ORIGINAL PAPER

BICOMPLEX ORESME NUMBERS

SELİME BEYZA ÖZÇEVİK[1], ABDULLAH DERTLİ[1],
SAVAŞ KARAAHMETOĞLU[2]

Manuscript received: 15.05.2024. Accepted paper: 19.12.2024;
Published online: 30.03.2025.

Abstract. *In this study, bicomplex Oresme numbers are defined, and based on this definition, a bicomplex Oresme sequence is generated. Fundamental equations of the sequence, such as the Binet formula, generating function, and Cassini identity, are derived. Subsequently, complex Oresme numbers are introduced, and matrices of bicomplex Oresme numbers are constructed using this definition.*

Keywords: *Bicomplex numbers; Oresme numbers; Binet formula.*

1. INTRODUCTION

In 1892, Corrado Segre, drawing inspiration from the works of Hamilton and Clifford, developed a novel algebraic structure, which he termed bicomplex numbers. He introduced the concept of multicomplex numbers and described an infinite family of algebras [1]. Bicomplex numbers are generally expressed as $a + bi + cj + dij$, where a, b, c, d are real numbers and $i^2 = j^2 = -1$. Thus, bicomplex numbers constitute a system with two distinct roots of -1 . In recent years, bicomplex algebra has become a prevalent research subject in physics and mathematics. Researchers have begun examining the algebraic, geometric, topological and dynamic properties of bicomplex numbers [2-4].

On the other hand, numerous sequences of numbers exist in the literature with various properties, among which the most prominent are the Fibonacci and Lucas number sequences. Studies regarding various applications of these number sequences are available [5-7]. Among these studies, Nurkan and Güven defined bicomplex Fibonacci and Lucas numbers in 2015 [8]. Furthermore, Halıcı defined bicomplex Horadam numbers and she examined various properties in 2019 [9].

The Oresme sequence, defined by Nicole Oresme in the 14th century, has the recurrence relation as follows:

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, O_0 = 0, O_1 = \frac{1}{2}, \text{ for } n \geq 0.$$

A few terms of this sequence are $0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \dots, \frac{n}{2^n}$. Over time, studies have been conducted on Oresme numbers, and various properties have been obtained [7].

This study explores the concept of bicomplex Oresme numbers by utilizing the recurrence relation and certain properties of the Oresme number sequence. Additionally, a

[1]Department of Mathematics, Ondokuz Mayıs University, Samsun, 55139, Turkey. E-mail: ozcevikbeyza8@gmail.com; abdullah.dertli@gmail.com.
[2] Akpinar Science High School, Samsun, 55760, Turkey. E-mail: sawasx@gmail.com.

https://doi.org/10.46939/J.Sci.Arts-25.1-a13

bicomplex Oresme sequence is defined using these numbers. Subsequently, some fundamental equations of the bicomplex Oresme sequence, such as the Binet formula and generating function are obtained.

Definition 1.1. Bicomplex numbers, being a subalgebra of quaternions, form a non-commutative structure. Let $\mathbb{C}_2$ be the set of bicomplex numbers. Then, $\mathbb{C}_2$ is defined as follows:

$$\mathbb{C}_2 = \{x = x_1 + x_2 i + x_3 j + x_4 ij \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\},$$

where $1, i, j, ij$ are units of bicomplex numbers and these units satisfy the following equations.

$$i^2 = j^2 = -1,\ ij = ji, (ij)^2 = 1\ .$$

Table 1 shows the multiplication of the units of bicomplex numbers is as follows.

Table 1. Multiplication of the units of bicomplex numbers.

