ORIGINAL PAPER

EUCLIDEAN SURFACES WITH CONFORMAL SECOND FUNDAMENTAL FORM

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Manuscript received: 21.05.2024. Accepted paper: 16.12.2024;
Published online: 30.03.2025.

Abstract. *In the present study, we consider surfaces in* E^{2+d} *with conformal second fundamental form. We give a result for Chen surfaces to have a conformal second fundamental form. We show that every Chen surface is h-conformal. Furthermore, we give necessary and sufficient conditions for general rotational surfaces in Euclidean 4-space to become h-conformal. Finally, we prove that the Vranceanu surface is h-conformal if and only if it is either flat or minimal.*

Keywords: *Second fundamental form; conformal; Chen surface; Casorati curvature.*

1. INTRODUCTION

Let *M* be an *n*-dimensional oriented submanifold in an *(n+d)*-dimensional Riemannian manifold *N*. We can choose a local orthonormal frame $\{e_1,...,e_n,n_1,...,n_d\}$ on *M* such that $e_1,...,e_n$ are tangent vectors of *M* and $n_1,...,n_d$ are normal vectors of *M*. Then the second fundamental form *h* of *M* is called conformal (i.e. *M* is *h*-conformal) if for any vector fields *v* and *w* on the normal bundle $T^{\perp}(M)$

$$\sum_{i,j=1}^{n}\langle h(e_i,e_j),v\rangle\langle h(e_i,e_j),w\rangle=\lambda^2\langle v,w\rangle,$$

holds, where $\langle,\rangle$ denotes the induced metric on *N* and λ is a smooth function on *M* (see, [1, 2]). In [2], S. Console gave a characterization of a compact connected surface in *(2+d)*-dimensional space form of constant curvature *c (n=2,3)* with parallel mean curvature vector field *H*. He showed that in any case *M* is a pseudo umbilical surface and it is either minimal in $R^{2+d}(c)$ or minimally immersed in a small hypersphere of $R^{2+d}(c)$. In [3], X. Mo considered a submanifold with a conformal second fundamental form.

The Casorati curvature of a *n*-dimensional submanifold *M* of a *(n+d)*-dimensional Riemannian manifold *N* is the normalized square of the length of the second fundamental form of the submanifold (see, [4]). The notion of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold. The geometrical meaning and importance of the Casorati curvature are discussed by several geometers in [5, 6]. The Casorati curvature was used to obtain optimal inequalities between intrinsic and extrinsic curvatures of submanifolds in real space forms and different ambient spaces.

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https://doi.org/10.46939/J.Sci.Arts-25.1-a11

The rotational surfaces in E^3 play an important role in surface modeling [7]. General rotational surfaces are the extension of these surfaces which were first introduced by C. Moore in 1919 (see, [8]). Further, many researchers studied this area (see [9-14]). The rotational surfaces in E^4 with constant curvatures are studied in [15, 16] independently. For higher dimensional cases one can find results for the rotational surfaces [17].

In [18] B Y. Chen defined the allied vector $a(v)$ of a normal vector field *v*. For the mean curvature vector field, the allied vector $a(H)$ is perpendicular to *H*. Furthermore, Chen defined *A*-surfaces for which $a(H)=0$ identically. These surfaces are also called Chen surfaces [19]. The class of Chen surfaces includes all minimal and pseudo-umbilical surfaces as well as all hyper-surfaces. These Chen surfaces are said to be trivial *A*-surfaces [20]. For more details, see also [21-24].

In the second part of this paper, we give some preliminaries of the submanifolds in E^{n+d}. Section 3 explains some geometric properties of conformality of *h* of a smooth surface *M* in Euclidean space E^{2+d}. We obtain a relation between the Casorati curvature *C* and the scalar function λ on *M*. Further, we consider Chen surfaces in E^4 to become *h*-conformal. In section 4 we obtain some results on generalized rotation surfaces in E^4 with conformal second fundamental form. Finally, we give some examples of Vranceanu surfaces to become *h*-conformal.

