ORIGINAL PAPER

ON A CLASS OF INTEGRALS INVOLVING APPELL FUNCTIONS

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Abstract. *In this paper, we establish some useful results in terms of the Kampe de Feriet function by evaluating different integrals of products of Appell functions with algebraic, Jacobi, and Legendre functions of common variables.*

Keywords: *Definite Integrals; special functions; Appell function; Jacobi function; Legendre function; Kampe de Feriet function.*

1. INTRODUCTION

Definite integrals, Hypergeometric functions, Appell functions, algebraic functions, Jacobi polynomials, and Legendre functions are frequently used in Mathematics and due to their extensive use in Engineering and Sciences, these are topics of great interest among researchers. Several formulas, extensions, applications, products, derivatives, finite or infinite sums, and integrals have been established by the researchers so far. Since these functions and polynomials have numerous applications in Engineering, Physics, and Mathematics, the advancement and expansion in this area is of great importance and is inevitable.

In recent decades, several applications in a variety of fields led to advancements in research of special functions [1]. Several relations, integrals, and forms of special functions and polynomials have been established in the past [2-6]. Numerous applications and extensions of those functions and polynomials have been studied lately [7-14].

The notations $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}^+$, $\mathbb{N}$ are used for the set of complex, real, positive real numbers, positive integrals, and natural numbers, respectively. For the study topics related to special functions and integrals, we refer to [2-6] and [15-20].

2. MATERIALS AND METHODS

To solve the problem arising in ancient times involving combinations and permutations, Factorial function $\alpha!$ was defined as

$$\alpha! = \alpha(\alpha-1)(\alpha-2)...(3)(2)1, \ \alpha \in \mathbb{N}, \tag{1}$$

which plays a vital role in special functions too.

Gamma function $\Gamma(p)$, an extension of factorial function, is defined as

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$$\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}dx,\ \ \mathrm{Re}(p)>0. \tag{2}$$

The Pochhammer symbol $(\alpha)_n$ is defined as

$$(\alpha)_n=\alpha(\alpha+1)(\alpha+2)...(\alpha+(n-1)),\ \alpha\in\mathbb{C},\ n\in\mathbb{N}. \tag{3}$$

The Beta function $B(p,q)$ is defined as

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,\ \ \mathrm{Re}(p)>0,\ \ \mathrm{Re}(q)>0 \tag{4}$$

(see Rainville [15]). One may obsever that

$$\Gamma(z+1)=z\Gamma(z), \tag{5}$$

$$\Gamma(z)=(z-1)!, \tag{6}$$

$$(1)_n=n!, \tag{7}$$

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \tag{8}$$

$$(\alpha)_{2n}=2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n, \tag{9}$$

$$(n+k)!=(n+1)_k\,n!=(n+1)_k(1)_n, \tag{10}$$

$$B(m,n)=\int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}}dt, \tag{11}$$

$$B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{12}$$

and

$$2^{m+n-1}B(m,n)=\int_{-1}^1 (1+t)^{m-1}(1-t)^{n-1}dt. \tag{13}$$

The Generalized Hypergeometric function ${}_pF_q\left(\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix};z\right)$ is given by

$${}_pF_q\left(\begin{matrix}a_1,a_2,...,a_p\\ b_1,b_2,...,b_q\end{matrix};z\right)=\sum_{j=0}^\infty \frac{(a_1)_j(a_2)_j...(a_p)_j}{(b_1)_j(b_2)_j...(b_q)_j}\frac{z^j}{j!},\ p,q\in\mathbb{Z}^+ \tag{14}$$

(see Rainville [15]). Jacobi function $P_n^{(\tau,\nu)}(x)$ in terms of Hypergeometric function is given by

$$P_n^{(\tau,\vartheta)}(x)=\frac{(\tau+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,1+\tau+\vartheta+n\\1+\tau\end{matrix};\frac{1-x}{2}\right),\ \ \tau,\vartheta>-1,n\in\mathbb{N} \tag{15}$$

and

$$\begin{aligned}&\int_{-1}^{1}x^{\alpha}(1-x)^{m}(1+x)^{u}P_n^{(m,v)}(x)dx\\&=(-1)^n2^{m+u+1}\frac{\Gamma(u+1)\Gamma(n+m+1)\Gamma(u+v+1)}{n!\Gamma(u+v+n+1)\Gamma(u+m+n+2)}\\&\times{}_3F_2\left(\begin{matrix}-\lambda,u+v+1,u+1\\u+v+n+1,u+m+n+2\end{matrix};1\right)\end{aligned} \tag{16}$$

(see Erdelyi *et al.* [20]).

The Legendre function $P_n^{\tau}(z)$ of the first kind is given by

$$P_n^{\tau}(z)=\frac{(\tau+1)}{n!}\left(\frac{z+1}{z+2}\right)^{\frac{\tau}{2}}{}_2F_1\left(\begin{matrix}-n,n+1\\1-\tau\end{matrix};\frac{1-z}{2}\right),\ \ |1-z|<2. \tag{17}$$

It is easy to see that

$$\int_0^1 u^{\omega-1}(1-u^2)^{\frac{\tau}{2}}P_v^{\tau}(u)du=\frac{(-1)^{\tau}2^{-\omega-i\xi-\tau}\sqrt{\pi}\Gamma(\omega)\Gamma(1+\tau+v)}{\Gamma(1-\tau+v)\Gamma\left(\frac{1+\omega+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+\tau+v}{2}\right)} \tag{18}$$

and

$$\int_0^1 u^{\omega-1}(1-u^2)^{\frac{-\tau}{2}}P_v^{\tau}(u)du=\frac{2^{\tau-\omega}\sqrt{\pi}\Gamma(\omega)}{\Gamma\left(\frac{1+\omega-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega-\tau-v}{2}\right)} \tag{19}$$

(see Erdelyi *et al.* [20]).

The Power series of the forms

$$F_1(\alpha,\beta,\beta^{'};\gamma;ht,s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(ht)^is^j}{i!\,j!},\ \ \max\left(|t|,|s|\right)<1, \tag{20}$$

$$F_2(\alpha,\beta,\beta^{'};\gamma,\gamma^{'};ht,s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_i(\gamma^{'})_j}\frac{(ht)^is^j}{i!\,j!},\ \ |t|+|s|<1, \tag{21}$$

$$F_3(\alpha,\alpha^{'},\beta,\beta^{'};\gamma;ht,s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_i(\alpha^{'})_j(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(ht)^is^j}{i!\,j!},\ \ \max\left(|t|,|s|\right)<1 \tag{22}$$

and

$$F_4(\alpha,\beta;\gamma,\gamma^{'};ht,s)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_{i+j}}{(\gamma)_i(\gamma^{'})_j}\frac{(ht)^is^j}{i!\,j!},\ \ \sqrt{|t|}+\sqrt{|s|}<1 \tag{23}$$

are called Appell functions and are denoted by the symbols mentioned above (see Appell [4]).

The double power series defined as

$$F^{p:q;r}_{s:t;u}\left(\begin{matrix} a_1,a_2,...,a_p : b_1,b_2,...,b_q; b'_1,b'_2,...,b'_r \\ c_1,c_2,...,c_s : d_1,d_2,...,d_t; d'_1,d'_2,...,d'_u \end{matrix}; x,y\right)$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a_1)_{i+j}(a_2)_{i+j}...(a_p)_{i+j}(b_1)_i(b_2)_i...(b_q)_i(b'_1)_j(b'_2)_j...(b'_r)_j}{(c_1)_{i+j}(c_2)_{i+j}...(c_s)_{i+j}(d_1)_i(d_2)_i...(d_t)_i(d'_1)_j(d'_2)_j...(d'_u)_j}\frac{x^i y^j}{i!\,j!} \tag{24}$$

is known as the generalized Kampe de Ferit function, according to Pandey [16].

