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SPECIAL CURVES ACCORDING TO TYPE-2 QUATERNIONIC FRAME IN $\mathbb{R}^4$

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Abstract. *Quaternions, which were defined by William Rowan Hamilton in 1843, are a number system in four-dimensional space and are analogous to complex numbers. However, quaternion multiplication is not commutative, distinguishing them from complex numbers. Quaternions are special mathematical tools used in computer science, robotics, and many other mathematical sciences. From this point of view, they also get attention in differential geometry. In particular, their characterizations given by the Serret-Frenet apparatus are challenging. For this reason, Bharathi and Nagaraj obtained Serret-Frenet formulas for spatial quaternionic and quaternionic curves in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$*, respectively. Inspired by this work, the geometers obtained quaternionic frames in 4-dimensional Semi-Euclidean space* $\mathbb{R}^4_2$*, dual quaternionic space, and lately type -2 quaternionic frame in* $\mathbb{R}^4$*. Their features are also examined. On the other hand, studying the characterizations of the curves is a significant research area for differential geometers because of having used various branches of applied sciences. Particularly, the features of the specially defined curves, such as Bertrand, Smarandache, rectifying, and osculating curves, are curiously studied. In four-dimensional spaces, rectifying curves are defined as a curve whose position vector fully lies in* $\{T,N_2,N_3\}$*. Similarly, the osculating curve defined as first and second kind became of having two different binormals* $\{T,N_1,N_2\}$ *and* $\{T,N_1,N_3\}$*, respectively. In this study, inspired by the definitions above, we define rectifying, osculating first kind, and osculating second kind curves according to the type-2 quaternionic frame in* $\mathbb{R}^4$*. Their characterizations are also examined.*

Keywords: *Quaternions; rectifying curve; osculating curve of the first kind; osculating curve of the second kind.*

1. INTRODUCTION

The quaternions were introduced by Irish mathematician Sir William R. Hamilton, who discovered that the appropriate generalization in which the real axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vector axes in 1843 [1]. While the practical use of quaternions was minimal compared to other methods until the mid-20th century, that has now changed. Recently, the theory of quaternion has developed rapidly, and many mathematicians are focusing on this field from different points of view. One of them is the quaternion valued function of a real variable Serret-Frenet formula studied by Baharathi and Nagaraj [2]. Inspired by this study, a new quaternionic framework was

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obtained by Aksoyak in $\mathbb{R}^4$ using a method similar to the method given by Baharathi and Nagaraj [3]. Also, another study on Serret-Frenet formulas of quaternionic curves in Semi-Euclidean space $\mathbb{E}_2^4$ can be found in [4] in detail. Another one is Keçilioglu and Ilarslan's study. They obtained some characterizations for (1,3) type quaternionic Bertrand curves in Euclidean 4-space using the curvature functions of the curve [5].

The Latin origin word "Osculation" also means "kiss" since the curves connect a different tangent. In [6], Ilarslan and Nesoviç gave the first and the second kinds of osculating curves in Minkowski space-time. In addition, Yılmaz and Külahcı studied the quaternionic curve in $\mathbb{Q}_4$ and obtained some characterization about osculating curves [7]. Bektaş et al. studied the quaternionic osculating curves in Euclidean and semi-Euclidean space [8]. Ilarslan and Nesoviç described some characterizations of null osculating curves in the Minkowski space-time [9]. In another study, the term of the rectifying curve is viewed as a space curve whose position vector always lies in its rectifying plane [10]. Chen and Dillen [11] achieved a relationship between the rectifying curves and the centrodes given by the end points of the Darboux vector of a space curve, which played a significant role in mechanics. Also, Güngör and Tosun determined the spatial quaternionic rectifying curves in $\mathbb{R}^3$and obtained some characteristics for these curves. In addition, they explored quaternionic rectifying curves in $\mathbb{R}^4$ [12].

In this study, firstly, we attain some characterizations of quaternionic osculating first kind and quaternionic osculating second kind curves in $\mathbb{R}^4$ and secondly, obtains some characterizations for quaternionic rectifying curves in $\mathbb{R}^4$ by the aid of these properties.

