. F (Left) and $\bar{F}$ (Right) surfaces.**

In addition, the surfaces generated with $\beta(u)$ curve is obtained as follows $G(u,\theta) = A(\theta)\,.\,\beta(u)$ and $\bar{G}(u,\theta) = \bar{A}^T(\theta).\,\beta(u)$. And if we draw these surfaces, we obtain get the following (Fig. 3):

**Figure 3. G (Left) and $\bar{G}$ (Right) surfaces**

Let us consider the umbrella matrix $A(\theta)$ obtained from the semi skew-symmetric matrix $S$ with row sums of zero. Next, we present the theorem regarding the $W$ Darboux matrix of the one-parameter umbrella motion defined by the matrix $A(\theta)$.

**Theorem 3.3.** Let $A(\theta)$ be an umbrella matrix in Lorentz space, representing an umbrella motion. Then, the instantaneous angular matrix of the umbrella motion (Darboux matrix) is given by

$$W = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

*Proof:* To find the instantaneous angular velocity matrix of the umbrella motion, we can use

$$W = \dot{A}(\theta)A^{-1}(\theta). \tag{6}$$

Thus, we find

$$\dot{A}(\theta)=\begin{bmatrix}0 & -sinh\theta-cosh\theta & sinh\theta+cosh\theta\\ -sinh\theta+cosh\theta & 0 & -cosh\theta+sinh\theta\\ -sinh\theta+cosh\theta & -sinh\theta-cosh\theta & 2sinh\theta\end{bmatrix}$$

and

$$A^{-1}(\theta)=\begin{bmatrix}1 & 1+sinh\theta-cosh\theta & -sinh\theta+cosh\theta-1\\ -sinh\theta-cosh\theta+1 & 1 & cosh\theta+sinh\theta-1\\ -sinh\theta-cosh\theta+1 & 1+sinh\theta-cosh\theta & 2cosh\theta-1\end{bmatrix}.$$

Then, substituting these values into Eq.(6), we obtain

$$W=\begin{bmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}.$$

The proof is complete.

**Special Case.** Let $A(\theta)$ be the umbrella matrix obtained from the skew-symmetric matrix S. The Darboux matrix corresponds to $A(\theta)$ is $W=S$.

## 4. DUAL UMBRELLA MATRICES IN LORENTZ SPACE

In this section, we analyze dual umbrella matrices in Lorentz space together with the geometry of motion. If we organize the umbrella matrix

$$A(\theta)=\begin{bmatrix}1 & 1-sinh\theta-cosh\theta & sinh\theta+cosh\theta-1\\ sinh\theta-cosh\theta+1 & 1 & cosh\theta-sinh\theta-1\\ sinh\theta-cosh\theta+1 & 1-sinh\theta-cosh\theta & 2cosh\theta-1\end{bmatrix},$$

obtained in the previous section with the dual-angle $\hat{\theta}=\theta+\varepsilon\theta^*$ and definition (2), we can write the equation

$$A(\hat{\theta})=\begin{bmatrix}1 & 1-sinh\theta-cosh\theta & sinh\theta+cosh\theta-1\\ sinh\theta-cosh\theta+1 & 1 & cosh\theta-sinh\theta-1\\ sinh\theta-cosh\theta+1 & 1-sinh\theta-cosh\theta & 2cosh\theta-1\end{bmatrix}$$
$$+\varepsilon\theta^*\begin{bmatrix}0 & -sinh\theta-cosh\theta & sinh\theta+cosh\theta\\ -sinh\theta+cosh\theta & 0 & -cosh\theta+sinh\theta\\ -sinh\theta+cosh\theta & -sinh\theta-cosh\theta & 2sinh\theta\end{bmatrix}.$$

Subsequently, we obtain

$$A(\hat{\theta})=A(\theta)+\varepsilon\theta^*\dot{A}(\theta).$$

Thus, for $\mathbf{1}=[1 \quad 1 \quad 1]^T\in\mathbb{R}_1^3$, we obtain the equation

$$A(\hat{\theta})\mathbf{1}=(A(\theta)+\varepsilon\theta^*\dot{A}(\theta))\mathbf{1}$$

$$= A(\theta)\mathbf{1} + \varepsilon\theta^* \dot{A}(\theta)\mathbf{1}$$
$$= \mathbf{1}$$

and we can provide the following definition for the $A(\hat{\theta})$ dual matrix.

