ORIGINAL PAPER

NUMERICAL ACCURACY OF THE TIME-FRACTIONAL DIFFERENTIAL EQUATIONS OF DIFFERENT ORDER BY USING HOMOTOPY PERTURBATION AND SUMUNDU TRANSFORM METHOD

AACHAL MISHRA[1], NAJMUDDIN AHMAD[2]

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Abstract. *In this paper we obtain the analytical solution of problems which is related to Time-fractional differential equation. We get the solution by using combined form of the Homotopy perturbation method with the Sumudu transform. The solutions we obtain are approximately given in form of Mittag-Leffler functions. The solutions are in series form and which is convergent. All calculations are performed by Matlab 2022 and Python.*

Keywords: *Fractional Power series; non-linear partial differential equation; fractional partial differential equations.*

1. INTRODUCTION

There are different kind of methods hasbeen developed to get solution of nonlinear fractional partial differential equations, that is the homotopy perturbation method, the variational iteration methods, homotopy perturbation transform methods, Adomian decomposition methods .Most of problems in physics and chemistry can be converted into the form of partial differential equations and which is of non- linear form can be solved by several methods. Difficulties are coming to solve nonlinear problems and to find an analytic solution. These methods were proposed to find approximate solutions of nonlinear equations. By combining the Homotopy in topology and classical perturbation techniques a method developed by He is called HPM and the method is very useful to solve many linear and nonlinear problems. In the recent years, this method is combined with the variational iteration method and Laplace transformation method to produce a highly effective technique for handling nonlinear terms is known as homotopy perturbation transform method (HPTM). Having scale and unit-preserving properties, the Sumudu transform may be used to solve problems without resorting to a new frequency domain [1].

Many researcher use homotopy perturbation transform method for different type of linear and nonlinear differential equations afterwards .Recently, another such a combination in which the sumudu transformation method and homotopy perturbation method are applied to solve nonlinear problems is known as homotopy perturbation Sumudu transform method (HPSTM). Many researcher use HPSTM to obtain the analytical exact and approximate solutions for linear and nonlinear partial differential equations. In this paper, HPSTM is applied to find the solution of some Time-fractional differential equations [2].

[1]Integral University, Department of Mathematics & Statistics, Lucknow-226024(UP), India. E-mail: aachalmishra98@gmail.com.
[2]Integral University, Department of Mathematics & Statistics, Lucknow-226024(UP), India. E-mail: mailto:najmuddinahmad33@gmail.com.

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2. METHODS

2.1. FRACTIONAL CALCULUS

There are several definitions of a fractional derivative of order e.g. Riemann-Liouville, Grunwald-Letnikow, Caputo and Generalized Functions Approach. The most commonly used definitions are the Riemann-Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

2.2. RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

Riemann-Liouville fractional derivative acquiring by Riemann in 1847 is defined as follows.

$$ {}_{a}^{RL}D_t^{\gamma}\mu(t) = \left(\frac{d}{dt}\right)^n \left({}_aD_t{}^{-(n-\gamma)}\right)\mu(t) $$
$$ = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{\mu(x)}{(t-x)^{\gamma-n+1}}dx\ , \ t > a $$

where $\gamma > 0$; this operator is an extension of Cauchy's integral from the natural number to real one. In addition, according to the above relation, if $0 < \gamma < 1$ then the Riemann-Liouville operator reduced to

$$ {}_{a}^{RL}D_t^{\gamma}\mu(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_a^t \frac{\mu(x)}{(t-x)^{\gamma}}dx $$

2.3. CAPUTO DERIVATIVE

Since, Riemann-Liouville fractional derivatives failed in the description and modeling of some complex phenomena, Caputo derivative was introduced in 1967. The Caputo derivative of fractional order γ ($n-1 \le \gamma <$ n) of function (t) defined as [3]

$$ {}_{a}^{c}D_t^{\gamma}\mu(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_a^t \frac{\mu(s)}{(t-s)^{\gamma}}ds. $$