2. PRELIMINARIES

In this section, we briefly introduce quaternion theory in Euclidean space. Detailed information can be found in [3] and [13]. The set of quaternions $\mathbb{Q}$ is determined by

$$Q = \{q = q_0 + q_1 i + q_2 j + q_3 k;\ q_i \in \mathbb{R},\ 0 \ll i \leq 3\}$$

where i, j, k are orthogonal unit spatial vectors in three-dimensional space so that

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -\mathrm{ji} = \mathrm{k},$$
$$\mathrm{jk} = -\mathrm{kj} = \mathrm{i},$$
$$\mathrm{ki} = -\mathrm{ik} = \mathrm{j}.$$

If we indicate by $S_q = q_0$ and $\vec{V}_q = q_1\, i + q_2 j + q_3\, k$ i; (where, respectively, the symbols indicate scalar and vectoral part q) we can write quaternion as $q = S_q + \vec{V}_q$. Using these basic products, we can now arrange the product of two quaternions to get

$$p \times q = S_p S_q - < \vec{V}_p, \vec{V}_q > + S_p \vec{V}_q + S_q \vec{V}_p + \vec{V}_p \wedge \vec{V}_q, \forall p, q \in \mathbb{Q}.$$

where we have used the dot and cross products in Euclidean space R^3 [14]: The conjugate of the quaternion q is denoted by q and denoted as

$$\alpha q = S_q - V_q = q_3\, k - q_0 - q_1\, i - q_2 j.$$

We recall that the Hamiltonian conjugation is an antiautomorphism of Q. This includes being satisfied

$$\alpha(p \times q) = \alpha p \times \alpha q \text{ for all p,q} \in \mathbb{Q}$$

Therefore, we define the symmetric real-valued, non-degenerate, bilinear form h as follows

$$h : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$$
$$(p, q) \to h(p, q) = (1/2)[(p \times \alpha q) + (q \times \alpha p)].$$

It is named the quaternion inner product. The norm of a real quaternion q is

$$\|q^2\| = h(q, q) = q \times \alpha q = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2$$

If $\|q\| = 1$, then q is called a unit quaternion. It is known that the groups of unit real quaternions and unitary matrices *SU* (2) are isomorphic. Therefore, spherical concepts in S^3 such as meridians of longitude and parallels of latitude are explained with assistance elements of *SU* (2)*:* Besides, the element of *SO* (3) can match with each element of S^3 [15].

The sphere S^3 Q in quaternionic calculus is like the unit circle S^1 C in complex calculus. Indeed, $S^3 = \{q \in \mathbb{Q}, \|q\| = 1\}$ constitutes a group under quaternionic multiplication. q is called spatial quaternion whenever $q + \alpha q = 0$ [2]*:* and a temporal quaternion whenever $q - \alpha q = 0$. Any q can be written as $q = (1/2)(q + \alpha q) + (1/2)(q - \alpha q)$ [14].

The Serret- Frenet formulas for quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ were given Baharathi and Nagaraj [2]. A new quaternionic framework obtained by Aksoyak in $\mathbb{R}^4$ using a method similar to the method given by Baharathi and Nagaraj as follows:

Theorem 2.1: (See [3]) Let $I = [0;1]$ indicate the unit interval in the real line $\mathbb{R}$ and

$$\beta : I \subset \mathbb{R} \to \mathbb{Q}$$
$$s \to \beta(s) = \beta_0(s) + \beta_1(s)i + \beta_2(s)j + \beta_3(s)k$$

be an arc-length curve in $\mathbb{R}^4$. Then, the Frenet equations of β are obtain by

$$\begin{bmatrix} T' \\ {N_1}' \\ {N_2}' \\ {N_3}' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & -\tau & 0 \\ 0 & \tau & 0 & (K - k) \\ 0 & 0 & -(K - k) & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{1}$$

where $T = \beta'$ is unit tangent N_1, N_2, N_3 are unit normal vectors of the curve β and $K = \|T\|$ is the principal curvature $-\tau$, is the torsion $(K - k)$and is the bitorsion of the curve β [3].

Using (1), we will consider the quaternionic osculating curve of the first kind and second kind in $\mathbb{R}^4$. Similarly, in $\mathbb{R}^4$, we obtain some characterizations for the quaternionic rectifying curve by virtue of (1).