$\cdot$	1	i	j	ij
1	1	i	j	ij
i	i	-1	ij	$-j$
j	j	ij	-1	$-i$
ij	ij	$-j$	$-i$	1

The sum of two bicomplex numbers $z_1 = a_1 + b_1 i + c_1 j + d_1 ij$ and $z_2 = a_2 + b_2 i + c_2 j + d_2 ij$ is as follows:

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)ij\ .$$

Furthermore, the multiplication of z_1 and z_2 is defined by

$$\begin{aligned} z_1.z_2 =& (a_1 + b_1 i + c_1 j + d_1 ij)(a_2 + b_2 i + c_2 j + d_2 ij) \\ =& (a_1 a_2 - b_1 b_2 - c_1 c_2 + d_1 d_2) + i(a_1 b_2 + b_1 a_2 - c_1 d_2 - c_2 d_1) \\ &+ j(a_1 c_2 - b_1 d_2 + c_1 a_2 - b_2 d_1) + ij(a_1 d_2 + b_1 c_2 + c_1 b_2 + a_2 d_1)\ . \end{aligned}$$

Any bicomplex number x has involutions in three different forms.

$$\boldsymbol{x = x_1 + x_2 i + x_3 j + x_4 ij},$$

$$\bar{x}_i = x_1 - x_2 i + x_3 j - x_4 ij,$$

$$\boldsymbol{\bar{x}_j = x_1 + x_2 i - x_3 j - x_4 ij},$$

$$\bar{x}_{ij} = x_1 - x_2 i - x_3 j + x_4 ij.$$

Norms are obtained from the definitions of involutions as follows:

$$N_i(x) = x\bar{x}_i = (x_1 + x_2 i + x_3 j + x_4 ij)(x_1 - x_2 i + x_3 j - x_4 ij),$$

$$N_j(x) = x\bar{x}_j = (x_1 + x_2 i + x_3 j + x_4 ij)(x_1 + x_2 i - x_3 j - x_4 ij),$$

$$N_{ij}(x) = x\bar{x}_{ij} = (x_1 + x_2 i + x_3 j + x_4 ij)(x_1 - x_2 i - x_3 j + x_4 ij).$$

The two idempotent elements found in $\mathbb{C}_2$ are as follows. These elements satisfy given equations:

$$e_1 = \frac{1+ij}{2},\ e_2 = \frac{1-ij}{2}$$

$$\boldsymbol{e_1 e_2 = 0,\ e_1 + e_2 = 1,\ e_1 - e_2 = ij.}$$

Furthermore, any x element in $\mathbb{C}_2$ can be expressed in terms of e_1 and e_2:

$$x = z_1 + z_2 j = \beta_1 e_1 + \beta_2 e_2 = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2\ ,$$

where z_1,z_2 are complex numbers, β_1 and β_2 are idempotent coefficients and this equation is called the idempotent representation of element x. Also, for the elements x_1, x_2 are written as

$$x_1 = \beta_1 e_1 + \beta_2 e_2\ , x_2 = \theta_1 e_1 + \theta_2 e_2$$

and the algebraic operations are defined as follows:

$$x_1 + x_2 = (\beta_1 + \theta_1)e_1 + (\beta_2 + \theta_2)e_2,$$

$$x_1 x_2 = (\beta_1 \theta_1)e_1 + (\beta_2 \theta_2)e_2,$$

$$\boldsymbol{x_1^n = \beta_1^n e_1 + \beta_2^n e_2},$$

where $\beta_1 = z_1 - iz_2$ and $\beta_2 = z_1 + iz_2$.

2. OBTAINING THE BICOMPLEX ORESME SEQUENCE VIA BICOMPLEX ORESME NUMBERS

This section defines bicomplex Oresme numbers and the bicomplex Oresme sequence and examines some properties. Let $\mathbb{C}$ be the set of complex numbers. It is defined as follows:

$$\mathbb{C} = \{ z = x + yi \mid x, y \in \mathbb{R} \}.$$

For any complex number z, its real and imaginary parts are denoted as $\mathrm{Re}(z) = x, \mathrm{Im}(z) = y$, respectively. The matrix representation of elements in set $\mathbb{C}$ is

$$M = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

which is isomorphic to the set of 2×2 real matrices. In other words, the transformation

$$\varphi_{\mathbb{C}}: \mathbb{C} \longrightarrow M_{\mathbb{C}}$$

is a field isomorphism between $\mathbb{C}$ and $M_{\mathbb{C}} = \left\{M = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in \mathbb{R}\right\}$.

