*ORIGINAL PAPER*

# THE WEIGHTED AND INTERVAL WEIGHTED ARITMETIC-GEOMETRIC INDEX WITH GRAPH OPERATIONS

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***Abstract.*** *In this paper, Aritmetic-Geometric index is explained for positive weighted and positive interval weighted graphs in terms of the edges and vertices. Also, the graph operations is defined for Aritmetic-Geometric index as graph union, intersection, sum, product and direct sum operations with the weights.*

***Keywords:*** *Arithmetic-geometric ındex; ınterval weights; graph operations.*

## 1. INTRODUCTION

In the 18th Century, the foundations of graph theory were laid with the solution produced by Euler, who was looking for a solution to the problem of the Königsberg bridge. Graph theory is considered as a sub-branch of mathematics. Graph theory has been used and developed in different disciplines over time. An example of these studies is seen in chemistry. In chemistry, the relationships of chemical molecules or compounds between elements can be modelled with graphs and topological index calculations can be made in this way. Topological calculations are used in mathematical chemistry.

Arithmetic-Geometric index was defined by Shegehalli V.S. and Rachanna Kanabur in 2015. Since the index was defined, many studies have been and are being carried out on it. Shegehalli V.S. and Rachanna Kanabur analysed the values of the arithmetic-geometric index for path graphs and special graphs in [1]. The spectral radius and energy of the defined matrices are shown and bounded in [2]. Since the definition of the arithmetic-geometric index, many studies have been carried out in different aspects. In this paper, Arithmetic-Geometric index is studied and Arithmetic-Geometric Index in positive weighted graphs and positive interval graphs are defined and showed with some graph operations.

A finite graph consists of a non-empty finite set of vertices V and a set of edges. The set of points $V = (1,2, \dots, n)$ and the set of edges $G(V, E)$ is called a graph. The graph formed by assigning real numbers or square matrices as weights to each edge of the graph is called a weighted graph. ([3,4])

Let $G$ be a weighted graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E$. Let $w_{ij}$ be the positive definite weight of edge $ij$. Also, $w_{ij} = w_{ji}$. If vertices $i$ and $j$ are adjacent, it is written $i{\sim}j$. The weight of vertex i is $w_i = \sum_{j:j\sim i} w_{ij}$ [5,6]. The Arithmetic-Geometric index is defined as $AG = AG(G) = \sum_{i\sim j} \frac{d_i + d_j}{2\sqrt{d_i d_j}}$ depending on the degrees $d_i$ and $d_j$ of the vertices $i$ and $j$. [3].

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## 2. ARITHMETIC-GEOMETRIC INDEX IN WEIGHTED AND INTERVAL WEIGHTED GRAPHS

In this section, it is shown how to define the Arithmetic-Geometric index in positive weighted and positive interval weighted graphs. After defining the index, graph operations will be performed on positive weighted and positive interval weighted graphs. It will be shown how the new result for the Arithmetic-Geometric index will be in the new graphs resulting from graph operations.

Before defining the interval weighted Arithmetic-Geometric index, the definition of interval will be explained. Since the Arithmetic-Geometric index is defined in positive numbers, the numbers in the interval will be defined in positive numbers.

Let $\underline{x}$ and $\overline{x}$ be positive real numbers satisfying the inequality $0 < \underline{x} \leq \overline{x}$. Then $x = [\underline{x}, \overline{x}] = \{y \in R^+ \colon\ 0 < \underline{x} \leq y \leq \overline{x}\}$ is called a positive interval. In the interval $[\underline{x}, \overline{x}]$, $\underline{x}$ is the lower bound and $\overline{x}$ is the upper bound. If $\underline{x} = \overline{x} = \tilde{x}$ then $x = [\tilde{x}, \tilde{x}]$ is a positive integer.

Let $x = [\underline{x}, \overline{x}]$ and $y = [\underline{y}, \overline{y}]$ be two positive intervals. The following properties are satisfied:

i. $x + y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$.

ii. $x - y = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$.

iii. $x.y = [\underline{x}.\underline{y}, \overline{x}.\overline{y}]$.

iv. $\frac{x}{y} = \left[\frac{\underline{\underline{x}}}{\overline{y}}, \frac{\overline{x}}{\underline{y}}\right], y \notin 0$.

