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MATRIX PROPERTIES OF FACTORIAL POLYNOMIALS AND APPLICATIONS TO OVERALL FUNCTIONAL INTEGRO DIFFERENTIAL EQUATIONS

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Abstract. *In this study, by using the falling and rising factorial polynomials and the Stirling numbers of first and second kinds; the operational matrix characterizations of these expressions along with their derivatives and integrals are derived. Also, a typical matrix-collocation method based on the matrix relations of the mentioned polynomials is offered for solving the linear Volterra type integro-differential equations with functional delays. The approximate solutions are achieved in terms of the falling and rising factorial polynomials, thereby in the finite power series form. Additionally, a convergence criterion and an error technique related with the residual functions, together with three illustrative examples are created to reveal the efficiency of the method.*

Keywords: *Factorial polynomials; functional delays; matrix-collocation method; residual error analysis; Volterra integro-differential equation with functional delay.*

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

The factorial polynomials and the Stirling numbers [1-13] play important role in the theory of finite differences, combinatorics, interpolation theory, spline theory and other branches of mathematics; also in the science and engineering fields. The numerical method we used in this study is established by the aid of the factorial polynomials, the Stirling numbers, and their matrix relations. Definitions and basic properties of the relevant expressions are presented below.

1.1. STIRLING NUMBERS

In mathematics and applied mathematics, the Stirling numbers originate from many combinatorics problems. They are revealed in the eighteenth century by James Stirling. There are two kinds of these numbers, denominated the Stirling numbers of the first and second kinds [3].

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The Stirling numbers of the first kind $S_1(n,k)$ can be interpreted as the number of permutations of n elements with disjoint cycles. The Stirling numbers of the first kind are defined by the generated function [1-5]

$$\frac{\left[\ln(1+x)\right]^k}{k!}=\sum_{n=0}^{\infty}S_1(n,k)\frac{x^n}{n!}, \qquad |x|<1. \tag{1}$$

From the above equation (1), one has obtained the recurrence relations, for $S_1(n,k)$:

$$S_1(n,k)=S_1(n-1,k-1)-(n-1)S_1(n-1,k)$$

where

$$S_1(n,0)=0 \quad (n\in\mathbb{N}),\ S_1(n,n)=1 \quad (n\in\mathbb{N}_0),$$

$$S_1(n,1)=(-1)^{n-1}(n-1)! \quad (n\in\mathbb{N}),\ S_1(n,k)=0 \quad (n<k \text{ or } k<0).$$

The Stirling numbers of the second kind $S_2(n,k)$ can be interpreted as the number of ways to partition a set of n elements into k nonempty subsets [4]. A general formula for the Stirling numbers of the second kind is given as [2, 4]

$$S_2(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n. \tag{2}$$

The Stirling numbers of the second kind are defined by the generating function [1-8]

$$\frac{\left(e^x-1\right)^k}{k!}=\sum_{n=0}^{\infty}S_2(n,k)\frac{x^n}{n!}. \tag{3}$$

From (2) or (3), the following recurrence relation [6-8] is obtained as

$$S_2(n+1,k)=kS_2(n,k)+S_2(n,k-1)$$

where

$$S_2(n,0)=0 \quad (n\in\mathbb{N}),\ S_2(n,n)=1 \quad (n\in\mathbb{N}),$$
$$S_2(n,1)=1 \quad (n\in\mathbb{N}),\ S_2(n,k)=0 \quad (n<k \text{ or } k<0).$$

1.2. FACTORIAL POLYNOMIALS

There are two types of factorial polynomials, called the falling and rising factorial polynomials [9, 10, 13]. It is known that, for $x\in\mathbb{R}$, these quantities are defined, respectively, as the falling factorial polynomials and the rising factorial polynomials [1, 9-13]

$$(x)_{\underline{n}}=\begin{cases} n\geq 1, & x(x-1)\ldots(x-n+1)=\prod_{k=0}^{n-1}(x-k) \\ n=0, & 1 \end{cases}, \tag{4}$$

$$(x)^{\overline{n}} = \begin{cases} n \geq 1, & x(x+1)\ldots(x+n-1) = \prod_{k=0}^{n-1}(x+k) \\ n = 0, & 1 \end{cases}. \tag{5}$$

Also, the Stirling numbers of the first and second kinds can be generated by [1, 2, 9, 10]

$$(x)_{\underline{n}} = \sum_{k=0}^{n} S_1(n,k)x^k,\ x \in \mathbb{R},\ n \in \mathbb{N}_0, \tag{6}$$

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_{\underline{k}},\ \ x \in \mathbb{R},\ n \in \mathbb{N}_0, \tag{7}$$

$$(x)^{\overline{n}} = \sum_{k=0}^{n} (-1)^{n-k} S_1(n,k)x^k,,\ x \in \mathbb{R},\ n \in \mathbb{N}_0 \tag{8}$$

where

$$S_1(n,0) = S_2(n,0) = \delta_{n,0},\ (x)_{\underline{0}} = (x)^{\overline{0}} = 1,\ (x)_{\underline{n}} = \binom{x}{n}.$$

1.3. PROBLEM DESCRIPTION

In this study, a revisited and modified matrix-collocation method is proposed, by using the operational matrix properties of the factorial polynomials and the Stirling numbers, for solving the generalized linear integro-differential equations with the functional delays in the form

$$\sum_{k=0}^{m_1}\sum_{j=0}^{m_2} P_{kj}(x)y^{(k)}(h_{kj}(x)) = f(x) + \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)y^{(r)}(g_{rs}(t))dt,\ x,t \in [a,b] \tag{9}$$

under the mixed conditions

$$\sum_{k=0}^{m-1}\sum_{l=0}^{L} a_{kjl}(x)y^{(k)}(\gamma_l) = \lambda_j,\ j = 0,1,2,\ldots,m-1 \tag{10}$$

where $f(x)$, $P_{kj}(x)$, $g_{rs}(t)$, $h_{kj}(x)$, $u_{rs}(x)$, $v_{rs}(x)$ and $K_{rs}(x,t)$ are functions defined on the interval a_{jkl}, γ_l and λ_j are appropriate constants. Additionally, the functional expressions $h_{kj}(x)$ and $g_{rs}(t)$ can be composed of the functions as follows:

$$h_{kj}(x) = x,\ g_{rs}(t) = t,\ h_{kj}(x) = \alpha_{kj}x + \beta_{kj},\ g_{rs}(t) = \lambda_{rs}t + \mu_{rs},\ h_{kj}(x) = x + \tau_{kj}(x),\ g_{rs}(t) = t + \gamma_{rs}(t)$$

where α_{kj}, β_{kj}, λ_{rs} and μ_{rs} are suitable constants, $\tau_{kj}(x)$ and $\gamma_{rs}(t)$ are the analytic functions on $a \leq x,t \leq b$.

The generalized functional integro-differential equations (9), which also involve Volterra and Fredholm forms, play an important role for modelling problems in the fields of applied mathematics and engineering such as physics, biology, electrodynamics, visco-elasticity, heat and mass transfer, economy, mechanics, and astronomy etc. [14-22]. These type equations are usually difficult to solve analytically; so there are particular methods that have solved them numerically. In the recent years, to solve these functional integro-differential equations, the following methods have been presented: Taylor collocation method [14,18], Dickson matrix-collocation method [15], Chelyshkov collocation method [16], Taylor polynomial method [17], Matching polynomial method [19], Morgan-Voyce matrix method [20], Hybrid Taylor-Lucas collocation method [22], Chebyshev interpolation method [23], Lagrange collocation method [24], Chebyshev collocation method [25], Laguerre polynomial approach [26], Variational iteration method [27], Legendre collocation method [28], Backward substitution method [29], Tau method [30], Bernoulli matrix collocation method [31], New Chelyshkov approach technique [32]. Collocation method based on Bernoulli operational matrix [33], Operational approach method [34], Homotopy perturbation method [35], Bessel polynomial approach [36], Legendre–Gauss collocation method [37], Adomian's decomposition method [38], and Mott polynomial method [39] and so on.

The purpose of this study is to investigate the numerical solutions of the $m-th$ order generalized functional integro-differential equation (9) with the conditions (10), in terms of the falling polynomials (4) and (5) defined by $(x)_{\underline{n}}$ and $(x)^{\overline{n}}$; in other words, in the truncated series forms respectively,

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} \underline{a}_n (x)_{\underline{n}}, \ a \le x \le b, \tag{11}$$

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} \overline{a}_n (x)^{\overline{n}}, \ a \le x \le b \tag{12}$$

where $\underline{a}_n$ and $\overline{a}_n$ are the unknown falling and rising polynomial coefficients. On the other hand, to indicate the efficiency of the modified matrix-collocation method, the error estimation and the convergence criterion based on the residual functions are developed.