More detailed information about complex numbers can be found in [10].

Definition 2.1. The Oresme complex number $\mathbb{C}O_n$ is defined as

$$\mathbb{C}O_n = \mathrm{O}_n + \mathrm{O}_{n+1} i$$

where O_n is nth Oresme number, $n \geq 0$.

For example, $\mathbb{C}O_0 = \frac{1}{2} i$, $\mathbb{C}O_1 = \frac{1}{2} + \frac{1}{2} i$, $\mathbb{C}O_2 = \frac{1}{2} + \frac{3}{8} i$. The set of matrix representations of Oresme complex numbers is

$$M_{\mathbb{C}O_n} = \begin{bmatrix} O_n & -O_{n+1} \\ O_{n+1} & O_n \end{bmatrix}$$

where $\mathrm{O}_n, \mathrm{O}_{n+1}$ are Oresme numbers.

Definition 2.2. The set of bicomplex Oresme numbers is defined as

$$\mathbb{BC}O_{\mathrm{n}} = \{\mathbb{C}O_n + \mathbb{C}O_{n+2}\mathrm{j} \mid \mathbb{C}O_n = O_n + iO_{n+1}, j^2 = -1\}$$

where O_n is nth Oresme number for $n \geq 0$. The few terms of this set are $O_0, O_1, O_2, \ldots$, where

$$\mathbb{BC}O_0 = \mathbb{C}O_0 + \mathbb{C}O_2 j = O_0 + iO_1 + jO_2 + ijO_3,$$

$$\mathbb{BC}O_1 = \mathbb{C}O_1 + \mathbb{C}O_3 j = O_1 + iO_2 + jO_3 + ijO_4,$$

$$\mathbb{BC}O_2 = \mathbb{C}O_2 + \mathbb{C}O_4 j = O_2 + iO_3 + jO_4 + ijo_5,$$

$$\vdots$$

$$\mathbb{BC}O_n = \mathbb{C}O_n + \mathbb{C}O_{n+2} j = O_n + iO_{n+1} + jO_{n+2} + ijO_{n+3}\,.$$

Bicomplex Oresme numbers can be expressed in another way as follows.

$$\mathbb{BC}O_{\mathrm{n}} = \{\partial_0, \partial_1, \partial_2, \ldots, \partial_n \ldots \mid \partial_n = O_n - jO_{n+2}\}$$

where

$$\partial_n = \theta_n e_1 + \gamma_n e_2,$$

$$\theta_n = O_n + iO_{n+2}\,,$$

$$\gamma_n = O_n - iO_{n+2}\,.$$

Using the ξ matrix transformation, we can express the matrix representation of bicomplex Oresme numbers as follows.

$$\xi : \mathbb{BC}O_n \longrightarrow M_{B\mathbb{C}_n}$$

where this transformation forms a ring isomorphism, and the set of resulting matrices

$$M_{B\mathbb{C}_n} = \left\{\begin{bmatrix} \mathbb{C}O_n & -\mathbb{C}O_{n+2} \\ \mathbb{C}O_{n+2} & \mathbb{C}O_n \end{bmatrix} : \mathbb{C}O_n, \mathbb{C}O_{n+2} \in \mathbb{BC}O_{\mathrm{n}}\right\}.$$

Corollary 2.3. Bicomplex Oresme numbers can be represented as 4×4 real matrices as follows:

$$M_{B\mathbb{C}_n} = \begin{bmatrix} O_n & -O_{n+1} & -O_{n+2} & O_{n+3} \\ O_{n+1} & O_n & -O_{n+3} & -O_{n+2} \\ O_{n+2} & -O_{n+3} & O_n & -O_{n+1} \\ O_{n+3} & O_{n+2} & O_{n+1} & O_n \end{bmatrix}.$$