**Definition 2.1.** Let $G = (V, E)$ be a positive weighted graph and $w_i$ and $w_j$ be the weights of vertices $i$ and $j$ respectively. Then, the weighted Arithmetic-Geometric index of graph $G$ is

$$AG_w(G) = \sum_{i,j \epsilon V(G), i \sim j} \frac{w_i + w_j}{2\sqrt{w_i w_j}}.$$

**Definition 2.2.** Let $G = (V, E)$ be a positive interval weighted graph and $\widetilde{w}_i$ and $\widetilde{w}_j$ be the interval weights of vertices $i$ and $j$ respectively. Then, the positive interval weighted Arithmetic-Geometric index of graph $G$ is

$$AG_{\widetilde{w}}(G) = \sum_{i,j \epsilon V(G), i \sim j} \frac{\widetilde{w}_i + \widetilde{w}_j}{2\sqrt{\widetilde{w}_i\, \widetilde{w}_j}}.$$

Interval is obtained in a more general expression by using the definition of interval with the definition of Arithmetic-Geometric index. That is, by using $\widetilde{w}_i = [\underline{i}, \overline{i}]$ and $\widetilde{w}_j = [\underline{j}, \overline{j}]$ intervals, $AG_{\widetilde{w}}$ is obtained with interval operation definitions. Hence, if it is divided

$$\widetilde{w}_i + \widetilde{w}_j = \left[\left(\underline{i} + \underline{j}\right), \left(\overline{i} + \overline{j}\right)\right] \text{ by } \frac{1}{2\sqrt{\widetilde{w}_i \widetilde{w}_j}} = \left[\left(\frac{1}{2\sqrt{\underline{i}.\underline{j}}}\right), \left(\frac{1}{2\sqrt{\overline{i}.\overline{j}}}\right)\right],$$

the result is

$$\frac{\widetilde{w}_i + \widetilde{w}_j}{2\sqrt{\widetilde{w}_i \widetilde{w}_j}} = \left[\left(\frac{\underline{i} + \underline{j}}{2\sqrt{\underline{i}.\underline{j}}}\right), \left(\frac{\overline{i} + \overline{j}}{2\sqrt{\overline{i}.\overline{j}}}\right)\right].$$

$AG_{\widetilde{w}}$ in the form

$$AG_{\widetilde{w}}(G) = \sum_{i,j\epsilon V(G), i\sim j} \frac{\widetilde{w}_i + \widetilde{w}_j}{2\sqrt{\widetilde{w}_i\,\widetilde{w}_j}} = \left[\left(\sum_{i,j\epsilon V(G), i\sim j} \frac{\underline{i}+\underline{j}}{2\sqrt{\underline{i}.\underline{j}}}\right), \left(\sum_{i,j\epsilon V(G), i\sim j} \frac{\overline{i}+\overline{j}}{2\sqrt{\overline{i},\overline{j}}}\right)\right]$$

is written more clearly.

## 3. GRAPH OPERATIONS ON THE ARITHMETIC-GEOMETRIC INDEX OF POSITIVE WEIGHTED AND POSITIVE INTERVAL WEIGHTED GRAPHS

In this section, some graph operations on positive weighted and positive interval weighted graphs for the Arithmetic-Geometric index are explained. The graph operations are studied as graph union, intersection, sum, product and direct sum operations.