The rest of the structure of this paper is organized as follows: The following section describes the matrix properties of factorial polynomials and the solution method for linear Volterra-type integro-differential equations involving functional delays. In Section 3, we present the convergence criterion for the falling and rising factorial polynomials solutions by using the residual function. Section 4 provides numerical examples that demonstrate the efficiency of the presented method. Finally, in Section 5, we present the paper conclusion.

2. MATRIX PROPERTIES OF FACTORIAL POLYNOMIALS AND SOLUTION METHOD

2.1. MATRIX RELATIONS BETWEEN THE FACTORIAL POLYNOMIALS AND THE STIRLING NUMBERS

The factorial polynomials and the Stirling numbers are important instruments for solving problems in mathematics and applied mathematics such as the problem (9) – (10). In this section, the operational matrix relations between the factorial polynomials and the Stirling numbers which are defined by the formulas (1) – (4), are generated by means of the relations (6) – (8) as follows.

For $n=0,1,...,N$, the matrix forms of the falling factorial polynomials $(x)_{\underline{n}}$ and the rising factorial polynomials $(x)^{\overline{n}}$, respectively, are obtained as

$$\begin{bmatrix} (x)_{\underline{0}} \\ (x)_{\underline{1}} \\ (x)_{\underline{2}} \\ \vdots \\ (x)_{\underline{N}} \end{bmatrix} = \begin{bmatrix} S_1(0,0) & 0 & 0 & \dots & 0 \\ S_1(1,0) & S_1(1,1) & 0 & \dots & 0 \\ S_1(2,0) & S_1(2,1) & S_1(2,2) & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ S_1(N,0) & S_1(N,1) & S_1(N,2) & \dots & S_1(N,N) \end{bmatrix} . \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^N \end{bmatrix}$$

or briefly,

$$\underline{\chi}^T(x) = \mathbf{S}_1^T \mathbf{X}^T(x) \Rightarrow \underline{\chi}(x) = \mathbf{X}(x)\mathbf{S}_1$$

$$\underline{\chi}(x) = \left[(x)_{\underline{0}} \quad (x)_{\underline{1}} \quad (x)_{\underline{2}} \quad \dots \quad (x)_{\underline{N}} \right], \; \mathbf{X}(x) = \left[1 \quad x \quad x^2 \quad \dots \quad x^N \right],$$

$$\boldsymbol{S}_1 = \begin{bmatrix} S_1(0,0) & S_1(1,0) & S_1(2,0) & \dots & S_1(N,0) \\ 0 & S_1(1,1) & S_1(2,1) & \dots & S_1(N,1) \\ 0 & 0 & S_1(2,2) & \dots & S_1(N,2) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & S_1(N,N) \end{bmatrix} \tag{13}$$

another different matrix form is obtained as

$$\begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^N \end{bmatrix} = \begin{bmatrix} S_2(0,0) & 0 & 0 & \dots & 0 \\ S_2(1,0) & S_2(1,1) & 0 & \dots & 0 \\ S_2(2,0) & S_2(2,1) & S_2(2,2) & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ S_2(N,0) & S_2(N,1) & S_2(N,2) & \dots & S_2(N,N) \end{bmatrix} . \begin{bmatrix} (x)_{\underline{0}} \\ (x)_{\underline{1}} \\ (x)_{\underline{2}} \\ \vdots \\ (x)_{\underline{N}} \end{bmatrix}$$

or briefly,

$$\mathbf{X}^T(x) = \mathbf{S}_2^T \underline{\chi}^T(x) \Rightarrow \mathbf{X}(x) = \underline{\chi}(x)\mathbf{S}_2;$$

$$\boldsymbol{S}_2 = \begin{bmatrix} S_2(0,0) & S_2(1,0) & S_2(2,0) & \dots & S_2(N,0) \\ 0 & S_2(1,1) & S_2(2,1) & \dots & S_2(N,1) \\ 0 & 0 & S_2(2,2) & \dots & S_2(N,2) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & S_2(N,N) \end{bmatrix} \tag{14}$$

and

$$\begin{bmatrix} (x)^{\overline{0}} \\ (x)^{\overline{1}} \\ (x)^{\overline{2}} \\ \vdots \\ (x)^{\overline{N}} \end{bmatrix} = \begin{bmatrix} S_1(0,0) & 0 & 0 & \dots & 0 \\ -S_1(1,0) & S_1(1,1) & 0 & \dots & 0 \\ S_1(2,0) & -S_1(2,1) & S_1(2,2) & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (-1)^N S_1(N,0) & (-1)^{N-1} S_1(N,1) & (-1)^{N-2} S_1(N,2) & \dots & (-1)^0 S_1(N,N) \end{bmatrix} . \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^N \end{bmatrix}$$

or briefly,

$$\left(\overline{\chi}(x)\right)^T = \left(\mathbf{S}_1^*\right)^T \mathbf{X}^T(x) \Rightarrow \overline{\chi}(x) = \mathbf{X}(x)\mathbf{S}_1^*$$

$$\overline{\chi}(x) = \left[(x)^{\overline{0}} \quad (x)^{\overline{1}} \quad (x)^{\overline{2}} \quad \dots \quad (x)^{\overline{N}} \right], \tag{15}$$

$$\boldsymbol{S}_1^* = \begin{bmatrix} S_1(0,0) & -S_1(1,0) & S_1(2,0) & \dots & (-1)^N S_1(N,0) \\ 0 & S_1(1,1) & -S_1(2,1) & \dots & (-1)^{N-1} S_1(N,1) \\ 0 & 0 & S_1(2,2) & \dots & (-1)^{N-2} S_1(N,2) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^0 S_1(N,N) \end{bmatrix}.$$

Consequently, the matrix relations between the $\boldsymbol{X}(x)$, $\underline{\chi}(x)$ and $\overline{\chi}(x)$ become

$$\underline{\chi}(x) = \boldsymbol{X}(x)\boldsymbol{S}_1,\ \boldsymbol{X}(x) = \underline{\chi}(x)\boldsymbol{S}_2 \Rightarrow \underline{\chi}(x) = \boldsymbol{X}(x)\boldsymbol{S}_2^{-1},\ \overline{\chi}(x) = \boldsymbol{X}(x)\boldsymbol{S}_1^*,\ \underline{\chi}(x) = \boldsymbol{X}(x)\boldsymbol{S}_1 = \boldsymbol{X}(x)\boldsymbol{S}_2^{-1} \Rightarrow \boldsymbol{S}_1 = \boldsymbol{S}_2^{-1} \tag{16}$$

2.2. DESCRIPTION OF BASIC MATRIX FORMS

Firstly, we write Eq. (9) as

$$D_{kj}(x) = P_{kj}(x)\, y^{(k)}\left(h_{kj}(x)\right),\ I_{rs}(x) = \int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)\, y^{(r)}\left(g_{rs}(t)\right) dt\,. \tag{17}$$

By using the matrix relations (13) – (15), the matrix forms of the falling factorial polynomial solution (11) and the rising factorial polynomial solution (12), respectively, are achieved as, for $n = 0,1,\dots,N$,

$$y_N(x) = \underline{\boldsymbol{\chi}}(x)\underline{\mathbf{A}} = \mathbf{X}(x)\mathbf{S}_1\underline{\mathbf{A}} = \mathbf{X}(x)\mathbf{S}_2^{-1}\underline{\mathbf{A}} \text{ and } y_N(x) = \overline{\boldsymbol{\chi}}(x)\overline{\mathbf{A}} = \mathbf{X}(x)\mathbf{S}_1^*\overline{\mathbf{A}};$$
$$\underline{\mathbf{A}} = \begin{bmatrix}\underline{a}_0 & \underline{a}_1 & \underline{a}_2 & \dots & \underline{a}_N\end{bmatrix}^T,\ \overline{\mathbf{A}} = \begin{bmatrix}\overline{a}_0 & \overline{a}_1 & \overline{a}_2 & \dots & \overline{a}_N\end{bmatrix}^T. \tag{18}$$

