ORIGINAL PAPER

EXPLORING SUMS OF SQUARES AND CUBES OF FIBONACCI NUMBERS IN DIOPHANTINE EQUATIONS

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Manuscript received: 25.12.2024; Accepted paper: 07.04.2025;
Published online: 30.06.2025.

Abstract. *This study investigates numbers that are powers of two and can be expressed as the sum of the squares or cubes of any two Fibonacci numbers, utilizing Matveev's theorem and the Dujella-Pethő reduction lemma. More precisely, all the solutions to the Diophantine equation* $F_m^x + F_n^x = 2^a$ *with* $x \in \{2,3\}$*, where m, n, and a are positive integers, are presented herein.*

Keywords: *Matveev's theorem; Fibonacci number; Diophantine equation; linear forms in logarithms; Dujella-Pethő reduction lemma.*

1. INTRODUCTION AND MOTIVATION

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci numbers given by the recursive formula $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with the initial terms $F_0 = 0$ and $F_1 = 1$. Also, Fibonacci numbers can be generated with Binet's formula in the following:

$$F_n = \frac{\gamma^n - \delta^n}{\sqrt{5}} \tag{1}$$

for all integers $n \geq 0$, where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\delta = \frac{1-\sqrt{5}}{2}$, which are the roots of the equation $x^2 - x - 1 = 0$. Moreover, it can be easily seen that $\gamma + \delta = 1$ and $\gamma\delta = -1$. Scientists have worked on Fibonacci numbers in many areas of mathematics and geometry. More accurate examples of Fibonacci numbers can be found by referring to the fundamental reference in [1].

In recent years, there has been a rise in the study of several exponential Diophantine equations involving Fibonacci numbers. The case where the sums or differences of exponential Fibonacci numbers are equal to a Fibonacci or a perfect power has been studied. In [2], Bravo and Luca examined the Diophantine equation $F_m + F_n = 2^a$ and found all of its solutions. Also, in [3], Bravo and Bravo solved the Diophantine equation $F_m + F_n + F_l = 2^a$. In [4], Marques and Togbé investigated that if *s* is an integer such that $F_m^s + F_{m+1}^s = F_r$ for all sufficiently large integers *m*, where *r* is a non-negative integer, then *s* is equal to only 1 or 2. After that, in [5], Luca and Oyono proved that the equation $F_m^s + F_{m+1}^s = F_r$ has no solutions when $m \geq 2$ and $s \geq 3$, which completely solved this problem. In [6], Hirata-Kohno and Luca investigated the Diophantine equation $F_n^x + F_{n+1}^x = F_m^y$ and found its solutions, which are the

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https://doi.org/10.46939/J.Sci.Arts-25.2-a07

extended versions of the previous equation. In [7], Patel and Chaves studied the Diophantine equation $F_{n+1}^{x}-F_{n-1}^{x}=F_{m}$ and found all of its solutions. The Diophantine equation $F_{n+1}^{x}-F_{n-1}^{x}=F_{m}^{y}$, which is the extended version of the previous equation, was examined by Gómez et al., and all solutions of this equation are shown in [8].

Many studies have proven similar relationships between integer sequences other than Fibonacci numbers [9-17].

The problem of finding all perfect powers in the Fibonacci sequence and the Lucas sequence had been an open problem for a long time until it was resolved in 2006 by Bugeaud, Mignotte, and Siksek in [18]. That is, the only perfect powers in the Fibonacci sequence are 0, 1, 8, and 144, and also the only perfect powers in the Lucas sequence are 1 and 4.

As a result of the literature review, it was observed that whether the sum of the x-th powers of any two Fibonacci numbers equals a power of two for $x\geq 2$ has not been investigated. Therefore, in this study, we determine all the solutions of the Diophantine equations

$$F_{m}^{2}+F_{n}^{2}=2^{a} \tag{2}$$

and

$$F_{m}^{3}+F_{n}^{3}=2^{a}, \tag{3}$$

where $a\geq 1$ and $1\leq m\leq n$.

2. BASIC TOOLS

This section of the paper provides fundamental definitions, results, and notations from algebraic number theory. One can find the following lemma in the book by Koshy [1].

Lemma 2.1. [1] The inequality

$$\gamma^{n-2}\leq F_{n}\leq \gamma^{n-1} \tag{4}$$

holds for all $n\geq 1$.

Let χ be an algebraic number of degree s and

$$a_{0}x^{s}+a_{1}x^{s-1}+\ldots+a_{s}=\sum_{j=0}^{s}a_{j}x^{s-j}$$

be its minimal polynomial in ℤ, where the a_{j} 's are relatively prime integers with $a_{0}>0$. The logarithmic height of χ is denoted by $h(\chi)$ and defined by

$$h(\chi)=s^{-1}\left(\log|a_{0}|+\sum_{i=1}^{s}\log\left(\max\left\{\left|\chi^{(i)}\right|,1\right\}\right)\right), \tag{5}$$

where $\chi^{(i)}$'s are the conjugates of χ.