Example 2.4. The matrix representation of the $Onth$ bicomplex Oresme number

$$O_0 = \frac{1}{2}i + \frac{2}{4}j + \frac{3}{8}ij = \frac{1}{2}i + \left(\frac{2}{4} + \frac{3}{8}i\right)j$$

is obtained using complex matrix representation as follows:

$$M_{B\mathbb{C}_0} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{2}{4} & \frac{3}{8} \\ \frac{1}{2} & 0 & -\frac{3}{8} & -\frac{2}{4} \\ \frac{2}{4} & -\frac{3}{8} & 0 & -\frac{1}{2} \\ \frac{3}{8} & \frac{2}{4} & \frac{1}{2} & 0 \end{bmatrix}.$$

Definition 2.5. The involutions of bicomplex Oresme number are as follows:

$$\overline{\mathbb{B}\mathbb{C}O_{n_i}} = O_n - O_{n+1}i + O_{n+2}j - O_{n+3}ij,$$

$$\overline{\mathbb{B}\mathbb{C}O_{n_j}} = O_n + O_{n+1}iO_{n+2}j - O_{n+3}ij,$$

$$\overline{\mathbb{B}\mathbb{C}O_{n_{ij}}} = O_n - O_{n+1}i - O_{n+2}j + o_{n+3}ij.$$

Considering that the set of bicomplex Oresme numbers is defined in two ways, conjugate forms can also be expressed as follows:

(i) $\left(\bar{\partial}_n\right)_i = \bar{\theta}_n e_1 + \bar{\gamma}_n e_2$,

(ii) $\left(\bar{\partial}_n\right)_j = \theta_n e_1 + \gamma_n e_2$,

(iii) $\left(\bar{\partial}_n\right)_{ij} = \bar{\theta}_n e_1 + \bar{\gamma}_n e_2$.

Definition 2.6. Norms can be obtained with the help of involutions as follows.

$$N_i(\mathbb{B}\mathbb{C}\mathrm{O}_n) = O_n\overline{O_{ni}} = (O_n + O_{n+1}i + O_{n+2}j + O_{n+3}ij)(O_n - O_{n+1}i + O_{n+2}j - O_{n+3}ij),$$

$$N_j(\mathbb{B}\mathbb{C}\mathrm{O}_n) = O_n\overline{O_{nj}} = (O_n + O_{n+1}i + O_{n+2}j + O_{n+3}ij)(O_n + O_{n+1}i - O_{n+2}j - O_{n+3}ij),$$

$$N_{ij}(\mathbb{B}\mathbb{C}\mathrm{O}_n) = O_n\overline{O_{nij}} = (O_n + O_{n+1}i + O_{n+2}j + O_{n+3}ij)(O_n - O_{n+1}i - O_{n+2}j + O_{n+3}ij).$$

When necessary operations are made, we obtain the following result.

Corollary 2.7. Norms of the bicomplex Oresme number O_n are as follows:

$$N_i(\mathbb{BC}\mathrm{O}_n) = O_n^2 + O_{n+1}^2 - O_{n+2}^2 - O_{n+3}^2 + 2O_nO_{n+2}j + 2O_{n+1}O_{n+3}j,$$

$$N_j(\mathbb{BC}\mathrm{O}_n) = O_n^2 - O_{n+1}^2 + O_{n+2}^2 - O_{n+3}^2 + 2O_nO_{n+1}i + 2O_{n+2}O_{n+3}i,$$

$$N_{ij}(\mathbb{BC}\mathrm{O}_n) = O_n^2 + O_{n+1}^2 + O_{n+2}^2 + O_{n+3}^2 + 2O_nO_{n+3}ij\ .$$

Considering the recurrence relation of the Oresme number sequence, the following bicomplex Oresme sequence can be obtained.

Definition 2.8. The bicomplex Oresme sequence is defined by the recurrence relation

$$O_{n+1}^* = O_n^* - \frac{1}{4}O_{n-1}^*$$

with initial conditions

$$O_0^* = \mathbb{C}O_0 + \mathbb{C}O_2 j = O_0 + O_1 i + O_2 j + O_3 ij,$$
$$O_1^* = \mathbb{C}O_1 + \mathbb{C}O_3 j = O_1 + O_2 i + O_3 j + O_4 ij,$$

for $n \geq 1$.