Also, the derivatives of the expressions (18) can be obtain in the following matrix forms

$$y_N^{(k)}(x) = \mathbf{X}^{(k)}(x)\mathbf{S}_1\underline{\mathbf{A}} = \mathbf{X}(x)\mathbf{B}^k\mathbf{S}_1\underline{\mathbf{A}} \text{ and } y_N(x) = \mathbf{X}^{(k)}(x)\mathbf{S}_1^*\overline{\mathbf{A}} = \mathbf{X}(x)\mathbf{B}^k\mathbf{S}_1^*\overline{\mathbf{A}};$$
$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},\ \mathbf{B}^0 = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{19}$$

and, by substituting $x \to h_{kj}(x)$ and $t \to g_{rs}(t)$ into (19),

$$y^{(k)}\left(h_{kj}(x)\right) = \mathbf{X}\left(h_{kj}(x)\right)\mathbf{B}^k\mathbf{S}_1\underline{\mathbf{A}}, \qquad y^{(k)}\left(h_{kj}(x)\right) = \mathbf{X}\left(h_{kj}(x)\right)\mathbf{B}^k\mathbf{S}_1^*\overline{\mathbf{A}}$$
$$y^{(r)}\left(g_{rs}(t)\right) = \mathbf{X}\left(g_{rs}(t)\right)\mathbf{B}^r\mathbf{S}_1\underline{\mathbf{A}}, \qquad y^{(r)}\left(g_{rs}(t)\right) = \mathbf{X}\left(g_{rs}(t)\right)\mathbf{B}^r\mathbf{S}_1^*\overline{\mathbf{A}} \tag{20}$$

where

$$\mathbf{X}\left(h_{kj}(x)\right) = \begin{bmatrix}1 & h_{kj}(x) & \left(h_{kj}(x)\right)^2 & \dots & \left(h_{kj}(x)\right)^N\end{bmatrix},\ \mathbf{X}\left(g_{rs}(t)\right) = \begin{bmatrix}1 & g_{rs}(t) & \left(g_{rs}(t)\right)^2 & \dots & \left(g_{rs}(t)\right)^N\end{bmatrix}.$$

By using the matrix forms (20) into the differential part $D_{kj}(x)$ and the integral part $I_{rs}(x)$, we get the following matrix forms;

$$\underline{\mathbf{D}}_{kj}(x) = \mathbf{P}_{kj}(x)\mathbf{X}\big(h_{kj}(x)\big)\mathbf{B}^k\mathbf{S}_1\underline{\mathbf{A}}, \quad \underline{\mathbf{I}}_{rs}(t) = \int_{u_{rs}}^{v_{rs}} \mathbf{X}(x)\mathbf{K}_{rs}\mathbf{X}^T(t)\mathbf{X}\big(g_{rs}(t)\big)\mathbf{B}^r\mathbf{S}_1\underline{\mathbf{A}}dt,$$
$$\bar{\mathbf{D}}_{kj}(x) = \mathbf{P}_{kj}(x)\mathbf{X}\big(h_{kj}(x)\big)\mathbf{B}^k\mathbf{S}_1^*\bar{\mathbf{A}}, \quad \bar{\mathbf{I}}_{rs}(t) = \int_{u_{rs}}^{v_{rs}} \mathbf{X}(x)\mathbf{K}_{rs}\mathbf{X}^T(t)\mathbf{X}\big(g_{rs}(t)\big)\mathbf{B}^r\mathbf{S}_1^*\bar{\mathbf{A}}dt, \tag{21}$$

where

$$\mathbf{K}_{rs}(x,t) = \left[k_{mn}^{rs}\right] = \mathbf{X}(x)\mathbf{K}_{rs}\mathbf{X}^T(t), \; k_{mn}^{rs} = \frac{1}{m!n!}\frac{\partial^{m+n}K_{rs}(0,0)}{\partial x^m \partial t^n}, \; m,n = 0,1,\ldots,N.$$

Thereafter, the matrix forms of the integral part $I_{rs}(x)$ in (21) can be rearranged as

$$\underline{\mathbf{I}}_{rs}(x) = \mathbf{X}(x)\mathbf{K}_{rs}\mathbf{Q}_{rs}(x)\mathbf{B}^r\mathbf{S}_1\underline{\mathbf{A}} \text{ and } \bar{\mathbf{I}}_{rs}(x) = \mathbf{X}(x)\mathbf{K}_{rs}\mathbf{Q}_{rs}(x)\mathbf{B}^r\mathbf{S}_1^*\bar{\mathbf{A}};$$
$$\mathbf{Q}_{rs}(x) = \left[\mathbf{q}_{mn}^{rs}(x)\right] = \int_{u_{rs}}^{v_{rs}} \mathbf{X}^T(t)\mathbf{X}\big(g_{rs}(t)\big)dt, \; \mathbf{q}_{mn}^{rs}(x) = \int_{u_{rs}}^{v_{rs}} t^m g_{rs}^n(t)dt, \; m,n = 0,1,\ldots,N. \tag{22}$$

2.3. ESTABLISHMENT OF SOLUTION METHOD

To construct the matrix-collocation method, we firstly substitute the matrix expressions (21) and (22) into Eq. (17) and then, by using the standard collocation points defined by $x_i = a + \frac{b-a}{N}i, \; i = 0,1,\ldots,N$. Thereby, we constitute the following main matrix equations with regard to the falling and rising factorial polynomials, respectively,

$$\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}\underline{\mathbf{D}}_{kj} - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\underline{\mathbf{I}}_{rs} = \mathbf{F} \;\Rightarrow\; \left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}\mathbf{P}_{kj}\mathbf{X}_{kj}\mathbf{B}^k - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\bar{\mathbf{X}}\bar{\mathbf{K}}_{rs}\bar{\mathbf{Q}}_{rs}\mathbf{B}^r\right\}\mathbf{S}_1\underline{\mathbf{A}} = \mathbf{F}$$

or briefly,

$$\underline{\mathbf{W}}\underline{\mathbf{A}} = \mathbf{F} \;\Leftrightarrow\; \left[\underline{\mathbf{W}};\mathbf{F}\right] \tag{23}$$

and

$$\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}\bar{\mathbf{D}}_{kj} - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\bar{\mathbf{I}}_{rs} = \mathbf{F} \;\Rightarrow\; \left\{\sum_{k=0}^{m_1}\sum_{j=0}^{m_2}\mathbf{P}_{kj}\mathbf{X}_{kj}\mathbf{B}^k - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\bar{\mathbf{X}}\bar{\mathbf{K}}_{rs}\bar{\mathbf{Q}}_{rs}\mathbf{B}^r\right\}\mathbf{S}_1^*\bar{\mathbf{A}} = \mathbf{F}$$

or briefly,

$$\bar{\mathbf{W}}\bar{\mathbf{A}} = \mathbf{F} \;\Leftrightarrow\; \left[\bar{\mathbf{W}};\mathbf{F}\right] \tag{24}$$

where

$$\underline{\boldsymbol{D}}_{kj}=\begin{bmatrix}\underline{\boldsymbol{D}}_{kj}(x_0)\\ \underline{\boldsymbol{D}}_{kj}(x_1)\\ \underline{\boldsymbol{D}}_{kj}(x_2)\\ \vdots\\ \underline{\boldsymbol{D}}_{kj}(x_N)\end{bmatrix},\ \bar{\boldsymbol{D}}_{kj}=\begin{bmatrix}\bar{\boldsymbol{D}}_{kj}(x_0)\\ \bar{\boldsymbol{D}}_{kj}(x_1)\\ \bar{\boldsymbol{D}}_{kj}(x_2)\\ \vdots\\ \bar{\boldsymbol{D}}_{kj}(x_N)\end{bmatrix},\ \underline{\boldsymbol{I}}_{rs}=\begin{bmatrix}\underline{\boldsymbol{I}}_{rs}(x_0)\\ \underline{\boldsymbol{I}}_{rs}(x_1)\\ \underline{\boldsymbol{I}}_{rs}(x_2)\\ \vdots\\ \underline{\boldsymbol{I}}_{rs}(x_N)\end{bmatrix},\ \bar{\boldsymbol{I}}_{rs}=\begin{bmatrix}\bar{\boldsymbol{I}}_{rs}(x_0)\\ \bar{\boldsymbol{I}}_{rs}(x_1)\\ \bar{\boldsymbol{I}}_{rs}(x_2)\\ \vdots\\ \bar{\boldsymbol{I}}_{rs}(x_N)\end{bmatrix},\ \bar{\boldsymbol{Q}}_{rs}=\begin{bmatrix}\bar{\boldsymbol{Q}}_{rs}(x_0)\\ \bar{\boldsymbol{Q}}_{rs}(x_1)\\ \bar{\boldsymbol{Q}}_{rs}(x_2)\\ \vdots\\ \bar{\boldsymbol{Q}}_{rs}(x_N)\end{bmatrix},\ \boldsymbol{F}=\begin{bmatrix}f(x_0)\\ f(x_1)\\ f(x_2)\\ \vdots\\ f(x_N)\end{bmatrix}.$$

$$\boldsymbol{P}_{kj}=diag\left[\mathrm{P}_{kj}(\mathrm{x}_0)\ \ \mathrm{P}_{kj}(\mathrm{x}_1)\ \ \mathrm{P}_{kj}(\mathrm{x}_2)\ \ \dots\ \ \mathrm{P}_{kj}(\mathrm{x}_N)\right],$$

$$\boldsymbol{X}_{kj}=\begin{bmatrix}\boldsymbol{X}(h_{kj}(x_0))\\ \boldsymbol{X}(h_{kj}(x_1))\\ \boldsymbol{X}(h_{kj}(x_2))\\ \vdots\\ \boldsymbol{X}(h_{kj}(x_N))\end{bmatrix}=\begin{bmatrix}1 & h_{kj}(x_0) & h_{kj}^{2}(x_0) & \dots & h_{kj}^{N}(x_0)\\ 1 & h_{kj}(x_1) & h_{kj}^{2}(x_1) & \dots & h_{kj}^{N}(x_1)\\ 1 & h_{kj}(x_2) & h_{kj}^{2}(x_2) & \dots & h_{kj}^{N}(x_2)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & h_{kj}(x_N) & h_{kj}^{2}(x_N) & \dots & h_{kj}^{N}(x_N)\end{bmatrix},$$

$$\bar{\boldsymbol{X}}=diag[\boldsymbol{X}(x_0)\ \ \boldsymbol{X}(x_1)\ \ \boldsymbol{X}(x_2)\ \ \dots\ \ \boldsymbol{X}(x_N)],\ \ \bar{\boldsymbol{K}}_{rs}=diag\left[\boldsymbol{K}_{rs}\ \ \boldsymbol{K}_{rs}\ \ \boldsymbol{K}_{rs}\ \ \dots\ \ \boldsymbol{K}_{rs}\right].$$