3. RESULTS AND DISCUSSION

3.1. RESULTS

Using some Algebraic functions alongside Appell functions, we evaluate the following integrals in this section:

1) Let

$$A_1=\int_0^1 t^{-\sigma}(1-t)^{\sigma-\varpi-1}F_1(\alpha;\beta,\beta';\gamma;ht,s)dt.$$

Then, using (20), we get

$$A_1=\int_0^1 t^{-\sigma}(1-t)^{\sigma-\varpi-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(ht)^i s^j}{i!\,j!}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\int_0^1 t^{i-\sigma}(1-t)^{\sigma-\varpi-1}dt.$$

Applying (4), (12), (8), and (24); it follows that

$$A_1=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times B(i-\sigma+1,\sigma-\varpi)$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times\frac{\Gamma(i-\sigma+1)\Gamma(\sigma-\varpi)}{\Gamma(i-\varpi+1)}$$
$$=B(1-\sigma,\sigma-\varpi)\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(1-\sigma)_i}{(1-\varpi)_i}\frac{h^i s^j}{i!\,j!}$$
$$=B(1-\sigma,\sigma-\varpi)\times F^{1:2;1}_{1:1;0}\left(\begin{matrix}\alpha:\beta,1-\sigma;\beta' \\ \gamma:1-\varpi;-\end{matrix};h,s\right).$$

2) Consider the integral

$$A_2=\int_0^1 t^{\sigma-1}(1-t)^{\varpi-1}F_1(\alpha,\beta,\beta';\gamma;ht,s)dt.$$

Then, using (20), we get

$$A_2=\int_0^1 t^{\sigma-1}(1-t)^{\varpi-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(ht)^i s^j}{i!\,j!}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\int_0^1 t^{\sigma+i-1}(1-t)^{\varpi-1}dt.$$

Applying (4), (12), (8), and (24), it follows that

$$A_2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times B(i+\sigma,\varpi)$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times\frac{\Gamma(i+\sigma)\Gamma(\varpi)}{\Gamma(i+\varpi+\sigma)}$$
$$=B(\sigma,\varpi)\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\sigma)_i}{(\sigma+\varpi)_i}\frac{h^i s^j}{i!\,j!}$$
$$=B(\sigma,\varpi)\times F^{1:2;1}_{1:1;0}\left(\begin{matrix}\alpha:\beta,\sigma;\beta' \\ \gamma:\sigma+\varpi;-\end{matrix};h,s\right).$$

3) Consider the integral

$$A_3=\int_1^{\infty} t^{-\sigma}(t-1)^{\varpi-1}F_1(\alpha,\beta,\beta';\gamma;\frac{h}{t},s)dt.$$

Then, using (20), we get

$$A_3=\int_1^{\infty} t^{-\sigma}(t-1)^{\varpi-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{\left(\frac{h}{t}\right)^i s^j}{i!\,j!}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\int_1^{\infty} t^{-i-\sigma}(t-1)^{\varpi-1}dt.$$

Let $u=t-1$. Then $u:0\to\infty$ as $t:1\to\infty$. Then, using (4), (12), (8) and (24), we obtain

$$A_3=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\int_0^{\infty} u^{\varpi-1}(1+u)^{-(i+\sigma)}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times B(\varpi,i+\sigma-\varpi)$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times\frac{\Gamma(i+\sigma-\varpi)\Gamma(\varpi)}{\Gamma(i+\sigma)}$$

$$= B(\varpi,\sigma-\varpi)\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\sigma-\varpi)_i}{(\sigma)_i}\frac{h^i s^j}{i!\,j!}$$
$$= B(\varpi,\sigma-\varpi)\times F^{1:2;1}_{1:1;0}\left(\begin{matrix}\alpha:\beta,\sigma-\varpi;\beta'\\ \gamma:\sigma;-\end{matrix};h,s\right).$$

4) Consider the integral

$$A_4=\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{\varpi}F_1(\alpha;\beta,\beta';\gamma;h(1-t)^{\kappa},s)dt.$$

Then, using (20), we get

$$A_4=\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{\varpi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(h(1-t)^{\kappa})^i s^j}{i!\,j!}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\int_{-1}^{1}(1-t)^{i\kappa+\sigma}(t+1)^{\varpi}dt.$$

Using (13), (12), (8) and (24); we conclude that

$$A_4=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times 2^{i\kappa+\sigma+\varpi+1}B(i\kappa+\sigma+1,\varpi+1)$$
$$=2^{\sigma+\varpi+1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(2^{\kappa}h)^i s^j}{i!\,j!}\times\frac{\Gamma(i\kappa+\sigma+1)\Gamma(\varpi+1)}{\Gamma(i\kappa+\sigma+\varpi+2)}.$$

Particularly at $\kappa=1$, we have

$$A_4=2^{\sigma+\varpi+1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(2h)^i s^j}{i!\,j!}\times\frac{\Gamma(i+\sigma+1)\Gamma(\varpi+1)}{\Gamma(i+\sigma+\varpi+2)}$$
$$=2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\sigma+1)_i}{(\sigma+\varpi+2)_i}\frac{(2h)^i s^j}{i!\,j!}$$
$$=2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)\times F^{1:2;1}_{1:1;0}\left(\begin{matrix}\alpha:\beta,\sigma+1;\beta'\\ \gamma:\sigma+\varpi+2;-\end{matrix};2h,s\right).$$

Similarly, for $\kappa=1$; we can evaluate the following integral results:

$$\int_0^1 t^{-\sigma}(1-t)^{\sigma-\varpi-1}F_2(\alpha,\beta,\beta';\gamma,\gamma';ht,s)dt=B(1-\sigma,\sigma-\varpi)\times F^{1:2;1}_{0:2;1}\left(\begin{matrix}\alpha:\beta,1-\sigma;\beta'\\ -:\gamma,1-\varpi;\gamma'\end{matrix};h,s\right),\tag{25}$$

$$\int_0^1 t^{\sigma-1}(1-t)^{\varpi-1}F_2(\alpha,\beta,\beta';\gamma,\gamma';ht,s)dt=B(\sigma,\varpi)\times F^{1:2;1}_{0:2;1}\left(\begin{matrix}\alpha:\beta,\sigma;\beta'\\ -:\gamma,\sigma+\varpi;\gamma'\end{matrix};h,s\right),\tag{26}$$

$$\int_1^\infty t^{-\sigma}(t-1)^{\varpi-1}F_2(\alpha,\beta,\beta';\gamma,\gamma';\frac{h}{t},s)dt = B(\varpi,\sigma-\varpi)\times F^{1:2;1}_{0:2;1}\left(\begin{matrix}\alpha:\beta,\sigma-\varpi;\beta' \\ -:\gamma,\sigma;\gamma'\end{matrix};h,s\right), \tag{27}$$

$$\int_{-1}^1 (1-t)^\sigma (t+1)^\varpi F_2(\alpha,\beta,\beta';\gamma,\gamma';h(1-t)^\kappa,s)dt$$