2.4. SUMUNDU TRANSFROM METHOD

Definition 1. The Sumundu transform of fractional order derivative is defined by [4-5]

$$ \mathrm{S}[D_t^{\gamma}(\mu)(x)] = \frac{1}{u^{\gamma}}S[(\mu)(x)] - \sum_{k=0}^{\infty}\frac{1}{u^{\gamma-k}}[\mu^{(k)}(x)]_{t=0,}\ n-1 < \gamma \le n, n \in N $$

2.5. HOMOTOPY PERTURBATION SUMUDU TRANSFORM METHOD (HPSTM)

To illustrate the basic idea of this method, we consider a general fractional nonlinear partial differential equation with initial conditions of the form

$$D_t^{\gamma}\omega(x,t) = \mathcal{L}\,\omega(x,t) + N\omega(x,t) + \xi(x,t) \text{ and } \gamma\epsilon(n-1,n] \quad (1)$$

and subject to initial condition

$$\frac{\partial^r \omega(x,0)}{\partial t^r} = \omega^r(x,0) = \mu_r(x) \quad (2)$$
$$r = 0,1,\dots\dots\dots.n-1$$

where, $D_t^{\gamma}\omega(x,t)$ is the Caputo fractional derivative, $\xi(x,t)$ is the source term, $\mathcal{L}$ is the linear operator and N is the general nonlinear operator.

Applying Sumundu transform on both side of the above equation and which is denoted by S we get

$$\mathrm{S}[D_t^{\gamma}\omega(x,t)] = S[\mathcal{L}\,\omega(x,t) + N\omega(x,t) + \xi(x,t)], \quad (3)$$

Using the property of transformation and the initial conditions in above equation we get

$$u^{-\gamma}\mathrm{S}\,[\omega(x,t)] - \sum_{k=0}^{n-1} u^{-(\gamma-k)}\ \omega^k(x,0) = S[\mathcal{L}\,\omega(x,t) + N\omega(x,t) + \xi(x,t)], \quad (4)$$

and

$$\mathrm{S}\,[\omega(x,t)] = \sum_{k=0}^{n-1} u^k\,\mu_k(x) + u^{\gamma}\,[\mathcal{L}\,\omega(x,t) + N\omega(x,t) + \xi(x,t)], \quad (5)$$

Let us consider the solution of above equation is of the form

$$\omega(x,t) = \sum_{m=0}^{\infty} p^m\,\omega_m(x,t) \quad (6)$$

where , $p \in [0,1]$ is the homotopy parameter.

$N\omega(x,t)$ is the nonlinear term and can be decomposed as

$$N\omega(x,t) = \sum_{m=0}^{\infty} p^m\,\omega_m(x) \quad (7)$$

where, H_i are He's polynomials, which can be calculated with the formula

$$H_m(\omega_{0,}\omega_{1,}\omega_{2,}\dots\dots\dots\dots\dots\dots\omega_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}[N\left(\sum_{i=0}^{\infty} p^i\omega_i\right)]_{p=0} \quad (8)$$
$$\mathrm{m}= 0, 1, 2, 3...$$

substituting eqs. (6) and (7) in eq. (5) we get

$$\sum_{m=0}^{\infty} p^m\, \omega_m(x,t) = S^{-1}[\sum_{k=0}^{n-1} u^k\, \mu_k(x)] + pS^{-1}[u^\gamma S[\mathcal{L}\,(\sum_{m=0}^{\infty} p^m\, \omega_m(x,t)) + \sum_{m=0}^{\infty} p^m\, H_m(\omega) + \xi(x,t)]], \tag{9}$$

Equating the terms with identical powers of p, we can obtain a series of equatons as the follows:

$$p^0: \omega_0(x,t) = S^{-1}[\sum_{k=0}^{n-1} u^k\, \mu_k(x)\ ],$$
$$\dots \tag{10}$$
$$p^n: \omega_n(x,t) = S^{-1}[u^\gamma S[\mathcal{L}\,(\sum_{m=0}^{\infty} p^m\, \omega_m(x,t)) + \sum_{m=0}^{\infty} p^m\, H_m(\omega) + \xi(x,t)]].$$