Moreover, the compact matrix forms of the conditions (10), by considering the matrix relations (19), become for $j=0,1,...,m-1;\ a\le\gamma_l\le b,$

$$\underline{\mathbf{U}}_j\underline{\mathbf{A}}=\boldsymbol{\lambda}_j\ \Leftrightarrow\ \left[\underline{\mathbf{U}}_j;\boldsymbol{\lambda}_j\right],\tag{25}$$

$$\bar{\mathbf{U}}_j\bar{\mathbf{A}}=\boldsymbol{\lambda}_j\ \Leftrightarrow\ \left[\bar{\mathbf{U}}_j;\boldsymbol{\lambda}_j\right]\tag{26}$$

where

$$\underline{\mathbf{U}}_j=\sum_{k=0}^{m-1}\sum_{l=1}^{L}a_{jkl}\mathbf{X}(\gamma_l)\mathbf{B}^k\mathbf{S}_1\ \text{ and }\ \bar{\mathbf{U}}_j=\sum_{k=0}^{m-1}\sum_{l=1}^{L}a_{jkl}\mathbf{X}(\gamma_l)\mathbf{B}^k\mathbf{S}_1^{*}.$$

In conclusion, in order to find the solutions of the problem (9) – (10) in terms of the falling factorial and the rising factorial polynomials, we replace the m rows of the augmented matrix in (25), and in (26), by the any of the m rows of the augmented matrix (23) and (24). Accordingly, we have the result augmented matrices as follows:

$$\left[\underline{\mathbf{W}}^{*};\mathbf{F}^{*}\right]\ \text{ or }\ \underline{\mathbf{W}}^{*}\underline{\mathbf{A}}=\mathbf{F}^{*}\ \Rightarrow\ \underline{\mathbf{A}}=\left(\underline{\mathbf{W}}^{*}\right)^{-1}\mathbf{F}^{*},\tag{27}$$

$$\left[\bar{\mathbf{W}}^{*};\mathbf{F}^{*}\right]\ \text{ or }\ \bar{\mathbf{W}}^{*}\bar{\mathbf{A}}=\mathbf{F}^{*}\ \Rightarrow\ \bar{\mathbf{A}}=\left(\bar{\mathbf{W}}^{*}\right)^{-1}\mathbf{F}^{*}\tag{28}$$

where

$$rank\left[\underline{\mathbf{W}}^{*};\mathbf{F}^{*}\right]=rank\left[\underline{\mathbf{W}}^{*}\right]=N+1\ \text{ and }\ rank\left[\bar{\mathbf{W}}^{*};\mathbf{F}^{*}\right]=rank\left[\bar{\mathbf{W}}^{*}\right]=N+1.$$

Thereby, by using the coefficient matrices (27) and (28), we obtain the polynomial solutions:

$$y_N(x)=\mathbf{X}(x)\mathbf{S}_1\underline{\mathbf{A}}\ \text{ and }\ y_N(x)=\mathbf{X}(x)\mathbf{S}_1^{*}\bar{\mathbf{A}}.$$

3. ERROR ANALYSIS AND CONVERGENCE CRITERION

3.1. RESIDUAL ERROR ANALYSIS

When the obtained numerical solution $y_N(x)$ is substituted in Eq. (9), the resulting equation must satisfy approximately, that is, for $x_i \in [a,b]$;

$$R_N(x) = \sum_{k=0}^{m_1}\sum_{j=0}^{m_2} P_{kj}(x) y_N^{(k)}(h_{kj}(x)) - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t) y_N^{(r)}(g_{rs}(t))dt - f(x) \cong 0 \tag{29}$$

and $R_N(x_i) \le 10^{-k_i}$ $(k_i \in \mathbb{Z}^+)$. If $\max 10^{-k_i} = 10^{-k}$ is prescribed, then the truncation limit N is increased until the difference $R_N(x_i)$ at each of the points becomes smaller than 10^{-k} [15,19,22,40]. Also, to validate the numerical accuracy we calculate the absolute error as

$$\text{Absolute error} = |u_N(x) - u_{\text{exact}}(x)|.$$

3.2. CONVERGENCE CRITERION

In this article, we investigate the convergence of the falling and rising factorial polynomials solutions by using the residual function of some model problems in Banach spaces. Previously, an efficient error analysis making use of the residual function was presented to improve the solutions [40, 41]. By means of residual function $R_N(x)$, the convergence of the rising factorial and falling factorial polynomials approach can be determined. $R_N(x)$ can be written in the Taylor expansion form;

$$R_N(x) = \Omega_0 x^0 + \Omega_1 x^1 + ... + \Omega_N x^N = \sum_{i=0}^{N} \Omega_i x^i$$

where

$$R_N(x):[a,b] \to \mathbb{R} \text{ or } R_N(x):[a+\varepsilon, b-\varepsilon] \to \mathbb{R};$$

ε is a sufficiently small value.

Theorem 3.1. The residual function sequence $\{R_N(x)\}_{N=2}^{\infty}$ is convergent in Banach space and the following inequality is satisfied, so that $0 < \tilde{s}_N < 1$ where $\tilde{s}_N$ is constant in Banach space;

$$\|R_{N+1}(x)\| \le \tilde{s}_N \|R_N(x)\|. \tag{30}$$

Proof: It must be shown that the residual function sequence $\{R_N(x)\}_{N=m}^{\infty}$ is a Cauchy sequence in Banach space. By considering $\|R_{N+1}(x)\|$, it can be written as

$$
\begin{aligned}
\left\|R_{N+1}(x)\right\| &= \sup\left\{\left|\sum_{i=0}^{N}\Omega_i x^i\right| : x\in[a,b] \vee x\in[a+\varepsilon,b-\varepsilon]\right\}\\
&\le \sup\left\{\sum_{i=0}^{N}\left|\Omega_i x^i\right| : x\in[a,b] \vee x\in[a+\varepsilon,b-\varepsilon]\right\}\\
&= \left|R_{N+1}(b)\right|
\end{aligned}
$$

so, at the point b, the inequality (30) can be rewritten as

$$\left\|R_{N+1}(b)\right\| \le \tilde{s}_N \left\|R_N(b)\right\|. \tag{31}$$

From the inequality (31),

$$\left|R_{N+1}(b)\right| \le \left|R_{N+1}(b)-R_N(b)\right| \le \left(\tilde{s}_N-1\right)\left|R_N(b)\right|. \tag{32}$$

Generalizing the inequality (32),

$$\left|R_{N+1}(b)-R_N(b)\right| \le \left(\tilde{s}_N-1\right)\left|R_N(b)\right| \le \left(\tilde{s}_N-1\right)^2\left|R_{N-1}(b)\right| \le \ldots \le \left(\tilde{s}_N-1\right)^{N-1}\left|R_m(b)\right|. \tag{33}$$

For $\forall N,K\in\mathbb{N}$ and $N\ge K$, we obtain,

$$
\begin{aligned}
\left|R_N(b)-R_K(b)\right| &\le \left|R_N(b)-R_{N-1}(b)+\ldots+R_{K+1}(b)-R_K(b)\right|\\
&\le \left|R_N(b)-R_{N-1}(b)\right|+\ldots+\left|R_{K+1}(b)-R_K(b)\right|\\
&\le \left(\tilde{s}_N-1\right)^{N-1}\left|R_m(b)\right|+\ldots+\left(\tilde{s}_N-1\right)^{K-1}\left|R_m(b)\right|\\
&= \frac{1-\left(\tilde{s}_N-1\right)^{N-2-K}}{1-\left(\tilde{s}_N-1\right)}\left(\tilde{s}_N-1\right)^{K-1}\left|R_m(b)\right|.
\end{aligned}
$$

If $\tilde{s}_N<1$ is taken in the above inequality, then $\lim\limits_{N,K\to\infty}\left|R_N(b)-R_K(b)\right|=0$. Thereby, $\left\{R_N(x)\right\}_{N=m}^{\infty}$ is a Cauchy sequence in Banach space and it is convergent.