3. MATERIALS AND METHODS

3.1. QUATERNIONIC OSCULATING CURVE OF THE FIRST KIND IN $\mathbb{R}^4$ FOR TYPE -2 QUATERNIONIC FRAME

In this section, we obtain the characterizations of the osculating curves of the first kind in $\mathbb{R}^4$ using type-2 quaternionic frame. Recall that an arbitrary curve $\beta(s)$ in $\mathbb{R}^4$ is named an osculating curve of the first and second kind if its position vector (with according to some chosen origin) compensated, respectively. Then we give the related equations below

$$\beta(s) = \lambda(s)T(s) + \mu(s)N_1(s) + \nu(s)N_2(s) \tag{2}$$

$$\beta(s) = \lambda(s)T(s) + \mu(s)N_1(s) + \nu(s)N_3(s) \tag{3}$$

for some diffrentiable functions $\lambda(s)$, $\mu(s)$, $\nu(s)$ in the arc-length function s.

Theorem 3.1. Let $\beta(s)$ be a unit speed quaternionic osculating curve. Then $\beta(s)$ congruent to a quaternionic osculating curve of the first kind if and only if its bitorsion $(K\text{-}k) = 0$ for each s.

Proof: Suppose that $\beta(s)$ is a unit speed quaternionic osculating curve of the first kind in $\mathbb{R}^4$. At that time, its position vector met (2). Differentiating both parts of (2) according to s and considering (1), we get $(K- k) = 0$. Conversely, suppose that $(K -k) = 0$. Using (1), since $h(\beta ,N_3)$ is a constant function, we conclude that N_3 is a constant vector. The position vector of can be decomposed as follows.

$$\beta = h(\beta,T)T + h(\beta,N_1)N_1 + h(\beta,N_2)N_2 + h(\beta,N_3)N_3$$

and $h(\beta,N_3)N_3$ is a constant vector, we obtain that β is the quaternionic osculating curve of first kind.

Theorem 3.2. Let β be a unit speed quaternionic osculating curve. Then $\beta(s)$ is a quaternionic osculating curve of the first kind if and only if its position vector met the equation

$$\beta(s) = \lambda(s)T(s) + (\lambda' - 1)N_1(s) + \left(\frac{-\lambda K - \lambda''}{\tau}\right)N_2(s)$$

Proof: Recalling (2) may write

$$\beta(s) = \lambda(s)T(s) + \mu(s)N_1(s) + \nu(s)N_2(s)$$

Using Theorem 3.1., we get

$$\beta = h(\beta,T)T + h(\beta,N_1)N_1 + h(\beta,N_2)N_2 + h(\beta,N_3)N_3$$

Then it follows that

$$\lambda(s) = h(\beta,T),\ \mu(s) = h(\beta,N_1),\ \nu(s) = h(\beta,N_2)$$

Differentiating the above equations and using (1), after necessary procedures, we obtain

$$\beta(s) = \lambda(s)T(s) + (\lambda' - 1)N_1(s) + \left(\frac{-\lambda K - \lambda''}{\tau}\right)N_2(s)$$

3.2. QUATERNIONIC OSCULATING CURVE OF THE SECOND KIND IN $\mathbb{R}^4$ FOR TYPE - 2 QUATERNIONIC FRAME

In this part, we indicate some characterizations for quaternionic osculating curve of the second kind in $\mathbb{R}^4$. Let β (*s*) be unit speed quaternionic osculating curve in $\mathbb{R}^4$, with non-zero curvatures $K(s)$, $-\tau(s)$, $(K-k)(s)$. Then the radius vector satisfies the following equation

$$\beta(s)=\lambda(s)T(s)+\mu(s)N_1(s)+\nu(s)N_3(s)$$

for some diffrentiable functions $\lambda(s)$, $\mu(s)$, $\nu(s)$ in the arc-length function *s*. Then we introduce the following theorems of the quaternionic osculating curve of the second kind.