**Theorem 3.1.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two positive weighted graphs. In this case, the Arithmetic Geometric index of the graph $G_1 \cup G_2$ is

$$AG_w(G_1 \cup G_2) = \sum_{\substack{(i,j)\notin V_1\cap V_2 \\ ij\in E}} \frac{w_i+w_j}{2\sqrt{w_i w_j}} + \sum_{\substack{ik\,\in E, i\,\in V_1\setminus V_1\cap V_2 \\ k\in V_1\cap V_2}} \frac{w_i+(w_k-w_{t_k})}{2\sqrt{w_i(w_k-w_{t_k})}} + \sum_{\substack{pj\,\in E, j\,\in V_2\setminus V_1\cap V_2 \\ p\in V_1\cap V_2}} \frac{(w_p-w_{t_p})+w_j}{2\sqrt{(w_p-w_{t_p})w_j}} + \sum_{\substack{(i,j)\,\in V_1\cap V_2 \\ ij\in E(G_1\cup G_2)}} \frac{(w_i-w_{t_i})+(w_j-w_{t_j})}{2\sqrt{(w_i-w_{t_i})(w_j-w_{t_j})}}.$$

Here, it is expressed as $w_i = w_{G_1}(i) + w_{G_2}(i)$ and $w_{t_i} = \sum_{\substack{i\sim j \\ ij\in E_1\cap E_2}} w_{ij}$.

*Proof:* Given the union of a graph, there are four different neighbourhoods between points. The each graph $G_1$ and $G_2$ have a neighbourhood of non-common points. In $G_1$, non-common points are neighboured by common points, and the same is true for $G_2$. Finally, common points have neighbourhood among themselves. Defining the index in the graph, these four cases are defined separately.

**Corollary 3.2.** The operations in Theorem 3.1 cannot be performed in interval arithmetic due to the definition of subtraction. That is

$$[a,b] - [a,b] = [a,b] + [-b,-a] = [a-b, b-a] \neq [0,0].$$

Therefore, the Arithmetic-Geometric index cannot be defined in positive interval weighted graphs.

**Theorem 3.3.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two positive weighted graphs. In this case, the Arithmetic-Geometric index of the graph $G_1 \cap G_2$ is

$$\boldsymbol{AG_w(G_1 \cap G_2)} = \sum_{\substack{k\sim l \\ (k,l)\,\in V_1\cap V_2}} \frac{(w_{t_k}) + (w_{t_l})}{2\sqrt{(w_{t_k})\,(w_{t_l})}}$$

Here, it is expressed as $w_{t_i} = \sum_{\substack{i\sim j \\ ij\in E_1\cap E_2}} w_{ij}$.

**Theorem 3.4.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two positive interval weighted graphs. Then the Arithmetic-Geometric index of the graph $G_1 \cap G_2$ is

$$AG_w(G_1 \cap G_2) = \sum_{\substack{k\sim l \\ (k,l)\in V_1\cap V_2}} \frac{(\widetilde{w}_{t_k}) + (\widetilde{w}_{t_l})}{2\sqrt{(\widetilde{w}_{t_k})\,(\widetilde{w}_{t_l})}}.$$

Here, it is expressed as $\widetilde{w}_{t_i} = \sum_{\substack{i\sim j \\ ij\in E_1\cap E_2}} \widetilde{w}_{ij}$.

*Proof:* The intersection of graphs leaves only common points and edges. If the weight of the points resulting from the intersection is defined as $(\widetilde{w}_{t_i})$, the index is defined only over the intersected points.

**Theorem 3.5.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two positive weighted graphs. Then the Arithmetic-Geometric index of the graph $G_1 + G_2$ is

$$\boldsymbol{AG_w} = \sum_{\mathrm{i}\sim j, i,j\in V_1} \frac{\boldsymbol{A_i + A_j}}{\boldsymbol{2\sqrt{A_iA_j}}} + \sum_{\mathrm{i}\in V_1, j\in V_2} \frac{\boldsymbol{A_i + B_j}}{\boldsymbol{2\sqrt{A_iB_j}}} + \sum_{i\sim j, i,j\in V_2} \frac{\boldsymbol{B_i + B_j}}{\boldsymbol{2\sqrt{B_iB_j}}}.$$

Here, it is expressed as

$$A_s = \left\{ s \in V_1 : w_s + \sum_{k\in V_2} w_k \right\}, B_s = \left\{ s \in V_2 : w_s + \sum_{t\in V_1} w_t \right\}, s = i, j.$$