4. NUMERICAL EXAMPLES

In this section, some numerical examples are given to show the effectiveness and reliability of the presented methods. All computations have been carried out via the Mathematica 12 software. Moreover, the numerical results are demonstrated in figures and tables.

Example 1 (Test problem). Let us consider linear functional Volterra type integro-differential equation,

$$y''(x)-e^x y'(x^2-x)+y(x^2)=f(x)+\int_x^{x+1} 2te^{-x^2}y'(t^2)dt;$$

$$f(x)=-2e^{-x^2}x(x+1)(2x+1)+e^x\left(1-2(x-1)x\right)+x^4-x^2+1$$

with the initial conditions $y(0)=y'(0)=-1$. The exact solution of the problem is $y(x)=x^2-x-1$. Firstly, let us compute the solution in terms of the falling factorial polynomials defined by

$$y(x)\cong y_N(x)=\sum_{n=0}^{2}\underline{a}_n\left(x\right)_{\underline{n}}\Rightarrow y_2(x)=\mathbf{X}(x)\mathbf{S}_1\underline{\mathbf{A}}$$

The set of the collocation points for $N=2$ is calculated as $\{x_0=0,\ x_1=1,\ x_2=2\}$ and according to Eq. (9), the given equation is explicitly written as

$$P_{00}(x)y^{(0)}(h_{00}(x))+P_{10}(x)y^{(1)}(h_{10}(x))+P_{20}(x)y^{(2)}(h_{20}(x))=f(x)+\int_{u_{10}}^{v_{10}}K_{10}(x,t)y^{(1)}(g_{10}(t))dt\ ;$$

$$P_{00}(x)=1,\ P_{10}(x)=-e^x,\ P_{20}(x)=1,\ h_{00}(x)=x^2,\ h_{10}(x)=x^2-x,\ h_{20}(x)=x,$$
$$u_{10}(x)=x,\ v_{10}(x)=x+1,\ K_{10}(x,t)=2te^{-x^2},\ g_{10}(t)=t^2.$$

By using Eq. (23), we can shortly write the fundamental matrix equation of the problem

$$\left\{\sum_{k=0}^{2}\sum_{j=0}^{0}\boldsymbol{P}_{kj}\boldsymbol{X}_{kj}\boldsymbol{B}^k-\sum_{r=0}^{1}\sum_{s=0}^{0}\overline{\boldsymbol{X}}\overline{\boldsymbol{K}}_{rs}\overline{\boldsymbol{Q}}_{rs}\boldsymbol{B}^r\right\}\boldsymbol{S}_1\underline{\boldsymbol{A}}=\boldsymbol{F}\Rightarrow\underline{\boldsymbol{W}}\underline{\boldsymbol{A}}=\boldsymbol{F}\Leftrightarrow[\underline{\boldsymbol{W}};\quad\boldsymbol{F}];$$

$$\underline{\boldsymbol{W}}=\left\{\boldsymbol{P}_{00}\boldsymbol{X}_{00}\boldsymbol{B}^0+\boldsymbol{P}_{10}\boldsymbol{X}_{10}\boldsymbol{B}+\boldsymbol{P}_{20}\boldsymbol{X}_{20}\boldsymbol{B}^2-\overline{\boldsymbol{X}}\overline{\boldsymbol{K}}_{10}\overline{\boldsymbol{Q}}_{10}\boldsymbol{B}^1\right\}\boldsymbol{S}_1\Rightarrow\underline{W}=\begin{bmatrix}1&-2&-1\\1&1-e&4-e\\1&19-e^2&232-5e^2\end{bmatrix}.$$

$$\boldsymbol{S}_1=\begin{bmatrix}S_1(0,0)&S_1(1,0)&S_1(2,0)\\0&S_1(1,1)&S_1(2,1)\\0&0&S_1(2,2)\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix},\ \mathbf{F}=\begin{bmatrix}2\\e-12e^{-1}+1\\-3e^2-60e^{-4}+13\end{bmatrix}.$$

Thereby, we obtain the augmented matrix form of the fundamental matrix equation as follows:

$$[\underline{\boldsymbol{W}};\quad\boldsymbol{F}]=\begin{bmatrix}1&-2&-1&;&2\\1&1-e&4-e&;&e-12e^{-1}+1\\1&19-e^2&232-5e^2&;&-3e^2-60e^{-4}+13\end{bmatrix}$$

from Eq. (25), the matrix forms for the initial conditions are computed as

$$y'(0)=-1\Rightarrow\boldsymbol{X}(0)\boldsymbol{B}\boldsymbol{S}_1\underline{\boldsymbol{A}}=-1\Rightarrow[0\quad1\quad1;\quad-1]$$

$$y(0)=-1\Rightarrow\boldsymbol{X}(0)\boldsymbol{S}_1\underline{\boldsymbol{A}}=-1\Rightarrow[1\quad0\quad0;\quad-1],$$

or briefly

$$[\underline{\boldsymbol{U}};\quad\lambda]=\begin{bmatrix}1&0&0;&-1\\0&1&1;&-1\end{bmatrix}.$$

Therefore, from Eq. (27), the new augmented matrix for the problem is calculated as

$$\left[\underline{\boldsymbol{W}}^*;\quad \boldsymbol{F}^*\right]=\begin{bmatrix}1 & -2 & -1; & 2\\ 1 & 0 & 0; & -1\\ 0 & 1 & 1; & -1\end{bmatrix}.$$

Solving this system, the unknown coefficients matrix $\underline{\boldsymbol{A}}$ is obtained as;

$$\underline{\boldsymbol{A}}=\begin{bmatrix}-1 & -2 & 1\end{bmatrix}^T.$$

Then, we obtain the exact solution $y_2(x)=\mathbf{X}(x)\mathbf{S}_1\underline{\mathbf{A}}\Rightarrow y_2(x)=x^2-x-1$. Similarly, we write the approximate solution based on the rising factorial polynomials in the form

$$y(x)\equiv y_N(x)=\sum_{n=0}^{N}\bar{a}_n(x)^{\bar{n}}\ \Rightarrow y_2(x)=\boldsymbol{X}(x)\boldsymbol{S}_1^*\bar{\boldsymbol{A}}$$

and from Eq. (24), we have the fundamental matrix equation

$$\left\{\sum_{k=0}^{2}\sum_{j=0}^{0}\boldsymbol{P}_{kj}\boldsymbol{X}_{kj}\boldsymbol{B}^k-\sum_{r=0}^{1}\sum_{s=0}^{0}\bar{\boldsymbol{X}}\bar{\boldsymbol{K}}_{rs}\bar{\boldsymbol{Q}}_{rs}\boldsymbol{B}^r\right\}\boldsymbol{S}_1^*\bar{\boldsymbol{A}}=\boldsymbol{F}\ \Rightarrow\bar{\boldsymbol{W}}\ \bar{\boldsymbol{A}}=\boldsymbol{F}\Leftrightarrow[\bar{\boldsymbol{W}};\quad \boldsymbol{F}].\ ;$$

$$\bar{\boldsymbol{W}}=\{\boldsymbol{P}_{00}\boldsymbol{X}_{00}\boldsymbol{B}^0+\boldsymbol{P}_{10}\boldsymbol{X}_{10}\boldsymbol{B}^1+\boldsymbol{P}_{20}\boldsymbol{X}_{20}\boldsymbol{B}^2-\bar{\boldsymbol{X}}\bar{\boldsymbol{K}}_{10}\bar{\boldsymbol{Q}}_{10}\boldsymbol{B}^1\}\boldsymbol{S}_1^*,$$

$$\boldsymbol{S}_1^*=\begin{bmatrix}S_1(0,0) & -S_1(1,0) & S_1(2,0)\\ 0 & S_1(1,1) & -S_1(2,1)\\ 0 & 0 & S_1(2,2)\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix},$$

Thereby, we obtain the augmented form for the fundamental matrix equation as follows

$$\left[\bar{\boldsymbol{W}};\boldsymbol{F}\right]=\begin{bmatrix}1 & -2 & 3 & ; & 2\\ 1 & 1-e & 2+e & ; & e-12e^{-1}+1\\ 1 & 19-e^2 & 194-3e^2; & & -3e^2-60e^{-4}+13\end{bmatrix}$$

In addition, from Eq. (26), we can write the augmented matrix form for the initial conditions as

$$y(0)=-1\Rightarrow X(0)\mathrm{S}_1^*\bar{\mathrm{A}}=-1\Rightarrow\begin{bmatrix}1 & 0 & 0; & -1\end{bmatrix},$$

$$y'(0)=1\Rightarrow\boldsymbol{X}(0)\boldsymbol{B}\boldsymbol{S}_1^*\bar{\boldsymbol{A}}=-1\Rightarrow\begin{bmatrix}0 & 1 & -1; & -1\end{bmatrix},$$

or briefly in the form

$$[\bar{\boldsymbol{U}};\boldsymbol{\lambda}]=\begin{bmatrix}1 & 0 & 0; & -1\\ 0 & 1 & -1; & -1\end{bmatrix}.$$

Hence, from Eq. (28), the new augmented matrix is calculated as

$$\left[\bar{\boldsymbol{W}}^*;\boldsymbol{F}^*\right]=\begin{bmatrix}1 & -2 & 3; & 2\\ 1 & 0 & 0; & -1\\ 0 & 1 & -1; & -1\end{bmatrix}$$

Solving this system, the unknown coefficients matrix is obtained as $\bar{\boldsymbol{A}}=\begin{bmatrix}-1 & 0 & 1\end{bmatrix}^T$. Consequently, we get the exact solution is based on the rising factorial polynomials $y_2(x)=x^2-x-1$.