**Definition 4.1.** *Let $A(\hat{\theta}) \in SO(3,\mathbb{D})$ be a dual orthogonal matrix in the Lorentz space.* For $\mathbf{1} = [1 \quad 1 \quad 1]^T \in \mathbb{R}^3_1$,

$$A(\hat{\theta})\mathbf{1} = \mathbf{1}$$

any matrix $A(\hat{\theta})$ satisfying this property is called a dual umbrella matrix, where $\hat{\theta} = \theta + \varepsilon\theta^*$ represents the dual angle and for

$$A(\hat{\theta}) = A(\theta) + \varepsilon A^*(\theta)$$

where $A(\theta)$ denotes the real part, and $A^*(\theta) = \theta^* \dot{A}(\theta)$ represents the dual part.

**Special Case.** The instantaneous angular velocity matrix of umbrella motion in the Lorentz space of the matrix $A(\theta)$ is $W = \dot{A}(\theta)A(\theta)^{-1}$ , and we can write it as

$$A(\hat{\theta}) = A(\theta) + \varepsilon\theta^* W A(\theta)$$

where $\hat{\theta} = \theta + \varepsilon\theta^*$ is the dual angle. Here, if we consider $B = \theta^* W$, we obtain the semi skew-symmetric matrix

$$B = \begin{bmatrix} 0 & -\theta^* & \theta^* \\ \theta^* & 0 & -\theta^* \\ \theta^* & -\theta^* & 0 \end{bmatrix}$$

in the Lorentzian space and from here we can write the equation

$$A(\hat{\theta}) = A(\theta) + \varepsilon B A(\theta). \tag{7}$$

Thus, the matrix $A(\hat{\theta})$ represents a $\mathbf{1}$-axis screw motion in Lorentz space. Here, $A(\theta)$ represents the rotational part of the general motion $Y = Ax + \vec{C}$ in space, and if we consider the translation vector $\vec{C}$ together with the semi skew-symmetric matrix $B$, the amount of slip is denoted by $\theta^*$, and $\vec{C} = \theta^*(1,1,1)$ is calculated.

Let us examine $\mathbf{1}$-axis screw motion with the following example.

**Example 4.1.** We can consider $\theta = ln2$ and $\theta^* = 1$ for $\hat{\theta} = \theta + \varepsilon\theta^*$. Then, considering Eq. 7, we obtain

$$A(\hat{\theta}) = \begin{bmatrix} 1 & -1 & 1 \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{3}{2} \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & -2 & 2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}, \tag{8}$$

Kinematically, this matrix rotates the points in space by a $\theta = ln2$ angle along the $\mathbf{1}$-axis and shifts $\theta^* = 1$ unit on the same axis. Now, let us consider the x-axis. When the matrix (8) is applied to this axis, we obtain the following line:

$$\left(\begin{bmatrix}1 & -1 & 1\\ \frac{1}{2} & 1 & -\frac{1}{2}\\ \frac{1}{2} & -1 & \frac{3}{2}\end{bmatrix}+\varepsilon\begin{bmatrix}0 & -2 & 2\\ \frac{1}{2} & 0 & -\frac{1}{2}\\ \frac{1}{2} & -2 & \frac{3}{2}\end{bmatrix}\right)\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\ \frac{1}{2}+\frac{\varepsilon}{2}\\ \frac{1}{2}+\frac{\varepsilon}{2}\end{bmatrix}.$$