$$= 2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)\times F^{1:2;1}_{0:2;1}\left(\begin{matrix}\alpha:\beta,\sigma+1;\beta' \\ -:\gamma,\sigma+\varpi+2;\gamma'\end{matrix};2h,s\right), \tag{28}$$

$$\int_0^1 t^{-\sigma}(1-t)^{\sigma-\varpi-1}F_3(\alpha,\alpha',\beta,\beta';\gamma;ht,s)dt$$

$$= B(1-\sigma,\sigma-\varpi)\times F^{0:3;2}_{1:1;0}\left(\begin{matrix}-:\alpha,\beta,1-\sigma;\alpha',\beta' \\ \gamma:1-\varpi;-\end{matrix};h,s\right), \tag{29}$$

$$\int_0^1 t^{\sigma-1}(1-t)^{\varpi-1}F_3(\alpha,\alpha',\beta,\beta';\gamma;ht,s)dt = B(\sigma,\varpi)\times F^{0:3;2}_{1:1;0}\left(\begin{matrix}-:\alpha,\beta,\sigma;\alpha',\beta' \\ \gamma:\sigma+\varpi;-\end{matrix};h,s\right), \tag{30}$$

$$\int_1^\infty t^{-\sigma}(t-1)^{\varpi-1}F_3(\alpha,\alpha',\beta,\beta';\gamma;\frac{h}{t},s)dt$$

$$= B(\varpi,\sigma-\varpi)\times F^{0:3;2}_{1:1;0}\left(\begin{matrix}-:\alpha,\beta,\sigma-\varpi;\alpha',\beta' \\ \gamma:\sigma;-\end{matrix};h,s\right), \tag{31}$$

$$\int_{-1}^1 (1-t)^\sigma (t+1)^\varpi F_3(\alpha,\alpha';\beta,\beta';\gamma;h(1-t)^\kappa,s)dt$$

$$= 2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)\times F^{0:3;2}_{1:1;0}\left(\begin{matrix}-;\alpha,\beta,\sigma+1;\alpha',\beta' \\ \gamma:\sigma+\varpi+2;-\end{matrix};2h,s\right), \tag{32}$$

$$\int_0^1 t^{-\sigma}(1-t)^{\sigma-\varpi-1}F_4(\alpha,\beta;\gamma,\gamma';ht,s)dt = B(1-\sigma,\sigma-\varpi)\times F^{2:1;0}_{0:2;1}\left(\begin{matrix}\alpha,\beta:1-\sigma;- \\ -:\gamma,1-\varpi;\gamma'\end{matrix};h,s\right), \tag{33}$$

$$\int_0^1 t^{\sigma-1}(1-t)^{\varpi-1}F_4(\alpha,\beta;\gamma,\gamma';ht,s)dt = B(\sigma,\varpi)\times F^{2:1;0}_{0:2;1}\left(\begin{matrix}\alpha,\beta:\sigma;- \\ -:\gamma,\sigma+\varpi;\gamma'\end{matrix};h,s\right), \tag{34}$$

$$\int_1^\infty t^{-\sigma}(t-1)^{\varpi-1}F_4(\alpha,\beta;\gamma,\gamma';\frac{h}{t},s)dt = B(\varpi,\sigma-\varpi)\times F^{2:1;0}_{0:2;1}\left(\begin{matrix}\alpha,\beta:\sigma-\varpi;- \\ -:\gamma,\sigma;\gamma'\end{matrix};h,s\right) \tag{35}$$

and

$$\int_{-1}^1 (1-t)^\sigma (t+1)^\varpi F_4(\alpha,\beta;\gamma,\gamma';h(1-t)^\kappa,s)dt$$

$$= 2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)\times F^{2:1;0}_{0:2;1}\left(\begin{matrix}\alpha,\beta:\sigma+1;- \\ -:\gamma,\sigma+\varpi+2;\gamma'\end{matrix};2h,s\right). \tag{36}$$

In this section, we evaluate some integral formulae involving the Appell functions with the Jacobi function:

5) Consider the integral

$$A_5=\int_{-1}^{1}t^{\lambda}(1-t)^{\sigma}(t+1)^{\varpi}P_i^{(\sigma,v)}(t)F_1(\alpha;\beta,\beta';\gamma;h(t+1)^{\kappa},s)dt.$$

Then, using (20), we get

$$A_5=\int_{-1}^{1}t^{\lambda}(1-t)^{\sigma}(t+1)^{\varpi}P_i^{(\sigma,v)}(t)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(h(1+t)^{\kappa})^i s^j}{i!\,j!}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\int_{-1}^{1}t^{\lambda}(1-t)^{\sigma}(t+1)^{\varpi+i\kappa}P_i^{(\sigma,v)}(t)dt.$$

Using (16), (7), (8), (12) and (24); it follows that

$$A_5=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times(-1)^i 2^{i\kappa+\sigma+\varpi+1}\frac{\Gamma(\varpi+i\kappa+1)\Gamma(i+\sigma+1)\Gamma(\varpi+i\kappa+v+1)}{i!\Gamma(\varpi+i\kappa+v+i+1)\Gamma(\varpi+i\kappa+\sigma+i+2)}$$
$$\times{}_3F_2\left(\begin{matrix}-\lambda,\varpi+i\kappa+v+1,\sigma+1\\ \varpi+i\kappa+v+i+1,\varpi+i\kappa+\sigma+i+2\end{matrix};1\right)$$
$$=2^{\sigma+\varpi+1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j(\sigma+1)_i}{(\gamma)_{i+j}}\frac{(-2^{\kappa}h)^i s^j}{i!\,j!}$$
$$\times\frac{\Gamma(\sigma+1)\Gamma(i\kappa+\varpi+1)\Gamma(\varpi+i\kappa+v+1)}{\Gamma(i\kappa+\varpi+v+i+1)\Gamma(i\kappa+i+\sigma+\varpi+2)}$$
$$\times{}_3F_3\left(\begin{matrix}-\lambda,\varpi+i\kappa+v+1,\sigma+1\\ \varpi+i\kappa+v+i+1,\varpi+i\kappa+\sigma+i+2,1\end{matrix};1\right).$$

Particularly for $\kappa=1$, we get

$$A_5=2^{\sigma+\varpi+1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j(\sigma+1)_i}{(\gamma)_{i+j}}\frac{(-2h)^i s^j}{i!\,j!}$$
$$\times\frac{\Gamma(\sigma+1)\Gamma(i+\varpi+1)\Gamma(\varpi+v+i+1)}{\Gamma(2i+\varpi+v+1)\Gamma(\sigma+\varpi+2i+2)}\times{}_3F_3\left(\begin{matrix}-\lambda,\varpi+i+v+1,\sigma+1\\ \varpi+v+2i+1,\varpi+\sigma+2i+2,1\end{matrix};1\right)$$
$$=2^{\sigma+\varpi+1}\times\Gamma(\sigma+1)\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j(\sigma+1)_i}{(\gamma)_{i+j}}\frac{(-2h)^i s^j}{i!\,j!}$$
$$\times\frac{(\varpi+1)_i\Gamma(\varpi+1)(\varpi+v+1)_i\Gamma(\varpi+v+1)}{\left(\frac{\varpi+v+1}{2}\right)_i\left(\frac{\varpi+v+2}{2}\right)_i\Gamma(\varpi+v+1)\left(\frac{\sigma+\varpi+2}{2}\right)_i\left(\frac{\sigma+\varpi+3}{2}\right)_i\Gamma(\sigma+\varpi+2)}$$
$$\times{}_3F_3\left(\begin{matrix}-\lambda,\varpi+i+v+1,\sigma+1\\ \varpi+v+2i+1,\varpi+\sigma+2i+2,1\end{matrix};1\right)$$
$$=2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)$$

$$\times F_{1:4;0}^{1:4;1}\left(\begin{array}{c}\alpha:\beta,\sigma+1,\varpi+1,\varpi+v+1;\beta^{'}\\ \gamma:\dfrac{\varpi+v+1}{2},\dfrac{\varpi+v+2}{2},\dfrac{\sigma+\varpi+2}{2},\dfrac{\sigma+\varpi+3}{2};-\end{array};-2h,s\right)$$
$$\times{}_3F_3\left(\begin{array}{c}-\lambda,\varpi+i+v+1,\sigma+1\\ \varpi+v+2i+1,\varpi+\sigma+2i+2,1\end{array};1\right).$$