Example 2. Let us now take $f(x)=e^x-e^{2x+1}+1$ in Example 1, then the exact solution of the equation under the initial conditions $y(0)=y'(0)=1$, becomes $y(x)=e^x$. In the present case, by using the results in Example 1, we investigate the approximate solution based on the falling factorial polynomials, defined by

$$y(x)\cong y_2(x)=\sum_{n=0}^{2}\underline{a}_n(x)_{\underline{n}}\quad\Rightarrow\quad y_2(x)=\mathbf{X}(x)\mathbf{S}_1\underline{\mathbf{A}},\ 0\le x\le\frac{1}{2}.$$

The set of the collocation points for $N=2$ is calculated as $\left\{x_0=0,\ x_1=\frac{1}{4},\ x_2=\frac{1}{2}\right\}$. By using Eq. (23), we can shortly write the fundamental matrix equation of the problem

$$\left\{\sum_{k=0}^{2}\sum_{j=0}^{0}\boldsymbol{P}_{kj}\boldsymbol{X}_{kj}\boldsymbol{B}^k-\sum_{r=0}^{1}\sum_{s=0}^{0}\bar{\boldsymbol{X}}\bar{\boldsymbol{K}}_{rs}\bar{\boldsymbol{Q}}_{rs}\boldsymbol{B}^r\right\}\boldsymbol{S}_1\underline{\boldsymbol{A}}=\boldsymbol{F}\Rightarrow\underline{\boldsymbol{W}}\ \underline{\boldsymbol{A}}=\boldsymbol{F}\Leftrightarrow\left[\underline{\boldsymbol{W}};\quad\boldsymbol{F}\right];$$

$$\underline{\boldsymbol{W}}=\begin{bmatrix}1 & -2 & -1\\ 1 & -\frac{43}{32}-\sqrt[4]{e} & -\frac{13}{8}-\frac{5}{8}\sqrt[4]{e}\\ 1 & -\frac{5}{4}-\sqrt{e} & -\frac{47}{16}-\frac{1}{2}\sqrt{e}\end{bmatrix},\ \boldsymbol{F}=\begin{bmatrix}2-e\\ 1+\sqrt[4]{e}-\sqrt{e^3}\\ 1+\sqrt[4]{e}-\sqrt{e^3}\end{bmatrix}.$$

The augmented matrix form of the fundamental matrix equation can be obtained as

$$[\underline{\boldsymbol{W}};\boldsymbol{F}]=\begin{bmatrix}1 & -2 & -1 & ; & 2-e\\ 1 & -\frac{43}{32}-\sqrt[4]{e} & -\frac{13}{8}-\frac{5}{8}\sqrt[4]{e} & ; & 1+\sqrt[4]{e}-\sqrt{e^3}\\ 1 & -\frac{5}{4}-\sqrt{e} & -\frac{47}{16}-\frac{1}{2}\sqrt{e} & ; & 1+\sqrt{e}-e^2\end{bmatrix}.$$

and the matrix forms for the initial conditions:

$$y(0)=1\Rightarrow\boldsymbol{X}(0)\boldsymbol{S}_1\ \underline{\boldsymbol{A}}=1\Rightarrow\begin{bmatrix}1 & 0 & 0; & 1\end{bmatrix}\text{and } y'(0)=1\Rightarrow\boldsymbol{X}(0)\boldsymbol{B}\boldsymbol{S}_1\ \underline{\boldsymbol{A}}=1\Rightarrow\begin{bmatrix}0 & 1 & 1; & 1\end{bmatrix}$$

or briefly

$$[\underline{\boldsymbol{U}};\quad \lambda]=\begin{bmatrix}1 & 0 & 0; & 1\\ 0 & 1 & 1; & 1\end{bmatrix}.$$

Therefore, from Eq. (27), the new augmented matrix and the coefficients matrix are calculated as

$$\left[\underline{\boldsymbol{W}}^*;\quad \boldsymbol{F}^*\right]=\begin{bmatrix}1 & -2 & -1; & 2-e\\ 1 & 0 & 0; & 1\\ 0 & 1 & 1; & 1\end{bmatrix}\Rightarrow \underline{\boldsymbol{A}}=\begin{bmatrix}1 & e-2 & 3-e\end{bmatrix}^T;$$

consequently, the approximate solution for $N=2$ becomes

$$y_2(x)=\boldsymbol{X}(x)\,\boldsymbol{S}_1\,\underline{\boldsymbol{A}}=1+x+(3-e)x^2\cong 1+x+0.2817181715409549\;x^2.$$

By using the same way, we have the approximate solution based on the rising factorial polynomials in the form

$$y(x)\equiv y_N(x)=\sum_{n=0}^{N}\bar{a}_n(x)^{\bar{n}},\;\;\Rightarrow y_2(x)=\boldsymbol{X}(x)\boldsymbol{S}_1^{*}\bar{\boldsymbol{A}};$$

$$\boldsymbol{S}_1^*=\begin{bmatrix}S_1(0,0) & -S_1(1,0) & S_1(2,0)\\ 0 & S_1(1,1) & -S_1(2,1)\\ 0 & 0 & S_1(2,2)\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}.$$

For this purpose, we obtain the fundamental matrix equation, from (24),

$$\bar{\boldsymbol{W}}\,\bar{\boldsymbol{A}}=\boldsymbol{F}\Leftrightarrow[\bar{\boldsymbol{W}};\quad \boldsymbol{F}]=\begin{bmatrix}1 & -2 & 3 & ; & 2-e\\ 1 & -\dfrac{43}{32}-\sqrt[4]{e} & \dfrac{17}{16}+\dfrac{11}{18}\sqrt[4]{e} & ; & 1+\sqrt[4]{e}-\sqrt{e^3}\\ 1 & -\dfrac{5}{4}-\sqrt{e} & -\dfrac{7}{16}+\dfrac{3}{2}\sqrt{e} & ; & 1+\sqrt{e}-e^2\end{bmatrix}$$

and the matrix equation for the initial condition, from (26),

$$y(0)=1\Rightarrow \boldsymbol{X}(0)\boldsymbol{S}_1^*\,\bar{\boldsymbol{A}}=1\Rightarrow\begin{bmatrix}1 & 0 & 0; & 1\end{bmatrix}$$

and

$$y'(0)=1\Rightarrow \boldsymbol{X}(0)\boldsymbol{B}\boldsymbol{S}_1^*\,\bar{\boldsymbol{A}}=1\Rightarrow\begin{bmatrix}0 & 1 & -1; & 1\end{bmatrix}$$

or briefly

$$[\boldsymbol{U};\quad \lambda]=\begin{bmatrix}1 & 0 & 0; & 1\\ 0 & 1 & -1; & 1\end{bmatrix}.$$

Therefore, from Eq. (28), the new augmented matrix and the unknown coefficients matrix and is calculated as

$$\left[\bar{W}^*;\quad F^*\right]=\begin{bmatrix}1 & -2 & 3; & 2-e\\ 1 & 0 & 0; & 1\\ 0 & 1 & -1; & 1\end{bmatrix}\Rightarrow \bar{A}=\begin{bmatrix}1 & 4-e & 3-e\end{bmatrix}^T.$$

In conclusion, we obtain the approximate solution

$$y_2(x)=\mathbf{X}(x)BS_1^*\bar{A}=1+x+(3-e)x^2\cong 1+x+0.2817181715409549x^2.$$

In Fig. 1, comparison of the exact solution and the approximate solutions based on factorial polynomials is given. From Fig. 1, it can be seen that the approximate solution based on rising factorial polynomials is close to the exact solution than the approximate solution based on falling factorial polynomials.

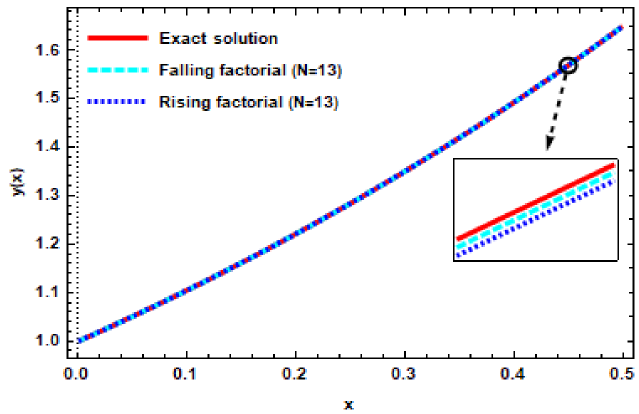

Figure 1. Comparison of the exact solution and the approximate solutions in Example 2.