2. BASIC CONCEPTS

Let *M* be a *n*-dimensional submanifold in *(n+d)*-dimensional Euclidean space E^{n+d}. We can choose an oriented local orthonormal frame $\{e_1,,...,e_n,n_1,...,n_d\}$ on *M* such that $e_1,,...,e_n$ are tangent to *M* and $n_1,,...,n_d$ are normal to *M*. Let ∇ and $\tilde{\nabla}$ be the covariant differentiations on *M* and E^{n+d}, respectively. For the tangent vector fields $e_i,e_j\in T_P(M)$, $1\le i,j\le n$ consider the second fundamental map $h:\chi(M)\times\chi(M)\to\chi^{\perp}(M)$;

$$h(e_i,e_j)=\tilde{\nabla}_{e_i}e_j-\nabla_{e_i}e_j. \tag{1}$$

This map is well-defined, symmetric, and bilinear. For the normal vector field $n_\alpha, 1\le\alpha\le d$ of *M*, the shape operator $A:\chi^{\perp}(M)\times\chi(M)\to\chi(M)$ is given by

$$A_{n_\alpha}e_i=-\tilde{\nabla}_{e_i}n_\alpha+D_{e_i}n_\alpha \tag{2}$$

where *D* denotes the normal connection of *M* in E^{n+d} and satisfies the following equation:

$$\left\langle A_{n_\alpha}e_i,e_j\right\rangle=\left\langle h(e_i,e_j),n_\alpha\right\rangle \tag{3}$$

The equation (1) is called Gaussian formula, and

$$h(e_i,e_j)=\sum_{\alpha=1}^{d}h_{ij}^{\alpha}n_\alpha, \tag{4}$$

where h_{ij}^{α} are the coefficients of the second fundamental form [18]. If the second fundamental form is equal to zero then *M* is called totally geodesic. The submanifold *M* is said to be totally umbilical if $h(e_i, e_j) = \langle e_i, e_j \rangle H$ for all vector fields are tangent to *M*.

Further, the Gaussian curvature and the mean curvature of *M* defined by

$$K = \sum_{\alpha=1}^{d} \det A_{n_\alpha} \tag{5}$$

and

$$\vec{H} = \frac{1}{n} \sum_{\alpha=1}^{d} trace(A_{n_\alpha}) n_\alpha . \tag{6}$$

respectively. (see, [25]). If the Gaussian curvature *K* (resp. mean curvature vector $\vec{H}$) is equal to zero then a surface *M* is said to be flat (resp. minimal) [18].

3. CONFORMALITY OF THE SECOND FUNDAMENTAL FORM

Let *M* be a *n*-dimensional smooth submanifold in *(n+d)*-dimensional Euclidean space E^{n+d}. The second fundamental form *h* of *M* is called conformal (i.e. *M* is *h*-conformal), if for any vector fields n_α and n_β normal to *M*,

$$\sum_{i,j=1}^{n} \langle h(e_i, e_j), n_\alpha \rangle \langle h(e_i, e_j), n_\beta \rangle = \lambda^2 \langle n_\alpha, n_\beta \rangle, \quad 1 < \alpha, \beta < d \tag{7}$$

where $\langle , \rangle$ denotes the inner product in E^{n+d} and λ is a suitable scalar function on *M* [2]. The *d*-symmetric matrices of order *n* which are determined by the second fundamental form *h* are

$$A_\alpha = A_{n_\alpha} = \; h_{ij}^{\alpha} \; . \tag{8}$$

The norm (the length) of *h* is given by

$$\|h\|^2 = \sum_{i,j,\alpha} \left(h_{ij}^{\alpha}\right)^2 = \sum_{\alpha} \|A_\alpha\|^2 \tag{9}$$

where the scalar product and the norm of matrices are defined in the usual way: if $A = \left(a_{ij}\right)$ and $B = \left(b_{ij}\right)$ then

$$\langle A, B \rangle = trace(A^{\mathrm{T}} B) = \sum_{i,j} a_{ij} b_{ij} . \tag{10}$$

Consequently, the conformality condition (7) reduces to

$$\langle A_\alpha, A_\beta \rangle = \lambda^2 \delta_\alpha^\beta \tag{11}$$

here δ_{α}^{β} is Kronecker delta and λ is a smooth function on M. In other words, the matrices (A_{α}) are orthogonal and they have the same length. In particular, (11) implies that, if h is conformal, then

$$\|h\|^2 = d\lambda^2 \tag{12}$$

where d is the codimension of the submanifold M (see, [2]).