**Theorem 3.6.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two positive interval weighted graphs. Then the Arithmetic-Geometric index of the graph $G_1 + G_2$ is

$$\boldsymbol{AG_{\widetilde{w}}} = \sum_{\mathrm{i}\sim j, i,j\in V_1} \frac{\boldsymbol{\widetilde{A}_i + \widetilde{A}_j}}{\boldsymbol{2\sqrt{\widetilde{A}_i\widetilde{A}_j}}} + \sum_{\mathrm{i}\in V_1, j\in V_2} \frac{\boldsymbol{\widetilde{A}_i + \widetilde{B}_j}}{\boldsymbol{2\sqrt{\widetilde{A}_i\widetilde{B}_j}}} + \sum_{i\sim j, i,j\in V_2} \frac{\boldsymbol{\widetilde{B}_i + \widetilde{B}_j}}{\boldsymbol{2\sqrt{\widetilde{B}_i\widetilde{B}_j}}}.$$

Here, it is expressed as $\tilde{A}_s = \{ s \in V_1 : \widetilde{w}_s + \sum_{k\in V_2} \widetilde{w}_k \}$, $\tilde{B}_s = \{ s \in V_2 : \widetilde{w}_s + \sum_{t\in V_1} \widetilde{w}_t \}$, $s = i, j$.

*Proof:* Let $V = V(G_1) \cup V(G_2)$ and $E = \{E(G_1) \cup E(G_2) \cup (u_1, u_1) : u_1 \in V(G_1), u_2 \in V(G_2)\}$. Also, let $A_i$ and $\tilde{A}_i$ be the weight of any vertex of graph $G_1$after addition, let $B_i$ and $\tilde{B}_i$ are the weight of any vertex of the graph $G_2$ after addition.

The graph sum $G_1 + G_2$ results in a complete bipartite graph. Since the Arithmetic-Geometric index is based on the neighbourhoods in the graph, the vertices of $G_1$ and $G_2$ have a neighbourhood among themselves. The Arithmetic-Geometric index is $\sum_{i\sim j, i,j\in V_1} \frac{A_i + A_j}{2\sqrt{A_iA_j}}$ and $\sum_{i\sim j, i,j\in V_1} \frac{\tilde{A}_i + \tilde{A}_j}{2\sqrt{\tilde{A}_i\tilde{A}_j}}$ depending on the neighbourhood formed by the vertices of the graph $G_1$with their vertices. When the same process is considered for the graph $G_2$, $\sum_{i\sim j, i,j\in V_2} \frac{B_i + B_j}{2\sqrt{B_iB_j}}$ and

$\sum_{i\sim j,i,j\in V_2}\frac{\tilde{B}_i+\tilde{B}_j}{2\sqrt{\tilde{B}_i\tilde{B}_j}}$ are obtained. Since all vertices of the graph $G_1$are neighbours of all vertices of the graph $G_2$, the Arithmetic-Geometric index $\sum_{i\in V_1,j\in V_2}\frac{A_i+B_j}{2\sqrt{A_iB_j}}$ and $\sum_{i\in V_1,j\in V_2}\frac{\tilde{A}_i+\tilde{B}_j}{2\sqrt{\tilde{A}_i\tilde{B}_j}}$ in $G_1+G_2$. When the three results are summed, the Arithmetic-Geometric index of the given graph $G_1+G_2$ is obtained. This procedure is valid for both weighted and interval weighted graphs.

**Theorem 3.7.** $G_1=(V_1,E_1)$ ve $G_2=(V_2,E_2)$ be two positive weighted graphs. In this case, the Arithmetic Geometric index of the graph $G_1\times G_2$ is

$$AG_w(G_1\times G_2)=\sum_{\substack{(u_i,v_j)\sim(u_k,v_l)\\(u_i,v_j),(u_k,v_l)\in V(G_1\times G_2)}}\frac{C_{u_i,v_j}+C_{u_k,v_l}}{2\sqrt{C_{u_i,v_j}C_{u_k,v_l}}}.$$

This is expressed as $C_{u_m,v_n}=\{(u_m,v_n)\in V_1xV_2: w(u_m)+w(v_n)\}$ for $m=i,n=j$ and $m=k,n=l$.