6) Consider the integral

$$A_6=\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{v}P_i^{(\tau,v)}(t)F_1(\alpha;\beta,\beta^{'};\gamma;h(1-t)^{\kappa},s)dt.$$

Using (20), we obtain

$$A_6=\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{v}P_i^{(\tau,v)}(t)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(h(1-t)^{\kappa})^i s^j}{i!\,j!}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\int_{-1}^{1}(1-t)^{\sigma+i\kappa}(t+1)^{v}P_i^{(\tau,v)}(t)dt.$$

Using (15), (14), (13), (12), (7), (8) and (24); it follows that

$$A_6=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{h^i s^j}{i!\,j!}\times\int_{-1}^{1}(1-t)^{\sigma+i\kappa}(t+1)^{v}\times\frac{(\tau+1)_i}{i!}\times{}_2F_1\left(\begin{array}{c}-i,1+\tau+v+i\\ \tau+1\end{array};\frac{1-t}{2}\right)dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{h^i s^j}{i!\,j!}\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{2^p\,p!}$$
$$\times\int_{-1}^{1}(1-t)^{\sigma+i\kappa+p+1-1}(t+1)^{v+1-1}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{h^i s^j}{i!\,j!}\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{2^p\,p!}$$
$$\times 2^{\sigma+i\kappa+v+1}\times 2^p\times B(1+\sigma+i\kappa+p,v+1)$$
$$=2^{\sigma+v+1}\Gamma(v+1)\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{(2^{\kappa}h)^i s^j}{i!\,j!}$$
$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{p!}\times\frac{\Gamma(1+\sigma+i\kappa+p)\Gamma(v+1)}{\Gamma(2+\sigma+i\kappa+p+v)}$$
$$=2^{\sigma+v+1}\Gamma(v+1)\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{(2^{\kappa}h)^i s^j}{i!\,j!}$$
$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p(1+\sigma+i\kappa)_p}{(1+\tau)_p(2+\sigma+i\kappa+v)_p}\frac{1}{p!}\times\frac{\Gamma(1+\sigma+i\kappa)}{\Gamma(2+\sigma+i\kappa+v)}$$
$$=2^{\sigma+v+1}\Gamma(v+1)\times F_{1:0;0}^{1:2;1}\left(\begin{array}{c}\alpha:\beta,\tau+1;\beta^{'};\\ \gamma:-;-;\end{array}2^{\kappa}h,s\right)\times{}_3F_2\left(\begin{array}{c}-i,1+\tau+v+i,1+\sigma+i\kappa\\ 1+\tau,2+\sigma+i\kappa+v\end{array};1\right)$$

$$\times\frac{\Gamma(1+\sigma+i\kappa)}{\Gamma(2+\sigma+i\kappa+v)}.$$

Particularly at $\kappa=1$, we get

$$A_6=2^{\sigma+v+1}\Gamma(v+1)\times F^{1:2;1}_{1:0;0}\left(\begin{matrix}\alpha:\beta,\tau+1;\beta';\\ \gamma:-;-;\end{matrix}2h,s\right)\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i\\ 1+\tau,2+\sigma+i+v\end{matrix};1\right)$$
$$\times\frac{\Gamma(1+\sigma+i)}{\Gamma(2+\sigma+i+v)}$$
$$=2^{\sigma+v+1}\times B(1+v,1+\sigma)\times F^{1:3;1}_{1:1;0}\left(\begin{matrix}\alpha:\beta,\tau+1,1+\sigma;\beta'\\ \gamma:2+\sigma+v;-\end{matrix};2^{\kappa}h,s\right)$$
$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i\\ 1+\tau,2+\sigma+i+v\end{matrix};1\right).$$

7) Consider the integral

$$A_7=\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{\omega}P_i^{(\tau,v)}(t)F_1(\alpha,\beta,\beta';\gamma;z(1-t)^{\kappa}(1+t)^{h},s)dt.$$

Then, using (20), we get

$$A_7=\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{\omega}P_i^{(\tau,v)}(t)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(z(1-t)^{\kappa}(1+t)^{h})^{i}s^{j}}{i!\,j!}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{z^is^j}{i!\,j!}\int_{-1}^{1}(1-t)^{\sigma+i\kappa}(t+1)^{\omega+ih}P_i^{(\tau,v)}(t)dt.$$

Using (15), (14), (13), (12), (7), (8), and (24); we have

$$A_7=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{z^is^j}{i!\,j!}$$
$$\times\int_{-1}^{1}(1-t)^{\sigma+i\kappa}(t+1)^{\omega+ih}\times\frac{(\tau+1)_i}{i!}\times{}_2F_1\left(\begin{matrix}-i,1+\tau+v+i\\ \tau+1\end{matrix};\frac{1-t}{2}\right)dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{z^is^j}{i!\,j!}\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{2^p\,p!}$$
$$\times\int_{-1}^{1}(1-t)^{\sigma+i\kappa+p+1-1}(t+1)^{\omega+ih+1-1}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{(1)_i}\frac{z^is^j}{i!\,j!}\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{2^p\,p!}$$
$$\times2^{\sigma+i\kappa++ih+1}\times2^{p}\times B(1+\sigma+i\kappa+p,\omega+ih+1)$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{(1)_i}\frac{z^is^j}{i!\,j!}$$

$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{p!}\times 2^{\sigma+i\kappa+v+1}\times\frac{\Gamma(1+\sigma+i\kappa+p)\Gamma(v+1)}{\Gamma(2+\sigma+i\kappa+p+v)}$$

$$=2^{\sigma+v+1}\Gamma(v+1)\times F_{1:1;0}^{1:2;1}\left(\begin{matrix}\alpha:\beta,\tau+1;\beta' \\ \gamma:1;-\end{matrix};2^{\kappa}h,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i\kappa \\ 1+\tau,2+\sigma+i\kappa+v\end{matrix};1\right)\times\frac{\Gamma(1+\sigma+i\kappa)}{\Gamma(2+\sigma+i\kappa+v)}.$$

Particularly for $\kappa=1$, we get

$$A_7=2^{\sigma+v+1}\Gamma(v+1)\times F_{1:1;0}^{1:2;1}\left(\begin{matrix}\alpha:\beta,\tau+1;\beta' \\ \gamma:1;-\end{matrix};2h,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i \\ 1+\tau,2+\sigma+i+v\end{matrix};1\right)\times\frac{\Gamma(1+\sigma+i)}{\Gamma(2+\sigma+i+v)}$$

$$=2^{\sigma+v+1}\times B(1+v,1+\sigma)\times F_{1:2;0}^{1:3;1}\left(\begin{matrix}\alpha:\beta,\tau+1,1+\sigma;\beta' \\ \gamma:1,2+\sigma+v;-\end{matrix};2h,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i \\ 1+\tau,2+\sigma+i+v\end{matrix};1\right).$$