Table 1 compares the absolute errors that are obtained by the approximate solutions based on factorial polynomials for different values of N. It can be seen that when N=15 the absolute error obtained by the falling factorial polynomials matrix collocation method are much smaller than the absolute error obtained by the rising factorial polynomials matrix collocation method.

Table 1. Absolute error for Example 2.

x	Falling factorial polynomials					Rising factorial polynomials			
	N=9	N=11	N=13	N=15		N=9	N=11	N=13	N=15
0.1	9.98E-07	1.48E-07	1.70E-08	2.63E-10		9.98E-07	1.48E-07	3.52E-08	7.96E-09
0.2	4.45E-06	6.61E-07	7.63E-08	1.17E-09		4.45E-06	6.63E-07	1.57E-07	3.55E-08
0.3	1.11E-05	1.65E-06	1.90E-07	2.93E-09		1.11E-05	1.65E-06	3.92E-07	8.86E-08
0.4	2.16E-05	3.21E-06	3.70E-07	5.71E-09		2.16E-05	3.22E-06	7.64E-07	1.72E-07
0.5	3.66E-05	5.43E-06	6.27E-07	9.66E-09		3.66E-05	5.44E-06	1.29E-06	2.92E-07

Fig. 2 shows the absolute error functions obtained by the falling factorial polynomials matrix collocation method and the rising factorial polynomials matrix collocation method for different values of N. It can be clearly seen that the accuracy improves with increasing N.

Using Eq. (29) and Theorem 3.1, the convergence of the numerical solutions for $x\in\left[0,\frac{1}{2}\right]$ is calculated as follows:

$$\|R_{N+1}(x)\| = \sup\left\{\left|\sum_{i=0}^{N}\Omega_i x^i\right| : x \in \left[0,\frac{1}{2}\right]\right\} \le \sup\left\{\sum_{i=0}^{N}\left|\Omega_i x^i\right| : x \in \left[0,\frac{1}{2}\right]\right\} = \left|R_{N+1}\left(\frac{1}{2}\right)\right|$$

and

$$\left|R_{N+1}\left(\frac{1}{2}\right)\right| \le \tilde{s}_N \left|R_N\left(\frac{1}{2}\right)\right|.$$

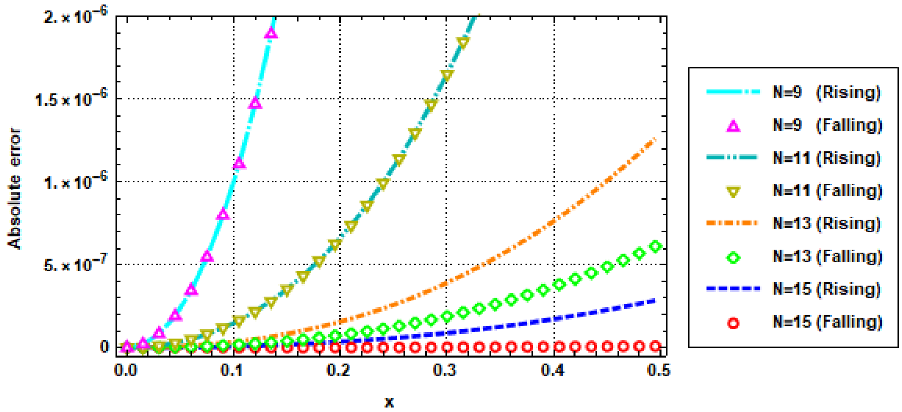

Figure 2. Absolute errors for Example 2.

The residual function sequence for the approximate solution based on the falling factorial polynomials can be computed as

$$\tilde{s}_N = \left\{\frac{\left|R_3\left(\frac{1}{2}\right)\right|}{\left|R_2\left(\frac{1}{2}\right)\right|}, \frac{\left|R_4\left(\frac{1}{2}\right)\right|}{\left|R_3\left(\frac{1}{2}\right)\right|}, \frac{\left|R_5\left(\frac{1}{2}\right)\right|}{\left|R_4\left(\frac{1}{2}\right)\right|}, \ldots, \frac{\left|R_{10}\left(\frac{1}{2}\right)\right|}{\left|R_9\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{11}\left(\frac{1}{2}\right)\right|}{\left|R_{10}\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{12}\left(\frac{1}{2}\right)\right|}{\left|R_{11}\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{13}\left(\frac{1}{2}\right)\right|}{\left|R_{12}\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{14}\left(\frac{1}{2}\right)\right|}{\left|R_{13}\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{15}\left(\frac{1}{2}\right)\right|}{\left|R_{14}\left(\frac{1}{2}\right)\right|}, \ldots\right\}$$

$$\tilde{s}_N = \{0.003679,\ 0.958115,\ 0.304495, \ldots,\ 0.419787,\ 0.267641,\ 0.205494,\ 0.614928,\ 0.197109,\ 0.079913, \ldots\}$$

Here, $\left\{R_N\left(\frac{1}{2}\right)\right\}_{N=2}^{\infty}$ satisfies the inequality $\left|R_{N+1}\left(\frac{1}{2}\right)\right| \le \tilde{s}_N \left|R_N\left(\frac{1}{2}\right)\right|$ with respect to $0 < \tilde{s}_N < 1$. Hence, the residual function sequence is convergent.

Similarly, the residual function sequence for the approximate solution based on the rising factorial polynomials can be obtained as

$$\tilde{s}_N = \left\{\frac{\left|R_3\left(\frac{1}{2}\right)\right|}{\left|R_2\left(\frac{1}{2}\right)\right|}, \frac{\left|R_4\left(\frac{1}{2}\right)\right|}{\left|R_3\left(\frac{1}{2}\right)\right|}, \frac{\left|R_5\left(\frac{1}{2}\right)\right|}{\left|R_4\left(\frac{1}{2}\right)\right|}, \ldots, \frac{\left|R_{10}\left(\frac{1}{2}\right)\right|}{\left|R_9\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{11}\left(\frac{1}{2}\right)\right|}{\left|R_{10}\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{12}\left(\frac{1}{2}\right)\right|}{\left|R_{11}\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{13}\left(\frac{1}{2}\right)\right|}{\left|R_{12}\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{14}\left(\frac{1}{2}\right)\right|}{\left|R_{13}\left(\frac{1}{2}\right)\right|}, \frac{\left|R_{15}\left(\frac{1}{2}\right)\right|}{\left|R_{14}\left(\frac{1}{2}\right)\right|}, \ldots\right\}$$

$$\tilde{s}_N = \{0.003679,\ 0.958115,\ 0.304495, \ldots, 0.419780,\ 0.268240,\ 0.121547,\ 0.999998,\ 0.606195,\ 0.369613, \ldots\}$$

so $0 < \tilde{s}_N < 1$ and the residual function sequence is convergent.

Example 3. Consider fourth order linear functional Volterra type integro-differential equation

$$y^{(4)}(x)+\sin(x)y''\left(x-\frac{1}{100}\right)-\cos(x)y(x)=f(x)+\int_{\cos(x)}^{2\cos(x)} e^{x+t}y\left(t-\frac{1}{100}\right)dt\ ,\ x\geq 0$$

with the initial conditions $y(0)=1$, $y'(0)=0$, $y''(0)=-1$ and $y'''(0)=0$. Here

$$f(x)=\cos(x)-\cos^2(x)-\sin(x)\cos\left(\frac{1}{100}-x\right)-\frac{1}{2}e^{x+\cos(x)}\left(\sin\left(\frac{1}{100}-\cos(x)\right)-\cos\left(\frac{1}{100}-\cos(x)\right)\right)$$
$$-\frac{1}{2}e^{x+\cos(x)}e^{\cos(x)}\left(\cos\left(\frac{1}{100}-2\cos(x)\right)-\sin\left(\frac{1}{100}-2\cos(x)\right)\right)$$

and the exact solution of the problem is $y(x)=\cos(x)$. According to Eq. (9), the given equation is explicitly written as

$$P_{00}(x)y^{(0)}\left(h_{00}(x)\right)+P_{20}(x)y^{(2)}\left(h_{20}(x)\right)+P_{40}(x)y^{(4)}\left(h_{40}(x)\right)=f(x)+\int_{u_{00}}^{v_{00}} K_{00}(x,t)y^{(0)}\left(g_{00}(t)\right)dt\ ;$$

$$P_{00}(x)=-\cos(x),\ P_{20}(x)=\sin(x),\ P_{40}(x)=1,\ h_{00}(x)=x,\ h_{20}(x)=x-\frac{1}{100},\ h_{40}(x)=x,$$

$$u_{00}(x)=\cos(x),\ v_{00}(x)=2\cos(x),\ K_{00}(x,t)=e^{x+t},\ g_{00}(t)=t-\frac{1}{100}.$$