This line becomes a line with a direction $(0,\frac{1}{2}+\frac{\varepsilon}{2},\frac{1}{2}+\frac{\varepsilon}{2})$ and a passing point $(1,\frac{1}{2}+\frac{\varepsilon}{2},\frac{1}{2}+\frac{\varepsilon}{2})$. In other words, the x-axis becomes

$$\begin{bmatrix}1\\ \frac{1}{2}+\frac{\varepsilon}{2}\\ \frac{1}{2}+\frac{\varepsilon}{2}\end{bmatrix}$$

axis due to a Lorentzian rotation of the $ln2$ angle along the **1** axis and a 1–unit displacement on the same axis. To generalize this situation, we may ask, “What would be the orbital surface formed by the continuous movement of a line look like?" To answer this question, let us consider the line $d = (1, \varepsilon, 0)$ and apply it to the $A(\hat{\theta})$ matrix, yielding the dual curve

$$A(\hat{\theta})\begin{bmatrix}1\\ \varepsilon\\ 0\end{bmatrix}=\begin{bmatrix}1+\varepsilon(1-sinh\theta-cosh\theta)\\ 1+sinh\theta-cosh\theta+\varepsilon(cosh\theta-sinh\theta+1)\\ 1+sinh\theta-cosh\theta+\varepsilon(1-sinh\theta-cosh\theta)\end{bmatrix}.$$

Let us denote this curve as

$$\alpha(\hat{\theta}) = \alpha(\theta) + \varepsilon\alpha^*(\theta),$$

where $\alpha(\theta)$ represents the real part and $\alpha^*(\theta)$ represents the dual part. If the spine curve of the $I(\theta, u)$ orbital surface of the line $d = (1, \varepsilon, 0)$ is

$$\alpha(\theta) \wedge_L \alpha^*(\theta) = (2cosh^2\theta - 2cosh\theta - 2sinh\theta cosh\theta, 1 + sinh\theta - cosh\theta, 1 + sinh\theta - 3cosh\theta)$$

and its direction is $\alpha(\theta)$, we obtain the surface as,

$$I(\theta, u) = (2cosh^2\theta - 2cosh\theta - 2sinh\theta cosh\theta, 1 + sinh\theta - cosh\theta, 1 + sinh\theta - 3cosh\theta) + u(1,1 - sinh\theta - cosh\theta, 1 - sinh\theta - cosh\theta).$$

This orbital surface denotes a ruled surface and is illustrated in Fig. 3.

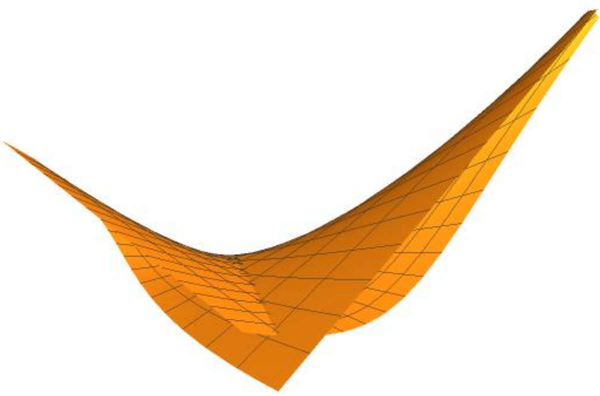

**Figure 3.** $\boldsymbol{I(\theta, u)}$ **surface**

Now, we will outline the results concerning infinitesimal motion in the Lorentz space.

**Special Case.** If we consider the Eq.(7) and the definition of infinitesimal motion, we obtain the following equation:

$$A(\hat{\theta}) = (I_3 + \varepsilon B)A(\theta)$$

This demonstrates that each dual umbrella matrix can be represented as a combination of its real part and the infinitesimal motion. Consequently, it can be deduced that the infinitesimal quantity "$\varepsilon$" and the dual number "$\varepsilon$" have identical meanings.

## 4. CONCLUSION

This study, which focuses on the geometric and kinematic properties through the analytical framework of dual numbers, has revealed the existence and derivation of umbrella matrices in the 3-dimensional Lorentz space. Our research has provided significant insights into the algebraic and geometric structure of these matrices, and the use of dual numbers has led to important implications regarding the potential applications of umbrella matrices. The findings establish a solid foundation for future studies, where these concepts will be further generalized, potentially expanding the scope of mathematical theory and umbrella matrices.

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