Definition 3.1. Let M be a n-dimensional smooth submanifold in $(n+d)$-dimensional Euclidean space E^{n+d}. The squared norm of the second fundamental form h of M is called the Casorati curvature of the submanifold M which is denoted by (see, [4])

$$C = \frac{1}{n}\sum_{i,j,\alpha}\left(h_{ij}^{\alpha}\right)^2. \tag{13}$$

Theorem 3.1. Let M be a n-dimensional smooth submanifold in $(n+d)$-dimensional Euclidean space E^{n+d}. If M is h-conformal then the Casorati curvature C of M becomes

$$C = \frac{d}{n}\lambda^2. \tag{14}$$

where d is the codimension of the submanifold M.

Proof: Comparing the equations (9) and (12) with (13) we obtain (14).

4. THE h-CONFORMAL SURFACES IN EUCLIDEAN SPACES

Let M be a local surface in E^{2+d} given with the surface patch $X(u,v):(u,v)\in D\subset E^2$. The tangent vectors of M at an arbitrary point $p = X(u,v)$ are $\{X_u, X_v\}$. The coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u\rangle,\ F = \langle X_u, X_v\rangle,\ G = \langle X_v, X_v\rangle \tag{15}$$

where $\langle , \rangle$ is the induced metric. We assume that $W^2 = EG - F^2 \neq 0$, i.e. $X(u,v)$ is regular. If we chose an orthonormal tangent frame field $\{e_1, e_2\}$

$$e_1 = \frac{X_u}{\sqrt{E}},$$
$$e_2 = \frac{\sqrt{E}}{W}\left(X_v - F\frac{X_u}{E}\right).$$

Then the coefficients of the second fundamental form are given by

$$
\begin{aligned}
h_{11}^{\alpha} &= \langle h(e_1,e_1), n_\alpha \rangle = \frac{c_{11}^{\alpha}}{E}, 1 \le \alpha \le d \\
h_{12}^{\alpha} &= \langle h(e_1,e_2), n_\alpha \rangle = \frac{1}{W}\left(c_{12}^{\alpha} - \frac{F}{E} c_{11}^{\alpha} \right), \\
h_{22}^{\alpha} &= \langle h(e_2,e_2), n_\alpha \rangle = \frac{1}{W^2}\left(Ec_{22}^{\alpha} - 2Fc_{12}^{\alpha} + \frac{F^2}{E} c_{11}^{\alpha} \right)
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
c_{11}^{\alpha} &= \langle h(X_u, X_u), n_\alpha \rangle, \\
c_{12}^{\alpha} &= \langle h(X_u, X_v), n_\alpha \rangle, \\
c_{22}^{\alpha} &= \langle h(X_v, X_v), n_\alpha \rangle.
\end{aligned}
$$

Further, the shape operator matrix of the surface $M \subset E^{2+d}$ becomes

$$
A_{n_\alpha} = \begin{pmatrix} h_{11}^{\alpha} & h_{12}^{\alpha} \\ h_{12}^{\alpha} & h_{22}^{\alpha} \end{pmatrix} \tag{17}
$$

The following result gives a characterization of *h*-conformal surfaces in E^{2+d}.

Theorem 4.1. Let *M* be a smooth surface in a Euclidean space E^{2+d} . Then *M* is *h*-conformal if and only if

$$
\begin{aligned}
&(h_{11}^{\alpha})^2 + 2(h_{12}^{\alpha})^2 + (h_{22}^{\alpha})^2 = \lambda^2, \quad for\ \alpha = \beta, \\
&h_{11}^{\alpha} h_{11}^{\beta} + 2h_{12}^{\alpha} h_{12}^{\beta} + h_{22}^{\alpha} h_{22}^{\beta} = 0, \quad otherwise,
\end{aligned} \tag{18}
$$

holds at every point on *M*, $1 \le \alpha, \beta \le d$, where λ is a real-valued function on *M*.

Proof: Substituting (16) into (7) and using (11) we get the result.

Definition 4.1. The allied vector field $a(\vec{H})$ of $\vec{H}$ is defined by the formula

$$
a(\vec{H}) = \frac{\|\vec{H}\|}{2} \{ trace(A_1 A_\alpha) \} n_\alpha \tag{19}
$$

where $\vec{H}$ is parallel to n_1 [18]. If the allied vector field vanishes identically, then the surface *M* is called *A*-surface (or Chen surface) of E^{2+d} [19].

In [26] one can see the proof of the following result.

