ORIGINAL PAPER

AN INEQUALITY WITH MOMENTS FOR CONCAVE FUNCTIONS

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Abstract. *We prove some inequalities for moments related to concave functions.*
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1. INTRODUCTION

The starting point of our work is the following inequality proposed in [1]:

Theorem 1 (Daróczy). Let $f: [0,1] \to (0, \infty)$ be a concave function with $f(0) = 1$. Then:

$$\int_0^1 xf(x)dx \leq \frac{2}{3}\left(\int_0^1 f(x)dx\right)^2. \tag{1}$$

The inequality (1) is a particular case of the following result that can be found in [2]:

Theorem 2. If the boundary of a convex domain contains a segment of length 1, then the distance of the weight point of the domain from that segment is at most $2/3$ times the area.

We prove in this work some inequalities for moments related to the inequality (1).

Let $f: [0,1] \to \mathbb{R}$ be a function. If f is Riemann integrable, then for every non-negative integer n, the quantity defined and denoted by

$$M_n = \int_0^1 x^n f(x)dx$$

is called the moment of order n of the function f.

Recall also that f is named concave (respectively convex) if the following inequality holds true, for every $x, y \in [0,1]$ and $a \in [0,1]$:

$$f(ax + (1-a)y) \geq (\leq) af(x) + (1-a)f(y).$$

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2. THE RESULTS

The main result is the following:

Theorem 3. Let f: $[0,1] \to \mathbb{R}$ be concave (respectively convex). Then the following inequality holds true for every integer n ≥ 0:

$$(n+3)M_{n+1} \leq (\geq)(n+2)M_n - \frac{f(0)}{(n+1)(n+2)}. \tag{2}$$

Proof: We will consider the case of a concave function f, only.

Let us take any numbers $x \in (0,1]$ and $t \in [0,x]$. As $\frac{t}{x} \in [0,1]$ and

$$t = \frac{t}{x} \cdot x + \left(1 - \frac{t}{x}\right) \cdot 0,$$

we obtain from the concavity of f:

$$f(t) \geq \frac{t}{x} f(x) + \left(1 - \frac{t}{x}\right) f(0).$$

By multiplying by t^n, we get:

$$t^n f(t) \geq \frac{f(x)}{x} \cdot t^{n+1} + t^n f(0) - \frac{t^{n+1}}{x} f(0).$$

By integration on $[0,x]$, we deduce:

$$F_n(x) := \int_0^x t^n f(t)dt \geq \frac{1}{n+2} f(x)x^{n+1} + \frac{f(0)}{(n+1)(n+2)} \cdot x^{n+1}. \tag{3}$$

This inequality is also true for $x = 0$. Since f is concave, the function F_n is continuous on $[0,1]$ and derivable on $(0,1)$. By using the integration by parts method, we deduce:

$$M_{n+1} = \int_0^1 x^{n+1} f(x)dx = \int_0^1 xF_n'(x)dx = xF_n(x)|_0^1 - \int_0^1 F_n(x)dx.$$

Thus

$$M_{n+1} = M_n - \int_0^1 F_n(x)dx.$$

By using (3), we obtain:

$$M_{n+1} \leq M_n - \frac{1}{n+2}\int_0^1 x^{n+1} f(x)dx - \frac{f(0)}{(n+1)(n+2}\int_0^1 x^{n+1}dx,$$

or

$$(n+3)M_{n+1} \le (n+2)M_n - \frac{f(0)}{(n+1)(n+2)},$$

which is the conclusion.
Corollary 1. Let $f: [0,1] \to \mathbb{R}$ be concave (respectively convex). Then the following inequality holds true for every integer $n \ge 0$:

$$(n+2)M_n \le (\ge) 2M_0 - \frac{n}{n+1} f(0).$$

Proof: By summing up the inequalities (2), we get:

$$(n+2)M_n \le 2M_0 - f(0) \sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} = 2M_0 - \frac{n}{n+1} f(0).$$

Corollary 2. Let $f: [0,1] \to \mathbb{R}$ be concave (respectively convex), with $f(0) > (<) 0$. Then the following inequality holds true for every integer $n \ge 0$:

$$M_n \le (\ge) \frac{n+1}{n(n+2)f(0)} M_0^2.$$

Proof: By using Corollary 1, we deduce that:

$$M_n \le \frac{2}{n+2} M_0 - \frac{n}{(n+1)(n+2)} f(0),$$

or

$$M_n \le \frac{2}{(n+2)f(0)} \left(M_0 f(0) - \frac{f^2(0)n}{2n+2} \right).$$

Then, by using the arithmetic-geometric mean inequality, we get:

$$\frac{2n+2}{4n} M_0^2 + f^2(0) \cdot \frac{n}{2n+2} \ge M_0 f(0).$$

Finally, we deduce that

$$M_n \le \frac{2}{(n+2)f(0)} \cdot \frac{2n+2}{4n} M_0^2,$$

or

$$M_n \le \frac{n+1}{n(n+2)f(0)} M_0^2.$$

The proof is completed. If we take $n = 1$ in Corollary 2, then we obtain inequality (1).

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