8) Consider the integral

$$A_8=\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\omega}P_i^{(\tau,v)}(t)F_1(\alpha,\beta,\beta';\gamma;z(1+t)^{-r},s)dt.$$

Then, using (20), we get

$$A_8=\int_{-1}^{1}(1-t)^{\xi}(t+1)^{\omega}P_i^{(\tau,v)}(t)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(z(1+t)^{-r})^i s^j}{i!\,j!}dt$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{z^i s^j}{i!\,j!}\times\int_{-1}^{1}(1-t)^{\xi}(t+1)^{\omega-ir}P_i^{(\tau,v)}(t)dt.$$

Using (15), (14), (13), (12), (7), (8) and (24); we get

$$A_8=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{z^i s^j}{i!\,j!}\int_{-1}^{1}(1-t)^{\xi}(t+1)^{\omega-ir}\frac{(\tau+1)_i}{i!}\times{}_2F_1\left(\begin{matrix}-i,1+\tau+v+i \\ \tau+1\end{matrix};\frac{1-t}{2}\right)dt$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{z^i s^j}{i!\,j!}$$

$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{2^p\,p!}\times\int_{-1}^{1}(1-t)^{\xi+p}(t+1)^{\omega-ir}dt$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{z^i s^j}{i!\,j!}$$

$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{2^p p!}\times 2^{\xi+\omega-ir+1}\times 2^p\times B(1+\xi+p,\omega-ir+1)$$

$$=2^{\xi+\omega+1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{(1)_i}\frac{(2^{-r}z)^i s^j}{i!\,j!}$$

$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{p!}\times\frac{\Gamma(1+\xi+p)\Gamma(\omega-ir+1)}{\Gamma(2+\xi+p+\omega-ir)}$$

$$=2^{\xi+\omega+1}\Gamma(1+\xi)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{(1)_i}\frac{(2^{-r}z)^i s^j}{i!\,j!}\times\frac{\Gamma(\omega-ir+1)}{\Gamma(2+\xi+\omega-ir)}$$

$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{p!}\frac{(1+\xi)_p}{(2+\xi+\omega-ir)_p}$$

$$=2^{\xi+\omega+1}\Gamma(1+\xi)F_{1:1;0}^{1:2;1}\left(\begin{matrix}\alpha:\beta,\tau+1;\beta'\\ \gamma:1;-\end{matrix};2^{-r}z,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega-ir\end{matrix};1\right)\times\frac{\Gamma(\omega-ir+1)}{\Gamma(2+\xi+\omega-ir)}.$$

Particularly at $r=-1$, we get

$$A_8=2^{\xi+\omega+1}\Gamma(1+\xi)F_{1:1;0}^{1:2;1}\left(\begin{matrix}\alpha:\beta,\tau+1;\beta'\\ \gamma:1;-\end{matrix};2z,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega+i\end{matrix};1\right)\times\frac{\Gamma(\omega+i+1)}{\Gamma(2+\xi+\omega+i)}$$

$$=2^{\xi+\omega+1}B(1+\xi,\omega+1)F_{1:2;0}^{1:3;1}\left(\begin{matrix}\alpha:\beta,\tau+1,\omega+1;\beta'\\ \gamma:1,2+\xi+\omega;-\end{matrix};2z,s\right)\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega+i\end{matrix};1\right).$$

9) Consider the integral

$$A_9=\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\omega}P_i^{(\tau,v)}(t)F_1(\alpha,\beta,\beta';\gamma;z(1-t)^l(1+t)^{-r},s)dt.$$

Then, using (20), we obtain

$$A_9=\int_{-1}^{1}(1-t)^{\xi}(t+1)^{\omega}P_i^{(\tau,v)}(t)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(z(1-t)^l(1+t)^{-r})^i s^j}{i!\,j!}dt$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{z^i s^j}{i!\,j!}\times\int_{-1}^{1}(1-t)^{\xi+il}(t+1)^{\omega-ir}P_i^{(\tau,v)}(t)dt.$$

Using (15), (14), (13), (12), (7), (8) and (24); we have

$$A_9=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{z^i s^j}{i!\,j!}$$

$$\times\int_{-1}^{1}(1-t)^{\xi+il}(t+1)^{\omega-ir}\frac{(\tau+1)_i}{i!}\times{}_2F_1\left(\begin{matrix}-i,1+\tau+v+i\\ \tau+1\end{matrix};\frac{1-t}{2}\right)dt$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{z^i s^j}{i!\,j!}$$

$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{2^p p!}\times\int_{-1}^{1}(1-t)^{\xi+il+p}(t+1)^{\omega-ir}dt$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{i!}\frac{z^i s^j}{i!\,j!}$$

$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{2^p p!}2^{\xi+il+\omega-ir+1}\times 2^p\times B(1+\xi+il+p,\omega-ir+1)$$

$$=2^{\xi+\omega+1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(\tau+1)_i}{(1)_i}\frac{(2^{l-r}z)^i s^j}{i!\,j!}$$

$$\times\sum_{p=0}^{\infty}\frac{(-i)_p(1+\tau+v+i)_p}{(1+\tau)_p}\frac{1}{p!}\times\frac{\Gamma(1+\xi+il+p)\Gamma(\omega-ir+1)}{\Gamma(2+\xi+il+p+\omega-ir)}$$

$$=2^{\xi+\omega+1}F^{1:2;1}_{1:1;0}\left(\begin{matrix}\alpha:\beta,\tau+1;\beta^{'}\\ \gamma:1;-\end{matrix};2^{l-r}z,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi+il\\ 1+\tau,2+\xi+il+\omega-ir\end{matrix};1\right)\times\frac{\Gamma(1+\xi+il)\Gamma(\omega-ir+1)}{\Gamma(2+\xi+il+\omega-ir)}.$$

Particularly for $l=1$ and $r=-1$; we get

$$A_9=2^{\xi+\omega+1}F^{1:2;1}_{1:1;0}\left(\begin{matrix}\alpha:\beta,\tau+1;\beta^{'}\\ \gamma:1;-\end{matrix};4z,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega+2i\end{matrix};1\right)\times\frac{\Gamma(1+\xi+i)\Gamma(\omega+i+1)}{\Gamma(2+\xi+\omega+2i)}$$

$$=2^{\xi+\omega+1}B(1+\xi,1+\omega)F^{1:4;1}_{1:3;0}\left(\begin{matrix}\alpha:\beta,\tau+1,1+\xi,1+\omega;\beta^{'}\\ \gamma:1,\dfrac{2+\xi+\omega}{2},\dfrac{3+\xi+\omega}{2};-\end{matrix};4z,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega+2i\end{matrix};1\right).$$