Theorem 4.2. [26] Let *M* be a smooth surface in E^4. Then *M* is a non-trivial Chen surface if and only if

$$
\varphi(u,v) H_1 H_2 + \psi(u,v)(H_2{}^2 - H_1{}^2) = 0 \tag{20}
$$

holds, where

$$\varphi(u,v) = (h_{11}^1)^2 - (h_{11}^2)^2 + (h_{22}^1)^2 - (h_{22}^2)^2 + 2(h_{12}^1)^2 - 2(h_{12}^2)^2, \tag{21}$$

$$\psi(u,v) = h_{11}^1 h_{11}^2 + h_{22}^1 h_{22}^2 + 2h_{12}^1 h_{12}^2 \tag{22}$$

and H_1 and H_2 are real valued functions on *M*,

$$H_1 = \frac{1}{2}(h_{11}^1 + h_{22}^1),\ H_2 = \frac{1}{2}(h_{11}^2 + h_{22}^2).$$

As a consequence of Theorem 4.1 and Theorem 4.2 we obtain the following result.

Corollary 4.1. Every *h*-conformal surface in E^4 is a Chen surface.

Proof: Assume that the surface *M* of E^4 is a *h*-conformal surface then by Theorem 4.1

$$\begin{aligned} &(h_{11}^1)^2 + 2(h_{12}^1)^2 + (h_{22}^1)^2 = \lambda^2, \\ &(h_{11}^2)^2 + 2(h_{12}^2)^2 + (h_{22}^2)^2 = \lambda^2, \\ &h_{11}^1 h_{11}^2 + 2h_{12}^1 h_{12}^2 + h_{22}^1 h_{22}^2 = 0. \end{aligned} \tag{23}$$

The equations of (23) imply that $\varphi(u,v) = 0$ and $\psi(u,v) = 0$. So, by Theorem 4.2 *M* is a Chen surface.

Remark 4.1. The converse statement of the Corollary 4.1 may not be true in general. For instance, see Example 5.1.

5. h-CONFORMAL ROTATIONAL SURFACES IN E⁴

Given a regular curve $\beta(v) = (x(v), y(v), z(v), w(v))$ in E^4, the general rotation surface *M* in E^4 over β is given by

$$\begin{aligned} X(u,v) = (&x(v)\cos au - y(v)\sin au, x(v)\sin au + y(v)\cos au, \\ &z(v)\cos bu - w(v)\sin bu, z(v)\sin bu + w(v)\cos bu) \end{aligned} \tag{24}$$

where *a* and *b* are rates of the rotation in the fixed planes over the rotation [8]. In the case $\beta(v) = (f(v), 0, g(v), 0)$ the general rotation surface *M* in E^4 is given by the parametrization

$$M : X(u,v) = (f(v)\cos au, f(v)\sin au, g(v)\cos bu, g(v)\sin bu) \tag{25}$$

where *f(v)* and *g(v)* are differentiable functions which are satisfy the conditions $a^2 f^2 + b^2 g^2 > 0,\ (f')^2 + (g')^2 > 0$. Also, *a, b* are positive constants [14].

We can choose an oriented local orthonormal frame $\{e_1, e_2, n_1, n_2\}$ on M in the following:

$$e_1 = \frac{X_u}{\|X_u\|}, e_2 = \frac{X_v}{\|X_v\|},$$
$$n_1 = \frac{1}{\sqrt{(f')^2 + (g')^2}}(-g'(v)\cos au, -g'(v)\sin au, f'(v)\cos bu, f'(v)\sin bu),$$
$$n_2 = \frac{1}{\sqrt{(af)^2 + (bg)^2}}(-bg(v)\sin au, bg(v)\cos au, af(v)\sin bu, -af(v)\cos bu).$$

Then by a direct computation, one can get the components of the second fundamental form as [14]:

$$\begin{aligned} h_{11}^1 &= \frac{a^2 fg' - b^2 f'g}{\sqrt{(f')^2 + (g')^2}\left((af)^2 + (bg)^2\right)}, \\ h_{22}^1 &= \frac{f'g'' - f''g'}{\left((f')^2 + (g')^2\right)^{3/2}}, \\ h_{12}^2 &= \frac{ab(f'g - fg')}{\sqrt{(f')^2 + (g')^2}\left((af)^2 + (bg)^2\right)}, \\ h_{12}^1 &= h_{11}^2 = h_{22}^2 = 0. \end{aligned} \tag{26}$$

So, substituting (26) into (20)-(22) one can get the following result.