**Theorem 3.8.** $G_1=(V_1,E_1)$ ve $G_2=(V_2,E_2)$ be two positive interval weighted graphs. In this case, the Arithmetic Geometric index of the graph $G_1\times G_2$ is

$$AG_{\widetilde{w}}(G_1\times G_2)=\sum_{\substack{(u_i,v_j)\sim(u_k,v_l)\\(u_i,v_j),(u_k,v_l)\in V(G_1\times G_2)}}\frac{\tilde{C}_{u_i,v_j}+\tilde{C}_{u_k,v_l}}{2\sqrt{C_{u_i,v_j}C_{u_k,v_l}}}.$$

This is expressed as $\tilde{C}_{u_m,v_n}=\{(u_m,v_n)\in V_1\times V_2: \widetilde{w}(u_m)+\widetilde{w}(v_n)\}$ for $m=i,n=j$ and $m=k,n=l$.

*Proof:* In the graph $G_1\times G_2$, for the set of vertices $u_i,v_j\in V_1$ and $u_k,v_l\in V_2$ için $u=(u_i,v_j)$, $v=(u_k,v_l)\in V_1\times V_2$. $w(u_m)$ represents the weight of the vertex $u_m$. The weight of any vertex $u_i,v_j\in V_1\times V_2$ of the graph $G_1\times G_2$ is $w(u_i)+w(v_j)$ for $m=i,n=j$.

**Theorem 3.9.** $G_1=(V_1,E_1)$ ve $G_2=(V_2,E_2)$ be two positive weighted graphs. Then the Arithmetic-Geometric index of the graph $G_1\oplus G_2$ is

$$AG_w(G_1\oplus G_2)=\sum_{\substack{ij\in E\\(i,j)\notin V_1\cap V_2}}\frac{w_i+w_j}{2\sqrt{w_iw_j}}+\sum_{\substack{ik\in E\\i\in V_1\cap V_2\\k\in V_2\setminus V_1}}\frac{(w_i-2w_{t_i})+w_j}{2\sqrt{(w_i-2w_{t_i})w_j}}$$
$$+\sum_{\substack{pj\in E\\j\in V_1\cap V_2\\p\in V_1\setminus V_2}}\frac{w_p+(w_j-2w_{t_j})}{2\sqrt{w_p(w_j-2w_{t_j})}}+\sum_{\substack{i,k\notin E_1\cap E_2\\(i,k)\in V_1\cap V_2}}\frac{(w_i-2w_{t_i})+(w_j-2w_{t_j})}{2\sqrt{(w_i-2w_{t_i})(w_j-2w_{t_j})}}.$$

It is expressed as $w_i=w_{G_1}(i)+w_{G_2}(i), w_{t_i}=\sum_{\substack{i\sim j\\ij\in E_1\cap E_2}}w_{ij}$.

*Proof:* Given a direct sum of graphs, there are four different neighbourhoods between points. The each graphs $G_1$ and $G_2$ have a neighbourhood of points that are not in common. The common points of $G_1$ have a neighbourhood with the common points, and the same is true for $G_2$. Finally, common points have a neighbourhood among themselves. When defining the index in the graph, these four cases are defined separately. These graphs have no weight on the intersecting edge.

**Corollary 3.10.** The operations performed in Theorem.3.10 cannot be performed in interval arithmetic due to the definition of subtraction operation. That is;

$$[x, y] - [x, y] = [x, y] + [-y, -x] = [x - y, y - x] \neq [0, 0].$$

Therefore, Arithmetic-Geometric index could not be defined in positive interval weighted graphs.

## 4. CONCLUSION

In this study, Aritmetic-Geometric index is investigated for positive weighted and positive interval weighted graphs in terms of the edges and vertices. In the sequel, the graph operations is described for Aritmetic-Geometric index as graph union, intersection, sum, product and direct sum operations by the help of the weights.

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