Similarly, for $\kappa=l=1$ and $r=-1$; we can evaluate the following integral results:

$$\int_{-1}^{1}t^{\lambda}(1-t)^{\sigma}(t+1)^{\varpi}P_i^{(\sigma,v)}(t)F_2(\alpha,\beta,\beta^{'};\gamma,\gamma^{'};h(t+1)^{\kappa},s)dt$$

$$=2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)\times F^{1:4;1}_{0:5;1}\left(\begin{matrix}\alpha:\beta,\sigma+1,\varpi+1,\varpi+v+1;\beta^{'}\\ -:\gamma,\dfrac{\varpi+v+1}{2},\dfrac{\varpi+v+2}{2},\dfrac{\sigma+\varpi+2}{2},\dfrac{\sigma+\varpi+3}{2};\gamma^{'}\end{matrix};-2h,s\right)$$

$$\times {}_3F_3\left(\begin{matrix}-\lambda,\varpi+i+v+1,\sigma+1\\ \varpi+v+2i+1,\varpi+\sigma+2i+2,1\end{matrix};1\right), \tag{37}$$

$$\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{v}P_i^{(\tau,v)}(t)F_2(\alpha,\beta,\beta^{'};\gamma,\gamma^{'};h(1-t)^{\kappa},s)dt$$

$$=2^{\sigma+v+1}\frac{\Gamma(v+1)\Gamma(1+\sigma)}{\Gamma(2+\sigma+v)}\times F_{0;2;1}^{1;3;1}\left(\begin{matrix}\alpha:\beta,\tau+1,1+\sigma;\beta^{'}\\ -:\gamma,2+\sigma+v;\gamma^{'}\end{matrix};2^{\kappa}h,s\right)$$

$$\times {}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i\\ 1+\tau,2+\sigma+i+v\end{matrix};1\right), \tag{38}$$

$$\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{\omega}P_i^{(\tau,v)}(t)F_2(\alpha;\beta,\beta^{'};\gamma,\gamma^{'};z(1-t)^{\kappa}(1+t)^{h},s)dt$$

$$=2^{\sigma+v+1}\times B(1+v,1+\sigma)\times F_{1:2;0}^{0:4;0}\left(\begin{matrix}-:\alpha,\beta,\tau+1,1+\sigma;-\\ \gamma:1,2+\sigma+v;-\end{matrix};2^{\kappa}h,s\right)$$

$$\times {}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i\\ 1+\tau,2+\sigma+i+v\end{matrix};1\right), \tag{39}$$

$$\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\omega}P_i^{(\tau,v)}(t)F_2(\alpha,\beta,\beta^{'};\gamma,\gamma^{'};z(1+t)^{-r},s)dt$$

$$=2^{\xi+\omega+1}B(1+\xi,\omega+1)F_{0:3;1}^{1:3;1}\left(\begin{matrix}\alpha:\beta,\tau+1,\omega+1;\beta^{'}\\ -:\gamma,1,2+\xi+\omega;\gamma^{'}\end{matrix};2z,s\right)\times {}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega+i\end{matrix};1\right), \tag{40}$$

$$\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\omega}P_i^{(\tau,v)}(t)F_2(\alpha;\beta,\beta^{'};\gamma,\gamma^{'};z(1-t)^{l}(1+t)^{-r},s)dt$$

$$=2^{\xi+\omega+1}B(1+\xi,1+\omega)F_{0:4;1}^{1:4;1}\left(\begin{matrix}\alpha:\beta,\tau+1,1+\xi,1+\omega;\beta^{'}\\ -:\gamma,1,\frac{2+\xi+\omega}{2},\frac{3+\xi+\omega}{2};\gamma^{'}\end{matrix};4z,s\right)$$

$$\times {}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega+2i\end{matrix};1\right), \tag{41}$$

$$\int_{-1}^{1}t^{\lambda}(1-t)^{\sigma}(t+1)^{\varpi}P_i^{(\sigma,v)}(t)F_3(\alpha,\alpha^{'},\beta,\beta^{'};\gamma;h(t+1)^{\kappa},s)dt$$

$$=2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)$$

$$\times F_{1:4;0}^{0:5;2}\left(\begin{matrix}-:\alpha,\beta,\sigma+1,\varpi+1,\varpi+v+1;\alpha^{'},\beta^{'}\\ \gamma:\frac{\varpi+v+1}{2},\frac{\varpi+v+2}{2},\frac{\sigma+\varpi+2}{2},\frac{\sigma+\varpi+3}{2};-\end{matrix};-2h,s\right)$$

$$\times {}_3F_3\left(\begin{matrix}-\lambda,\varpi+i+v+1,\sigma+1\\ \varpi+v+2i+1,\varpi+\sigma+2i+2,1\end{matrix};1\right), \tag{42}$$

$$\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{v}P_i^{(\tau,v)}(t)F_3(\alpha,\alpha',\beta,\beta';\gamma;h(1-t)^{\kappa},s)dt$$

$$=2^{\sigma+v+1}\times B(1+v,1+\sigma)\times F_{1:1;0}^{0:4;2}\left(\begin{matrix}-:\alpha,\beta,\tau+1,1+\sigma;\alpha',\beta' \\ \gamma:2+\sigma+v;-\end{matrix};2h,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i \\ 1+\tau,2+\sigma+i+v\end{matrix};1\right), \tag{43}$$

$$\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{\omega}P_i^{(\tau,v)}(t)F_3(\alpha,\alpha'\beta,\beta';\gamma;z(1-t)^{\kappa}(1+t)^{h},s)dt$$

$$=2^{\sigma+v+1}\times B(1+v,1+\sigma)\times F_{1:2;0}^{0:4;2}\left(\begin{matrix}-:\alpha,\beta,\tau+1,1+\sigma;\alpha',\beta' \\ \gamma:1,2+\sigma+v;-\end{matrix};2h,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i \\ 1+\tau,2+\sigma+i+v\end{matrix};1\right), \tag{44}$$

$$\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\omega}P_i^{(\tau,v)}(t)F_3(\alpha,\alpha',\beta,\beta';\gamma;z(1+t)^{-r},s)dt$$

$$=2^{\xi+\omega+1}B(1+\xi,\omega+1)F_{1:2;0}^{0:4;2}\left(\begin{matrix}-:\alpha,\beta,\tau+1,\omega+1;\alpha',\beta' \\ \gamma:1,2+\xi+\omega;-\end{matrix};2z,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi \\ 1+\tau,2+\xi+\omega+i\end{matrix};1\right), \tag{45}$$

$$\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\omega}P_i^{(\tau,v)}(t)F_3(\alpha,\alpha',\beta,\beta';\gamma;z(1-t)^{l}(1+t)^{-r},s)dt$$

$$=2^{\xi+\omega+1}B(1+\xi,1+\omega)F_{1:3;0}^{0:5;2}\left(\begin{matrix}-:\alpha,\beta,\tau+1,1+\xi,1+\omega;\alpha',\beta' \\ \gamma:1,\dfrac{2+\xi+\omega}{2},\dfrac{3+\xi+\omega}{2};-\end{matrix};4z,s\right)$$

$$\times{}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi \\ 1+\tau,2+\xi+\omega+2i\end{matrix};1\right), \tag{46}$$

$$\int_{-1}^{1}t^{\lambda}(1-t)^{\sigma}(t+1)^{\varpi}P_i^{(\sigma,v)}(t)F_4(\alpha,\beta;\gamma,\gamma';h(t+1)^{\kappa},s)dt$$

$$=2^{\sigma+\varpi+1}\times B(\sigma+1,\varpi+1)$$

$$\times F_{0:5;1}^{2:3;0}\left(\begin{matrix}\alpha,\beta:\sigma+1,\varpi+1,\varpi+v+1;- \\ -:\gamma,\dfrac{\varpi+v+1}{2},\dfrac{\varpi+v+2}{2},\dfrac{\sigma+\varpi+2}{2},\dfrac{\sigma+\varpi+3}{2};\gamma'\end{matrix};-2h,s\right)$$