The solution above equation is given by-

$$\omega(x,t) = \lim_{p\to 1} \sum_{m=0}^{\infty} p^m\, \omega_m(x,t) = \omega_0 + \omega_1 + \omega_2 \dots\dots\dots\dots \tag{11}$$

3. NUMERICAL METHODS FOR DIFFERENTIAL EQUATION

Many researchers have looked at the fractional differential equation from two aspect, first theoretical aspect of finding solutions and second aspect is analytical and numerical method for finding solutions. It is known that there are many fields of applications where we can use the fractional calculus as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, viscoelasticity, heat conduction, electricity mechanics etc [6-10].

Example 1. Consider the time-fractional differential equation

$$D^\gamma \omega(x,t) + \omega\omega_x - \omega\omega_{xxx} + \omega_{xxxxx} = 0$$
$$\omega(x,0) = e^x, \gamma \in (0,1)$$

Exact solution for the above given equation for $\gamma = 1$ is $e^{x+t} erfc(\sqrt{t})$. By applying above way of solving equation we get

$$\mathrm{S}[D^\gamma \omega(x,t)] = -S[\omega\omega_x - \omega\omega_{xxx} + \omega_{xxxxx}]$$

and also we can write

$$\sum_{m=0}^{\infty} p^m\, \omega_m(x,t) =$$
$$S^{-1}[\sum_{k=0}^{n-1} u^k\, \mu_k(x)] + pS^{-1}[uS[\mathcal{L}\,(\sum_{m=0}^{\infty} p^m\, \omega_m(x,t)) + \sum_{m=0}^{\infty} p^m\, H_m(\omega) + \xi(x,t)]],$$
$$\sum_{m=0}^{\infty} p^m\, \omega_m(x,t) = e^x + p\, S^{-1}[u^\gamma(-S[(\sum_{m=0}^{\infty} p^m \frac{\partial^4}{\partial x^4}\, \omega_m(x,t)) + \sum_{m=0}^{\infty} p^m\, H_m(\omega)]]$$
$$p^0: \omega_0(x,t) = e^x: \omega_1(x,t) = -S^{-1}[u^{1/2}(S[\left(\frac{\partial^4}{\partial x^4}\, \omega_0(x,t)\right) + H_0(\omega)]] = \frac{-e^x}{\Gamma(\frac{3}{2})} t^\gamma$$
$$p^2: \omega_2(x,t) = -S^{-1}[u^{1/2}(S[(e^x) + \omega_1(\omega_{1_x} H_1(\omega)]] = \frac{e^x}{\Gamma(2)} t^{2\gamma}$$
$$p^3: \omega_3(x,t) = -S^{-1}[u^{1/2}(S[(e^x) + \omega_2(\omega_{2_x} H_1(\omega)]] = \frac{-e^x}{\Gamma(\frac{3}{2}+1)} t^{3\gamma}$$
$$\vdots$$

$$\omega(x,t) = e^{x} + \frac{-e^{x}}{\Gamma(2)} t^{\gamma} + \frac{e^{x}}{\Gamma(3)} t^{2\gamma} + \frac{-e^{x}}{\Gamma(4)} t^{3\gamma} - \ \ldots\ldots..$$

For $\gamma = 1$ solution of this equation is e^{x-t}. Here, we can see different data given for $x = t, x > t$ and $x < t$ and get that for better result we need to take the values only $x > t$ and $x < t$. We get much variation between exact and approximate solution for $x = t$ [11].