Theorem 3.2.1. Let $\beta(s)$ be a unit speed quaternionic osculating curve in $\mathbb{R}^4$, with non- zero curvatures $K(s)$, $-\tau(s)$, $(K-k)(s)$.Then $\beta(s)$is equal to a quaternionic osculating curve of a second kind if and only if

$$\left(\frac{1}{K}\left(\frac{K-k}{\tau}\right)'\right)' + \left(\frac{K-k}{\tau}\right)K = \frac{1}{\alpha}, a \in \mathbb{R} \tag{4}$$

Proof: Let $\beta(s)$ be a unit speed quaternionic osculating curve and $K(s)$, $-\tau(s)$ and $(K-k)(s)$ with non-zero curvatures. Then, the position vector $\beta(s)$ of the curve $\beta(s)$ satisfies the following equations

$$\beta(s) = \lambda(s)T(s) + \mu(s)N_1(s) + \nu(s)N_3(s)$$

where $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are particular differentiable functions. Differentiating the above equations and using (1), we get

$$T(s) = (\lambda' - \mu K)T(s) + (\lambda K + \mu')N_1(s) + \big(-\mu\lambda - \nu(K-k)\big)N_2(s) + \nu' N_3(s)$$

It follows that

$$\begin{gathered} \lambda' - \mu K = 1 \\ \lambda K + \mu' = 0 \\ -\mu\lambda - \nu(K-k) = 0 \\ \nu' = 0 \end{gathered} \tag{5}$$

and

$$\begin{gathered} \lambda(s) = \alpha\frac{1}{K}\left(\frac{K-k}{\tau}\right)' \\ \mu(s) = -\alpha\left(\frac{K-k}{\tau}\right) \\ \nu(s) = a \end{gathered} \tag{6}$$

where $a \in \mathbb{R}$. Here, the functions λ(s), $\mu(s)$, and $\nu(s)$ are arbitrary functions, and K (s), -$\tau(s)$ and (K -k)(s) are the curvature functions of the quaternionic osculating curve of the second kind. In fact, by using the first equation in (5) and (6), we conclude that curvatures K (s), -$\lambda(s)$ and (K- k)(s) satisfy (4).

Conversely, assume that the curvatures K (s), -λ (s) and (K- k)(s) of a unit speed quaternionic osculating curve of the second kind in $\mathbb{R}^4$ satisfies (4). Let us define a vector by $\gamma(s) \in \mathbb{R}^4$ given by

$$\gamma(s) = -\beta(s) + \alpha \frac{1}{K}\left(\frac{K-k}{\tau}\right)' T - \alpha\left(\frac{K-k}{\tau}\right) N_1(s) + aN_3(s)$$

Using (1) and (4), we find $\gamma'(s)$ = 0, which means that $\gamma(s)$ is a constant vector. This shows that β(s) is equal to a quaternionic osculating curve of the second kind.

Theorem 3.2.2. Let β (s) be a unit speed quaternionic osculating curve of the second kind in $\mathbb{R}^4$ with non-zero curvatures K (s), -$\tau(s)$, (K -k)(s). Then, the following expressions hold:

(i) The curvatures K (s), -$\lambda(s)$ and (K-k)(s) satisfies the following equation

$$\mathrm{K{=}k(s){+}\tau}\begin{Bmatrix} -\frac{1}{a}\int \sin\left(\int \left(K(s)d(s)\right)d(s) + c_1\right)\cos\left(\int K(s)d(s)\right) \\ + \left[\frac{1}{a}\int \cos\left(\int K(s)d(s)\right)d(s) + c_2\right]\sin\left(\int K(s)d(s)\right) \end{Bmatrix} \tag{7}$$

(ii) The tangential component and principal normal component of the radius vector of the quaternionic osculating curve of the second kind are given by

$$\begin{aligned} \langle \beta(s), T(s)\rangle &= \alpha \frac{1}{K}\left(\frac{K-k}{\tau}\right)' \\ \langle \beta(s), N_1(s)\rangle &= -\alpha\left(\frac{K-k}{\tau}\right), \end{aligned} \tag{8}$$

respectively.

(iii) The second binormal component of the radius vector of the quaternionic osculating curve of the second kind is a non-zero constant vector.

Conversely, if β (s) is a unit speed quaternionic curve of the second kind in $\mathbb{R}^4$ with non-zero curvatures K (s), -$\tau(s)$ and (K- k)(s) and either of equations (i), (ii) or (iii) is valid, then $\beta(s)$ is a quaternionic osculating curve of the second kind.