$${}_3F_3\left(\begin{matrix}-\lambda,\varpi+i+v+1,\sigma+1 \\ \varpi+v+2i+1,\varpi+\sigma+2i+2,1\end{matrix};1\right), \tag{47}$$

$$\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{v}P_i^{(\tau,v)}(t)F_4(\alpha;\beta;\gamma,\gamma';h(1-t)^{\kappa},s)dt$$

$$=2^{\sigma+v+1}\times B(1+v,1+\sigma)\times F_{0;2;1}^{2;2;0}\left(\begin{matrix}\alpha,\beta:\tau+1,1+\sigma;- \\ -:\gamma,2+\sigma+v;\gamma'\end{matrix};2^{\kappa}h,s\right)$$

$$\times {}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i\\ 1+\tau,2+\sigma+i+v\end{matrix};1\right), \tag{48}$$

$$\int_{-1}^{1}(1-t)^{\sigma}(t+1)^{\varpi}P_i^{(\tau,v)}(t)F_4(\alpha,\beta;\gamma,\gamma';z(1-t)^{\kappa}(1+t)^{h},s)dt$$
$$=2^{\sigma+v+1}\times B(1+v,1+\sigma)\times F_{0:3;1}^{2:2;0}\left(\begin{matrix}\alpha,\beta:\tau+1,1+\sigma;-\\ -:\gamma,1,2+\sigma+v;\gamma'\end{matrix};2h,s\right)$$
$$\times {}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\sigma+i\\ 1+\tau,2+\sigma+i+v\end{matrix};1\right), \tag{49}$$

$$\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\varpi}P_i^{(\tau,v)}(t)F_4(\alpha,\beta;\gamma,\gamma';z(1+t)^{-r},s)dt$$
$$=2^{\xi+\omega+1}B(1+\xi,\omega+1)F_{0:3;1}^{2:2;0}\left(\begin{matrix}\alpha,\beta:\tau+1,\omega+1;-\\ -:\gamma,1,2+\xi+\omega;\gamma'\end{matrix};2z,s\right)$$
$$\times {}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega+i\end{matrix};1\right) \tag{50}$$

and

$$\int_{-1}^{1}(1-t)^{\xi}(1+t)^{\varpi}P_i^{(\tau,v)}(t)F_4(\alpha,\beta;\gamma,\gamma';z(1-t)^{l}(1+t)^{-r},s)dt$$
$$=2^{\xi+\omega+1}B(1+\xi,1+\omega)F_{0:4;1}^{2:3;0}\left(\begin{matrix}\alpha,\beta:\tau+1,1+\xi,1+\omega;-\\ -:\gamma,1,\dfrac{2+\xi+\omega}{2},\dfrac{3+\xi+\omega}{2};\gamma'\end{matrix};4z,s\right)$$
$$\times {}_3F_2\left(\begin{matrix}-i,1+\tau+v+i,1+\xi\\ 1+\tau,2+\xi+\omega+2i\end{matrix};1\right). \tag{51}$$

Multiplied with the Legendre function, here we calculate some integrals involving the Appell function in this section:

10) Consider the integral

$$A_{10}=\int_0^1 t^{\varpi-1}(1-t^2)^{\frac{\tau}{2}}P_v^{\tau}(t)F_1(\alpha,\beta,\beta';\gamma;zt^{\xi},s)dt.$$

Then, using (20), we get

$$A_{10}=\int_0^1 t^{\varpi-1}(1-t^2)^{\frac{\tau}{2}}P_v^{\tau}(t)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(zt^{\xi})^i s^j}{i!\,j!}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{z^i s^j}{i!\,j!}\times\int_0^1 t^{\varpi+i\xi-1}(1-t^2)^{\frac{\tau}{2}}P_v^{\tau}(t)dt.$$

Using (18), (9), (8) and (24); we conclude that

$$A_{10} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{z^i s^j}{i!\,j!}\frac{(-1)^{\tau}2^{-\omega-i\xi-\tau}\sqrt{\pi}\Gamma(\omega+i\xi)\Gamma(1+\tau+v)}{\Gamma(1-\tau+v)\Gamma\left(\frac{1+\omega+i\xi+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+i\xi+\tau+v}{2}\right)}$$

$$= (-1)^{\tau}2^{-\omega-\tau}\sqrt{\pi}\frac{\Gamma(1+\tau+v)}{\Gamma(1-\tau+v)}$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(2^{-\xi}z)^i s^j}{i!\,j!}\frac{\Gamma(\omega+i\xi)}{\Gamma\left(\frac{1+\omega+i\xi+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+i\xi+\tau+v}{2}\right)}.$$

Particularly for $\xi = 2$, we get

$$A_{10} = (-1)^{\tau}2^{-\omega-\tau}\sqrt{\pi}\frac{\Gamma(1+\tau+v)}{\Gamma(1-\tau+v)}$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{\left(\frac{z}{4}\right)^i s^j}{i!\,j!}\frac{\Gamma(\omega+2i)}{\Gamma\left(\frac{1+\omega+2i+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+2i+\tau+v}{2}\right)}$$

$$= \frac{(-1)^{\tau}2^{-\omega-\tau}\sqrt{\pi}\Gamma(1+\tau+v)\Gamma(\omega)}{\Gamma(1-\tau+v)\Gamma\left(\frac{1+\omega+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+\tau+v}{2}\right)}$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{\left(\frac{z}{4}\right)^i s^j}{i!\,j!}\frac{\left(\frac{\omega}{2}\right)_i\left(\frac{\omega+1}{2}\right)_i}{\left(\frac{1+\omega+\tau-v}{2}\right)_i\left(\frac{2+\omega+\tau+v}{2}\right)_i}$$

$$= \frac{(-1)^{\tau}2^{-\omega-\tau}\sqrt{\pi}\Gamma(1+\tau+v)\Gamma(\omega)}{\Gamma(1-\tau+v)\Gamma\left(\frac{1+\omega+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+\tau+v}{2}\right)}$$

$$\times F_{1:3;0}^{1:3;1}\left(\begin{array}{c}\alpha:\beta,\frac{\omega}{2},\frac{\omega+1}{2};\beta' \\ \gamma:1,\frac{\omega+\tau-v+1}{2},\frac{\omega+\tau+v+2}{2};-\end{array};\frac{z}{4},s\right).$$

11) Consider the integral

$$A_{11} = \int_0^1 t^{\omega-1}(1-t^2)^{\frac{-\tau}{2}}P_v^{\tau}(t)F_1(\alpha,\beta,\beta';\gamma;zt^{\xi},s)dt.$$

Then, using (20), we get

$$A_{11} = \int_0^1 t^{\omega-1}(1-t^2)^{\frac{-\tau}{2}}P_v^{\tau}(t)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_{i+j}}\frac{(zt^{\xi})^i s^j}{i!\,j!}dt$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{z^i s^j}{i!\,j!}\times\int_0^1 t^{\omega+i\xi-1}(1-t^2)^{\frac{-\tau}{2}}P_v^{\tau}(t)dt.$$

Using (19), (9), (8), and (24); we conclude that

$$A_{11}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{z^i s^j}{i!\,j!}\frac{2^{\tau-\omega-i\xi}\sqrt{\pi}\Gamma(\omega+i\xi)}{\Gamma\left(\frac{1+\omega+i\xi-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+i\xi-\tau-v}{2}\right)}$$

$$=2^{\tau-\omega}\sqrt{\pi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{(2^{-\xi}z)^i s^j}{i!\,j!}\frac{\Gamma(\omega+i\xi)}{\Gamma\left(\frac{1+\omega+i\xi-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+i\xi-\tau-v}{2}\right)}.$$