Following the procedure in Section 2, the approximate solutions are obtained for different values of *N*. The fundamental matrix equation of the problem concerning the falling factorial polynomial matrix collocation method can be written as

$$\underline{\mathbf{W}}=\left\{\mathbf{P}_{00}\mathbf{X}_{00}\mathbf{B}^0+\mathbf{P}_{20}\mathbf{X}_{20}\mathbf{B}^2+\mathbf{P}_{40}\mathbf{X}_{40}\mathbf{B}^4-\overline{\mathbf{X}}\overline{\mathbf{K}_{00}}\overline{\mathbf{Q}_{00}}\mathbf{B}^0\right\}\mathbf{S}_1\,.$$

The fundamental matrix equation of the problem concerning the rising factorial polynomial matrix collocation method can be given as

$$\overline{\mathbf{W}}=\left\{\mathbf{P}_{00}\mathbf{X}_{00}\mathbf{B}^0+\mathbf{P}_{20}\mathbf{X}_{20}\mathbf{B}^2+\mathbf{P}_{40}\mathbf{X}_{40}\mathbf{B}^4-\overline{\mathbf{X}}\overline{\mathbf{K}_{00}}\overline{\mathbf{Q}_{00}}\mathbf{B}^0\right\}\mathbf{S}_1^{\,*}\,.$$

In Fig. 3, comparison of the exact solution and the approximate solutions is given. From Fig. 3, it is obvious that the approximate solution based on rising factorial polynomials is close to the exact solution than the approximate solution based on falling factorial polynomials.

In Table 2, the absolute errors of presented method have been compared with the absolute error of Mott matrix collocation method [39]. It is seen from Table 2 that the approximate solutions obtained by presented methods give better results than the solutions obtained by Mott matrix collocation method.

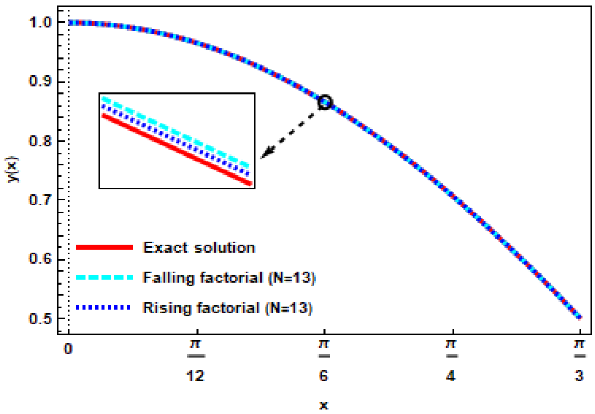

Figure 3. Comparison of the exact solution and the approximate solutions in Example 3.

Table 2. Absolute error for Example 3.

x	Falling factorial polynomials			Rising factorial polynomials			Mott matrix collocation method
	N=9	*N=11*	*N=12*	*N=9*	*N=11*	*N=12*	*N=9* [39]
0.2	7.53E-09	1.35E-11	9.11E-12	7.53E-09	1.35E-11	9.08E-12	9.16E-09
0.4	1.23E-07	2.17E-10	1.49E-10	1.23E-07	2.16E-10	1.49E-10	1.50E-07
0.6	6.30E-07	1.07E-09	7.68E-10	6.30E-07	1.07E-09	7.66E-10	7.66E-07
0.8	1.97E-06	3.26E-09	2.42E-09	1.97E-06	3.26E-09	2.41E-09	2.40E-06
1	4.71E-06	7.56E-09	5.78E-09	4.71E-06	7.55E-09	5.76E-09	5.71E-06

In Fig. 4, we compare the absolute error functions obtained by the falling factorial polynomial matrix collocation method and the rising factorial polynomials matrix collocation method for different values of N.

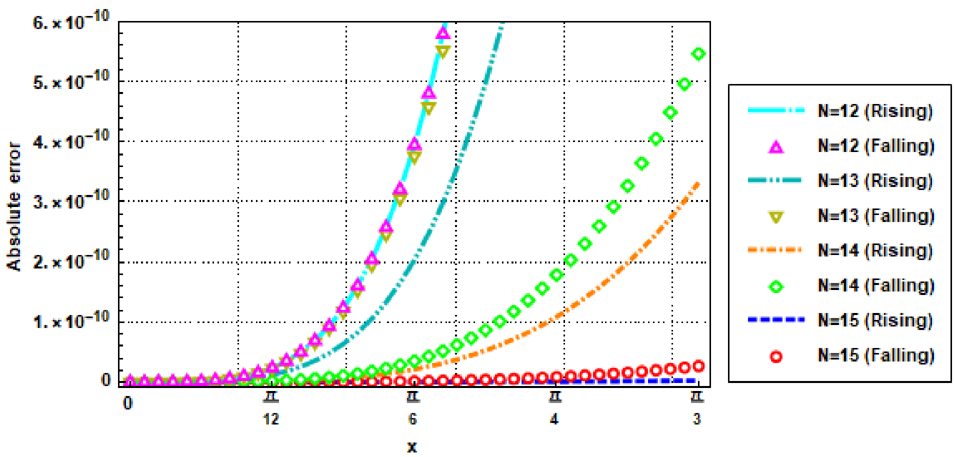

Figure 4. Absolute errors for Example 3.

Using Eq. (29) and Theorem 3.1, the convergence of the approximate solutions of Example 3 for $x \in \left[0, \frac{\pi}{3}\right]$ is computed as

$$\left\| R_{N+1}(x) \right\| = \sup \left\{ \left| \sum_{i=0}^{N} \Omega_i x^i \right| : x \in \left[0, \frac{\pi}{3}\right] \right\} \le \sup \left\{ \sum_{i=0}^{N} \left| \Omega_i x^i \right| : x \in \left[0, \frac{\pi}{3}\right] \right\} = \left| R_{N+1}\left(\frac{\pi}{3}\right) \right|$$

and

$$\left| R_{N+1}\left(\frac{\pi}{3}\right) \right| \le \tilde{s}_N \left| R_N\left(\frac{\pi}{3}\right) \right|.$$

The residual function sequence for the approximate solution based on the falling factorial polynomials can be computed as

$$\tilde{s}_N = \left\{ ..., \frac{\left|R_{11}\left(\frac{\pi}{3}\right)\right|}{\left|R_{10}\left(\frac{\pi}{3}\right)\right|}, \frac{\left|R_{12}\left(\frac{\pi}{3}\right)\right|}{\left|R_{11}\left(\frac{\pi}{3}\right)\right|}, \frac{\left|R_{13}\left(\frac{\pi}{3}\right)\right|}{\left|R_{12}\left(\frac{\pi}{3}\right)\right|}, \frac{\left|R_{14}\left(\frac{\pi}{3}\right)\right|}{\left|R_{13}\left(\frac{\pi}{3}\right)\right|}, \dots \right\} = \{..., 0.112078, 0.024348, 0.032571, 0.849287, ...\}$$

so, $0 < \tilde{s}_N < 1$ and the residual function sequence is convergent. In a similar way, the residual function sequence for the approximate solution based on the rising factorial polynomials can be obtained as

$$\tilde{s}_N = \left\{ ..., \frac{\left|R_{11}\left(\frac{\pi}{3}\right)\right|}{\left|R_{10}\left(\frac{\pi}{3}\right)\right|}, \frac{\left|R_{12}\left(\frac{\pi}{3}\right)\right|}{\left|R_{11}\left(\frac{\pi}{3}\right)\right|}, \frac{\left|R_{13}\left(\frac{\pi}{3}\right)\right|}{\left|R_{12}\left(\frac{\pi}{3}\right)\right|}, \frac{\left|R_{14}\left(\frac{\pi}{3}\right)\right|}{\left|R_{13}\left(\frac{\pi}{3}\right)\right|}, \dots \right\} = \{..., 0.112066, 0.024397, 0.175996, 0.181868, ...\}$$

so, $0 < \tilde{s}_N < 1$. The residual functions sequence is a Cauchy sequence and therefore it is convergent.

5. CONCLUSIONS

We have applied the factorial matrix collocation method to some numerical examples. We have also solved rapidly and efficiently these examples without requiring detailed procedure. As seen from figures and tables, the different kinds of error analysis based on the factorial polynomial solutions of numerical examples are presented. The investigation of convergence via the residual function has been provided for linear model problems. Accordingly, the factorial polynomial solutions approach to the exact solution, as N is increased. This situation reflects on the residual functions of problems. Hence, it can be seen from the comparisons and error analysis that the present method is very consistent and reliable to solve difficult problems. Also, this method can be extended to other problems.

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