Proof: Let us first assume that β (s) is a unit speed quaternionic osculating curve of the second kind in $\mathbb{R}^4$ with non-zero curvatures K (s), -$\tau(s)$ and (K- k)(s) .The position vector of the curve satisfies the equation (4). If we state y (s) = $\frac{(K-k)}{\tau}$ and q (s) = $\frac{1}{K}$, then equation (4) can be determined as

$$\frac{d}{ds}\left(q(s)\frac{dy}{ds}\right) + \frac{y(s)}{q(s)} = \frac{1}{\alpha}\ , \alpha \in \mathbb{R}$$

If we change the variables in the overhead expression as $t(s) = \frac{1}{q(s)}ds$ then it satisfies

$$\frac{d^2y}{d^2t}+y=\frac{1}{\alpha K}\ ,\alpha\in\mathbb{R}$$

The solution of this differential equation is

$$y=\left(-\frac{1}{\alpha}\int\frac{\sin t}{K}dt+c_1\right)\cos t+\left(\frac{1}{\alpha}\int\frac{\cos t}{K}dt+c_2\right)\sin t$$

where c_1 and $c_2 \in \mathbb{R}$. Let's say that $y\ (s)=\frac{(K-k)}{\tau}$ and $dt\ =\ K\ (s)ds$, we obtain

$$\mathrm{K}=\mathrm{k(s)}+\tau\begin{Bmatrix}-\frac{1}{a}\int\sin\left(\int\big(K(s)d(s)\big)d(s)+c_1\right)\cos\left(\int K(s)d(s)\right)\\+\left|\frac{1}{a}\int\cos\left(\int K(s)d(s)\right)d(s)+c_2\right|\sin\left(\int K(s)d(s)\right)\end{Bmatrix}$$

Thus, we prove expression (i). By using equations (3) and (6), we can state the radius vector of the curve as follows:

$$\beta(s)=\ \alpha\frac{1}{K}\left(\frac{K-k}{\tau}\right)'T(s)-\ \alpha\left(\frac{K-k}{\tau}\right)N_1(s)+aN_3(s) \tag{9}$$

Then we obtain $\langle\beta(s),T(s)\rangle=\ \alpha\frac{1}{K}\left(\frac{K-k}{\tau}\right)'$, $\langle\beta(s),N_1(s)\rangle=-\ \alpha\left(\frac{K-k}{\tau}\right)$ and $\langle\beta(s),N_3(s)\rangle=\alpha,\alpha\in\mathbb{R}$. Therefore, we proved the expressions (ii) and (iii). Conversely, consider that expression (i) is valid. Then the curvature functions $K\ (s)$, $-\tau(s)$ and $(K-\ k)(s)$ provides the equation

$$\mathrm{K}=\mathrm{k(s)}+\tau\begin{Bmatrix}-\frac{1}{a}\int\sin\left(\int\big(K(s)d(s)\big)d(s)+c_1\right)\cos\left(\int K(s)d(s)\right)\\+\left|\frac{1}{a}\int\cos\left(\int K(s)d(s)\right)d(s)+c_2\right|\sin\left(\int K(s)d(s)\right)\end{Bmatrix}$$

If we differentiate the above equation two times according to *s*, we get

$$\left(\frac{1}{K}\left(\frac{K-k}{\tau}\right)'\right)'+\left(\frac{K-k}{\tau}\right)K=\frac{1}{\alpha},a\in\mathbb{R}$$

Then $\beta(s)$ equals a quaternionic osculating curve of the second kind. Then, suppose that expression (ii) is valid. By taking the derivative of $\langle\beta(s),N_1(s)\rangle=-\ \alpha\left(\frac{K-k}{\tau}\right)$, according to *s* and using (1), we get,

$$-Kh\big(\beta(s),T(s)\big)-\tau\big(\beta(s),N_2(s)\big)=-\alpha\left(\frac{K-k}{\tau}\right)'$$

Taking into account of $\langle\beta(s),T(s)\rangle=\ \alpha\frac{1}{K}\left(\frac{K-k}{\tau}\right)'$ and $\tau\neq0$, we obtain $\langle\beta(s),N_1(s)\rangle=0$, which means that β(s) equals the quaternionic osculating curve of the second kind. If expression (iii) is valid, then we have $\langle\beta(s),N_3(s)\rangle=\alpha\ ,a\in\mathbb{R}$. By differentiating the last equation in terms of *s* and using (4), we obtain

$$(\text{K-k})\text{h}(\beta(s), N_2(s)) = 0$$

It pursues that $h(\beta(s), N_2(s)) = 0$ and the curve is a quaternionic osculating curve of the second kind. This completes the proof.