Particularly for $\xi=2$, we have

$$A_{11}=2^{\tau-\omega}\sqrt{\pi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{\left(\frac{z}{4}\right)^i s^j}{i!\,j!}\frac{\Gamma(\omega+2i)}{\Gamma\left(\frac{1+\omega+2i-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+2i-\tau-v}{2}\right)}$$

$$=\frac{2^{\tau-\omega}\sqrt{\pi}\Gamma(\omega)}{\Gamma\left(\frac{1+\omega-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega-\tau-v}{2}\right)}$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}(\beta)_i(\beta^{'})_j}{(\gamma)_{i+j}}\frac{\left(\frac{z}{4}\right)^i s^j}{i!\,j!}\frac{\left(\frac{\omega}{2}\right)_i\left(\frac{\omega+1}{2}\right)_i}{\left(\frac{1+\omega-\tau-v}{2}\right)_i\left(\frac{2+\omega-\tau-v}{2}\right)_i}$$

$$=\frac{2^{\tau-\omega}\sqrt{\pi}\Gamma(\omega)}{\Gamma\left(\frac{1+\omega-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega-\tau-v}{2}\right)}\times F_{1:3;0}^{1:3;1}\left(\begin{array}{c}\alpha:\beta,\frac{\omega}{2},\frac{\omega+1}{2};\beta^{'}\\ \gamma:1,\frac{\omega-\tau-v+1}{2},\frac{\omega-\tau-v+2}{2};-\end{array};\frac{z}{4},s\right).$$

Similarly, for $\xi=2$; we can evaluate the following integral results

$$\int_0^1 t^{\omega-1}(1-t^2)^{\frac{\tau}{2}}P_v^{\tau}(t)F_2(\alpha,\beta,\beta^{'};\gamma,\gamma^{'};zt^{\xi},s)dt$$

$$=\frac{(-1)^{\tau}2^{-\omega-\tau}\sqrt{\pi}\Gamma(1+\tau+v)\Gamma(\omega)}{\Gamma(1-\tau+v)\Gamma\left(\frac{1+\omega+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+\tau+v}{2}\right)}$$

$$\times F_{0:4;1}^{1:3;1}\left(\begin{array}{c}\alpha:\beta,\frac{\omega}{2},\frac{\omega+1}{2};\beta^{'}\\ -:\gamma,1,\frac{\omega+\tau-v+1}{2},\frac{\omega+\tau+v+2}{2};\gamma^{'}\end{array};\frac{z}{4},s\right),\tag{52}$$

$$\int_0^1 t^{\omega-1}(1-t^2)^{\frac{-\tau}{2}} P_v^{\tau}(t) F_2(\alpha,\beta,\beta';\gamma,\gamma';zt^{\xi},s)dt = \frac{2^{\tau-\omega}\sqrt{\pi}\Gamma(\omega)}{\Gamma\left(\frac{1+\omega-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega-\tau-v}{2}\right)}$$

$$\times F^{1:3;1}_{0:4;1}\left(\begin{matrix}\alpha:\beta,\frac{\omega}{2},\frac{\omega+1}{2};\beta' \\ -:\gamma,1,\frac{\omega-\tau-v+1}{2},\frac{\omega-\tau-v+2}{2};\gamma'\end{matrix};\frac{z}{4},s\right), \tag{53}$$

$$\int_0^1 t^{\omega-1}(1-t^2)^{\frac{\tau}{2}} P_v^{\tau}(t) F_3(\alpha,\alpha',\beta,\beta';\gamma;zt^{\xi},s)dt$$

$$= \frac{(-1)^{\tau}2^{-\omega-\tau}\sqrt{\pi}\Gamma(1+\tau+v)\Gamma(\omega)}{\Gamma(1-\tau+v)\Gamma\left(\frac{1+\omega+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+\tau+v}{2}\right)}$$

$$\times F^{0:4;2}_{1:3;0}\left(\begin{matrix}-:\alpha,\beta,\frac{\omega}{2},\frac{\omega+1}{2};\alpha',\beta' \\ \gamma:1,\frac{\omega+\tau-v+1}{2},\frac{\omega+\tau+v+2}{2};-\end{matrix};\frac{z}{4},s\right), \tag{54}$$

$$\int_0^1 t^{\omega-1}(1-t^2)^{\frac{-\tau}{2}} P_v^{\tau}(t) F_3(\alpha,\alpha',\beta,\beta';\gamma;zt^{\xi},s)dt$$

$$= \frac{2^{\tau-\omega}\sqrt{\pi}\Gamma(\omega)}{\Gamma\left(\frac{1+\omega-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega-\tau-v}{2}\right)} \times F^{0:4;2}_{1:3;0}\left(\begin{matrix}-:\alpha,\beta,\frac{\omega}{2},\frac{\omega+1}{2};\alpha',\beta' \\ \gamma:1,\frac{\omega-\tau-v+1}{2},\frac{\omega-\tau-v+2}{2};-\end{matrix};\frac{z}{4},s\right), \tag{55}$$

$$\int_0^1 t^{\omega-1}(1-t^2)^{\frac{\tau}{2}} P_v^{\tau}(t) F_4(\alpha,\beta;\gamma,\gamma';zt^{\xi},s)dt$$

$$= \frac{(-1)^{\tau}2^{-\omega-\tau}\sqrt{\pi}\Gamma(1+\tau+v)\Gamma(\omega)}{\Gamma(1-\tau+v)\Gamma\left(\frac{1+\omega+\tau-v}{2}\right)\Gamma\left(\frac{2+\omega+\tau+v}{2}\right)}$$

$$\times F^{2:2;0}_{0:4;1}\left(\begin{matrix}\alpha,\beta:\frac{\omega}{2},\frac{\omega+1}{2};- \\ -:\gamma,1,\frac{\omega+\tau-v+1}{2},\frac{\omega+\tau+v+2}{2};\gamma'\end{matrix};\frac{z}{4},s\right) \tag{56}$$

and

$$\int_0^1 t^{\omega-1}(1-t^2)^{\frac{-\tau}{2}} P_v^{\tau}(t) F_4(\alpha,\beta;\gamma,\gamma';zt^{\xi},s)dt = \frac{2^{\tau-\omega}\sqrt{\pi}\Gamma(\omega)}{\Gamma\left(\frac{1+\omega-\tau-v}{2}\right)\Gamma\left(\frac{2+\omega-\tau-v}{2}\right)}$$

$$\times F^{2:2;0}_{0:4;1}\left(\begin{matrix}\alpha,\beta:\frac{\omega}{2},\frac{\omega+1}{2};- \\ -:\gamma,1,\frac{\omega-\tau-v+1}{2},\frac{\omega-\tau-v+2}{2};\gamma'\end{matrix};\frac{z}{4},s\right). \tag{57}$$

3.2. DISCUSSION

Definite integrals of four Appell using Algebraic functions, Jacobi function, and Legendre function as products in integrand have been calculated in this paper using Pochhammer symbols, Factorial function Gamma, Beta, and Hypergeometric functions in the process. For most of the integrals, special cases have also been discussed.

4. CONCLUSION

Integrals calculated in this work are limited and are analogous to the work of Suther *et al.* (2020). We can extend and further investigate integrals containing similar double and higher power series using these results. The material is original and not previously published/submitted elsewhere.

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