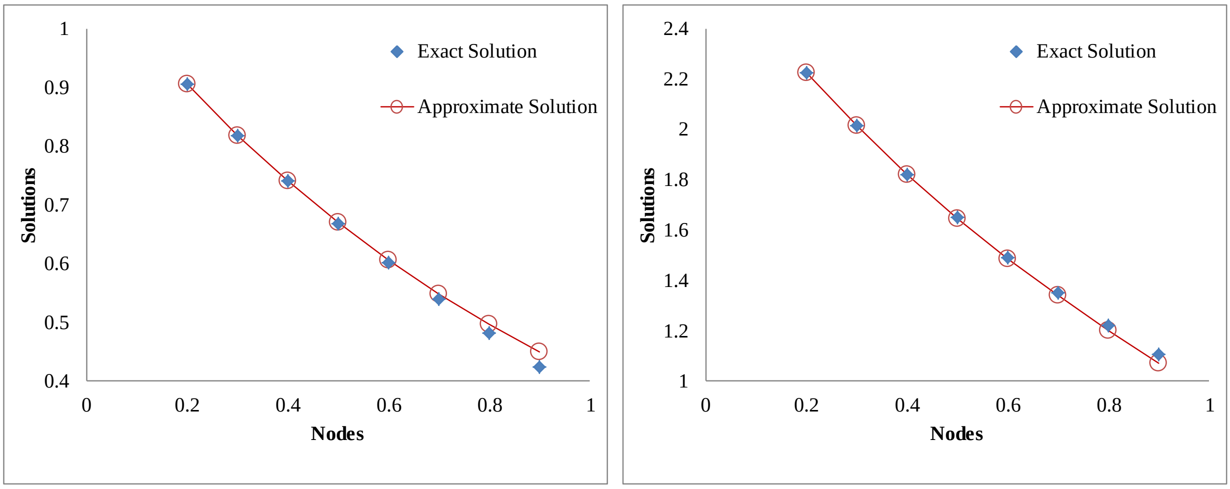
Table 1. Exact & Approximate solution along with error vector for $x < t$.

Nodes($x = 0.1 < t$)	Exact Solution	Approximate Solution	Error Vectors
0.2	0.904800000000000	0.9048000000000000	0.00000000000000
0.3	0.818700000000000	0.8184000000000000	0.00030000000000
0.4	0.739700000000000	0.7408000000000000	0.0011000000000
0.5	0.667700000000000	0.6703000000000000	0.0026000000000
0.6	0.601200000000000	0.6065000000000000	0.0053000000000
0.7	0.539100000000000	0.548800000000000	0.0097000000000
0.8	0.480400000000000	0.496600000000000	0.0162000000000
0.9	0.423800000000000	0.4493000000000000	0.0255000000000

Table 2. Exact & Approximate solution along with error vector for $x > t$.

Nodes($x = .9 > t$),Values for t only	Exact Solution	Approximate Solution	Error Vectors
0.1	2.225500000000000	2.2255000000000000	0.00000000000000
0.2	2.013800000000000	2.0136000000000000	0.00020000000000
0.3	1.822100000000000	1.8213000000000000	0.01800000000000
0.4	1.648700000000000	1.6463000000000000	0.00240000000000
0.5	1.491800000000000	1.4860000000000000	0.00580000000000
0.6	1.349900000000000	1.3380000000000000	0.01100000000000
0.7	1.221400000000000	1.1999000000000000	0.02141000000000
0.8	1.105200000000000	1.0691000000000000	0.03610000000000

The above table shows low variation between exact and approximate solutions.



a) **b)**

Figure 1. Comparison between exact and approximate solution for: a) *x* < *t*; b) *x* > *t*.

4. CONCLUSION

In this work, the HPSTM is applied to solving both the linear and nonlinear type of Fractional Differential Equations. From the numerical result, we can find that the Conformable fractional differential transform method is an efficient algorithm. We can see here is least difference between the exact solution and the solution by this method .We use only first several terms to approximate the exact solutions, the numerical result has high precision. In general, some non-linear partial differential equations are hard to deal with this method is a powerful tool to cope with this problem.

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