3.3. *QUATERNIONIC RECTIFYING CURVE IN* $\mathbb{R}^4$ *FOR TYPE -2 QUATERNIONIC FRAME*

In this part, we first obtain some properties of a quaternionic rectifying curve in $\mathbb{R}^4$ by using the components. Then, we characterize the quaternionic rectifying curve in $\mathbb{R}^4$ in terms of their curvatures. Let $\beta(s)$ be unit speed quaternionic rectifying curve in $\mathbb{R}^4$, with non zero curvatures K (s); $-\tau(s)$ and $(K-k)(s)$. Then, the position vector of β (s) satisfies the following equation

$$\beta(s) = \lambda(s)T(s) + \mu(s)N_2(s) + \nu(s)N_3(s) \tag{10}$$

for some diffrentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$. For the value of second curvature is zero and non zero, we get two cases regarding the quaternionic rectifying curve.

Case I. $-\tau(s) = 0$. If we differentiate (10) and use equation (1), we get

$$\begin{gathered} \lambda' = 1 \\ \lambda K = 0 \\ \mu' = \nu(K-k) \\ (K-k) = \nu' \end{gathered} \tag{11}$$

where $\lambda = h(\beta, N_1)$, $\mu = h(\beta, N_2)$; and $\nu = h(\beta, N_3)$ are the components of the position vector of the curve, which are named the first, the second, and the third principal normal components, respectively.

From the first equation of (11), we get that $\lambda = s + c$, and from the second equation of (11), we have $\lambda K = 0$. Since K is a constant, $\lambda = 0$ which is a contradiction. Son-$\tau(s)=0$ does not hold since $\beta(s)$ is a quaternionic rectifying curve.

Case II. $-\tau(s) \neq 0$. If we differentiate (10) using equation (1), we get the following system of equations

$$\begin{gathered} \lambda' = 1 \\ \lambda K + \mu\tau = 0 \\ \mu' - \nu(K-k) = 0 \\ \mu(K-k) + \nu' = 0 \end{gathered} \tag{12}$$

By using the third of the last equation, we have

$$\nu' = \left(\frac{\mu'}{(K-k)}\right)'$$

Using the last equations, after the necessary calculations, we ottain

$$\mu(K-k)+\left(\frac{\mu'}{(K-k)}\right)'=0$$
$$\frac{-\lambda K}{\tau}(K-k)+\left(\frac{\mu'}{(K-k)}\right)=0 \tag{13}$$

if $\frac{(K-k)}{\tau}$ is a constant, i.e., $\frac{(K-k)}{\tau}$ = $d\in\mathbb{R}-\{0\}$which concludes two possible cases.

Case I. $\tau = a\in\mathbb{R}-\{0\}$. Then $(K\text{-}k)=\alpha d\in\mathbb{R}-\{0\}$; which means that β is a helix. Moreover, (13) becomes

$$(ad)+\left(\frac{\mu'}{ad}\right)'=0$$

After routine calculations, this equation turns into

$$\frac{\mu''}{\mu}=const=c\ c\in\mathbb{R}$$

which means

$$\mu=c_1e^{\sqrt{cx}}+c_2e^{-\sqrt{cx}}$$

Case II. $\tau\neq const\in\mathbb{R}-\{0\}$. Then we can get the following theorem.