Theorem 5.1. Every general rotational surface given with the parametrization (25) is a Chen surface.

Proof: Let M be a general rotational surface given with the parametrization (25). By the use of (26) one can get $H_2 = 0$ and $\psi(u,v) = 0$. So, from Theorem 4.2 M becomes a Chen surface.
As a result of (26) with (23) one can get the following theorem.

Theorem 5.2. Let M be a general rotational surface given with the parametrization (25). Then the second fundamental form of M is conformal if and only if

$$(h_{11}^1)^2 - 2(h_{12}^2)^2 + (h_{22}^1)^2 = 0 \tag{27}$$

holds at every point in M.

Example 5.1. For the case $f(v) = v, g(v) = 1$ and $a=1$ the surface patch (25) turns into

$$M : X(u,v) = (v\cos u, v\sin u, \cos bu, \sin bu),\ b \in R. \tag{28}$$

This surface is obtained starting from helicoids $Y(u,v)=(v\cos u, v\sin u, bu)$ of E^3 by rolling up E^3 along their axis on the hypercylinder $\{(x_1,x_2,x_3,x_4)\in R; x_3{}^2+x_4{}^2=1\}$ in E^4 [20]. If $b=\pm\sqrt{2}$, then M is a h-conformal surface with $\lambda=\pm\dfrac{2}{u^2+2}$.

So as a result of Theorem 5.1 every general rotational surface is a Chen surface, but converse of this may not be true in general given for in this example.

Definition 5.1. Vranceanu surfaces in E^4 are defined by the following parametrization;

$$f(v)=r(v)\cos v, g(v)=r(v)\sin v, \; a=b=1 \tag{29}$$

where *r(v)* is a real-valued non-zero function [27].

Using (26) and (29) we obtain the coefficients of the second fundamental form of the Vranceanu surface as following:

$$\begin{aligned} h_{11}^1 &= \frac{1}{\sqrt{r^2(v)+(r'(v))^2}}, \\ h_{22}^1 &= \frac{2(r'(v))^2 - r(v)r''(v) + r^2(v)}{\left(r^2(v)+(r'(v))^2\right)^{3/2}}, \\ h_{12}^2 &= -\frac{1}{\sqrt{r^2(v)+(r'(v))^2}}, \\ h_{12}^1 &= h_{11}^2 = h_{22}^2 = 0. \end{aligned} \tag{30}$$

Thus by the use of (4)-(6) together with (30) the Gaussian curvature and mean curvature of the Vranceanu surface M become

$$K=\frac{(r'(v))^2 - r(v)r''(v)}{\left(r^2(v)+(r'(v))^2\right)^2} \tag{31}$$

and

$$H=\frac{3(r'(v))^2 + 2r^2(v) - r(v)r''(v)}{\left(r^2(v)+(r'(v))^2\right)^2} \tag{32}$$

respectively. Consequently, M is a flat or minimal surface if and only if

$$r(v)=c_1e^{c_2v} \tag{33}$$

or

$$r(v)=\pm\frac{1}{\sqrt{c_1\sin(2v)-c_2\cos(2v)}} \tag{34}$$

hold respectively, where c_1 and c_2 are real constants (see, [9]).

We prove the following result.

Theorem 5.3. Let M be a Vranceanu surface given with the parametrization (29). If the second fundamental form of M is conformal then either M is flat or minimal surface in E^4.

Proof: Let M be a Vranceanu surface given with the parametrization (29) in E^4. If M is h-conformal then by the use of (23) with (30) we get $h_{11}^1 = \pm h_{22}^1$. For the case $h_{11}^1 = h_{22}^1$ M is a flat surface, otherwise, M is a minimal surface.

6. CONCLUSIONS

The second fundamental form h of the surfaces in the Euclidean spaces characterizes the surface itself. If h is vanishing identically, then the surface is called totally geodesic. In the meantime, if the covariant derivative of h vanishes, M is called the parallel surface. The conformality of the second fundamental form of surfaces plays an important role in differential geometry. Because we realize that the conformality of the second fundamental form h is related to the Casorati curvature of the surface M. In this article, we have shown that all h-conformal surfaces are Chen surfaces. However, the converse is not valid. To support this proposition, we obtain some results of rotational surfaces in E^4. In the future study, it is possible to concentrate on h-conformal rotation surfaces together with Chen surfaces in high-dimensional Euclidean spaces.

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