A few terms of this sequence are

$$\frac{2}{4}j, \frac{1}{2} + \frac{3}{8}j, \frac{2}{4} + \frac{4}{16}j, \dots .$$

Theorem 2.9. The Binet formula for the bicomplex Oresme sequence is as follows:

$$O_n^* = \frac{n}{2^n} + \frac{n+1}{2^{n+1}}i + \frac{n+2}{2^{n+2}}j + \frac{n+3}{2^{n+3}}ij\ .$$

Proof: It can be easily obtained from the Oresme number's Binet formula.

Theorem 2.10. The generating function for the bicomplex Oresme sequence is defined as follows:

$$GO_n^*(x) = \frac{1}{4-4x-x^2}\left(\frac{7}{2} + j(2-2x)\right).$$

Proof: The generating function for the bicomplex Oresme sequence is $\mathrm{GO}_n^*(x)$ and this function can be written as

$$GO_n^*(x) = \sum_{n=0}^{\infty} {O_n}^* x^n\ .$$

Then,

$$\sum_{n=0}^{\infty} {O_{n+2}}^* x^n = \sum_{n=0}^{\infty} {O_{n+1}}^* x^n - \frac{1}{4}\sum_{n=0}^{\infty} {O_n}^* x^n,$$

$$\sum_{n=2}^{\infty} {O_n}^* x^{n-2} = \sum_{n=1}^{\infty} {O_n}^* x^{n-1} - \frac{1}{4}\sum_{n=0}^{\infty} {O_n}^* x^{n},$$

it can be written. Then, we get,

$$\frac{1}{x^2}\left(-{O_0}^* - {O_1}^* + \sum_{n=0}^{\infty} {O_n}^* x^n\right) = \frac{1}{x}\left(-{O_0}^* + \sum_{n=0}^{\infty} {O_n}^* x^n\right) - \frac{1}{4}\sum_{n=0}^{\infty} {O_n}^* x^n .$$

Finally, we obtain

$$GO_n^*(x) = \frac{1}{4-4x-x^2}\left(\frac{7}{2} + j(2-2x)\right)$$

with the help of necessary operations.

Theorem 2.11. The Cassini identity for the bicomplex Oresme number is as follows:

$${O_n}^* {O_{n+2}}^* - {O_{n+1}^*}^2 = -3\left(\frac{15}{2^{2n+6}} + \frac{5}{2^{2n+4}} + \frac{1}{2^{2n+2}}\right).$$

Proof: It can be seen by induction on n.

3. CONCLUSION

In this study, the bicomplex Oresme numbers are defined, and a numerical example is given. Thus, various topics such as algebra, number theory and geometry, which have significant importance in mathematics, are evaluated together, including bicomplex numbers, number sequences and matrix properties. For future studies, other applications of bicomplex numbers and matrices can be evaluated based on the results obtained here.

REFERENCES

[1] Segre, C., *Mathematische Annalen*, **40**(3), 413, 1892.
[2] Rochon, D., Tremblay, S. *Advances in Applied Clifford Algebras*, **14**(2), 231, 2004.
[3] Vignaux, J., *Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.*, **27**(6), 641, 1938.
[4] Gervais Lavoie, R., Marchildon, L., Rochon, D. *Advances in Applied Clifford Algebras*, **21**(3), 561, 2011.
[5] Aydın, F. T., *Chaos, Solitons & Fractals*, **106**, 147, 2018.
[6] Çelemoğlu, Ç., *Mathematics*, **12**(3), 461, 2024.
[7] Cook, C. K., *Proceedings of The Tenth International Research Conference on Fibonacci Numbers and Their Applications*, Springer, **9**, 87, 2004.
[8] Nurkan, S. K., Güven, I. A., *arXiv preprint*, arXiv:150803972, 2015.

[9] Halıcı, S., *Models and Theories in Social Systems*, 509, 2019.
[10] Başkan, T., Kompleks Fonsiyonsiyonlar Teorisi, *Dora yayıncılık*, Bursa, 2000.