Theorem 3.3.1. Let β(s) be unit speed quaternionic curve parameterized by arc length with curvatures *K=1,* $\tau=\frac{1-k}{d}$, $d\in\mathbb{R}-\{0\}$, $1-k\neq const\in\mathbb{R}-\{0\}$. Then β(s) is congruent to a quaternionic rectifying curve if it satisfies the following differential equation

$$(s+c)+\left(\frac{Kd}{(1-k)^2}+\frac{k'd(s+c)}{(1-k)^3}\right)\frac{1}{d}=0$$

Proof: Assume that $\tau=\frac{1-k}{d}$, β(s) is congruent to a quaternionic rectifying curve. Using (13), we have

$$\mu=-\left(\frac{\mu'}{(K-k)}\right)'\frac{1}{(K-k)} \tag{14}$$

From the first equation of (12), we obtain

$$\lambda=(s+c) \tag{15}$$

Using (12), we get

$$\nu=\frac{\mu}{(1-k)}$$

Then, using the second equation of (12), we get

$$\lambda K + \mu\frac{1-k}{d} = 0$$

Using (14), (15), and the second equation of (12), we conclude

$$(s + c) + \left(\frac{Kd}{(1-k)^2} + \frac{k'd(s+c)}{(1-k)^3}\right)\frac{1}{d} = 0$$

which completes the proof.

Now suppose that $\frac{K-k}{d}$ is not equal to a constant. $\tau \neq$ *const* and $1 - k \neq$ *const*. Then we can get the following theorem.

Theorem 3.3.2. Let β(s) be a unit speed quaternionic curve parametrized by arc length with curvatures $K = 1$, $\tau \neq$ *const* and $1 - k \neq$ *const*. Then β(s) is congruent to a quaternionic rectifying curve if τ and $1 - k$ satisfies the following differential equation

$$\left[\left(\frac{1}{\tau(1-k)}\right)' - \left(\frac{\lambda\tau'}{\tau^2(1-k)}\right)'\right] = -\frac{\lambda(1-k)}{\tau}$$

Proof: Using the second equation of (12), we find

$$\lambda K + \mu\tau = 0$$

From this equation, we may write

$$\mu = \frac{-\lambda K}{\tau}$$

Note that

$$\mu = \left[\left(\frac{1}{\tau(1-k)}\right)' - \left(\frac{\lambda\tau'}{\tau^2(1-k)}\right)'\right]\frac{1}{(1-k)}$$

Hence we get

$$\left[\left(\frac{1}{\tau(1-k)}\right)' - \left(\frac{\lambda\tau'}{\tau^2(1-k)}\right)'\right] = -\frac{\lambda(1-k)}{\tau}$$

4. RESULTS AND DISCUSSION

4.1. RESULTS

In the context of type-2 quaternionic frames, the distinction between rectifying curves and osculating curves is primarily based on their geometric properties and the way they relate to the tangent and normal vectors of the curve. From the rectifying curve point of view, the entire trajectory of the curve is constrained to a specific plane, allowing it to have a geometric relationship with tangent and second binormal vector.

In contrast, osculating curves are defined in terms of the curve's curvature properties. This distinction reflects the different ways that osculating curves "kiss" or approximate the original curve at particular points, involving the interaction of different geometric properties such as curvature and torsion.

In summary, the key distinction lies in the relationship to specific normal vectors and the spatial constraints on the position vector of each type of curve. Rectifying curves are confined to a plane, whereas osculating curves describe a more complex relationship involving the curvature structure of the original curve. □

4.2. DISCUSSION

This paper discusses special curves related to type-2 quaternionic frames in four-dimensional space $\mathbb{R}^4$. It begins with an overview of quaternions, a mathematical system invented by W. R. Hamilton, and their applications to various fields, including differential geometry. The study aims to define and characterize rectifying and osculating curves within a new quaternionic framework called type-2, building upon previous work in the area (For detailed information, we refer to [16]). It also highlights the importance of these curves in geometrical studies and applied sciences, making them a significant topic for further research.

5. CONCLUSION

This study contains special curves according to the type-2 quaternionic frame field in $\mathbb{R}^4$. Namely, by defining the osculating and rectifying curves, which are called special curves, some characterizations were examined in $\mathbb{R}^4$ using to type-2 framework. It is thought that this method can be applied to a large variety of curves. This study fills a major gap in the literature, and some of its findings may be of interest to researchers who study special curves. The purpose of this study is to offer an alternative viewpoint and is anticipated to offer valuable information to scholars who are interested in exploring this subject.

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