There are also numerous properties related to logarithmic height mentioned in the references. These properties are as follows:

$$h(\chi_{1}+\chi_{2})\leq h(\chi_{1})+h(\chi_{2})+\log 2, \tag{6}$$

$$h\left(\chi_{1}\chi_{2}^{\pm 1}\right)\leq h(\chi_{1})+h(\chi_{2}), \tag{7}$$

$$h\left(\chi^r\right)=|r|\,h(\chi). \tag{8}$$

Let $\chi_1,\chi_2,\ldots,\chi_r$ be non-zero real algebraic numbers in a number field $\mathbb{T}$ of degree $d_{\mathbb{T}}$, and let $t_1,t_2,\ldots,t_r$ be non-zero rational integer numbers. Also, let

$$\Lambda=\chi_1^{t_1}\chi_2^{t_2}\cdots\chi_r^{t_r}-1 \text{ and } B\geq\max\left\{|t_1|,|t_2|,\ldots,|t_r|\right\}.$$

Let $A_1,A_2,\ldots,A_r$ be the positive real numbers such that

$$A_j\geq\max\left\{d_{\mathbb{T}}h\left(\chi_j\right),\left|\log\chi_j\right|,0.16\right\} \tag{9}$$

for all $j=1,2,\ldots,r.$ Based on the notations mentioned above, an important theorem established by Matveev [19] will be presented as follows:

Theorem 2.1. [19] If $\Lambda\neq 0$ and $\mathbb{T}$ is a real algebraic number field, then,

$$\log\left(|\Lambda|\right)>-1.4\times 30^{r+3}\times r^{4.5}\times d_{\mathbb{T}}^2\times\left(1+\log d_{\mathbb{T}}\right)\times\left(1+\log B\right)\times A_1\times A_2\times\ldots\times A_r.$$

To reduce the bounds from applying Theorem 2.1., the following Lemma was developed by Dujella and Pethő [20].

Lemma 2.2. [20] Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational number τ such that $q>6M$, and let X, Y, μ be positive real numbers with $X>0$ and $Y>1$. Let $\varepsilon:=\|\mu q\|-M\|\tau q\|$, where $\|\cdot\|$ is the distance from the nearest integer. If $\varepsilon>0$, then there is no integer solution to the inequality

$$0<k\tau-n+\mu<XY^{-\nu}$$

with

$$k\leq M \text{ and } \nu\geq\frac{\log\left(Xq/\varepsilon\right)}{\log Y}.$$

3. MAIN RESULTS

The fundamental result of the paper is given below.

Theorem 3.1. Let m, n, and a be integers such that $a\geq 1$ and $1\leq m\leq n$. Then,

- Eq. (2) is satisfied only for the triples of

$$(m,n,a)\in\left\{(1,1,1),(1,2,1),(2,2,1),(3,3,3),(6,6,7)\right\}. \tag{10}$$

- Eq. (3) is satisfied only for the triples of

$$(m,n,a)\in\left\{(1,1,1),(1,2,1),(2,2,1),(3,3,4),(6,6,10)\right\}. \tag{11}$$

Proof of Eq. (2): First of all, we prove Eq. (2). Assume that Eq. (2) holds. First, if $n = m$, then Eq. (2) becomes $F_n^2 = 2^{a-1}$. By applying Theorem 1 in [18], we find that the solutions to this equation are $(n,a) \in \{(1,1),(2,1),(3,3),(6,7)\}$, which are exactly the solutions given in (10) of Theorem 3.1. Therefore, we assume that $m < n$.

If *m* and *n* are consecutive integers and $2^a = F_m^2 + F_n^2 = F_{n-1}^2 + F_n^2 = F_{2n-1}$, then we are looking for Fibonacci numbers that are perfect powers. The only such Fibonacci numbers, as shown in [18], are 0, 1, 8, and 144. Also, we know that $F_3 = 2$. Therefore, the only solution to this equation is $(m,n,a) \in \{(1,2,1)\}$ such that $1 \le m = n-1 < n$ and $a \ge 1$.

Now we get a relation between *n* and *a*, considering Lemma 2.1. and Eq. (2), we can write

$$2^a = F_m^2 + F_n^2 \le \gamma^{2(m-1)} + \gamma^{2(n-1)} < \gamma^{2(n-1)} + \gamma^{2(n-1)} = 2\cdot\gamma^{2(n-1)} < 2^{2n}, \tag{12}$$

where we used the fact that $\gamma < 2$. Hence, $a < 2n$, which will be used many times later in the proof.

Applying the Binet's formula in Eq. (1) to Eq. (2) yields

$$F_m^2 + F_n^2 = 2^a \Rightarrow \left(\frac{\gamma^m - \delta^m}{\sqrt{5}}\right)^2 + \left(\frac{\gamma^n - \delta^n}{\sqrt{5}}\right)^2 = 2^a \tag{13}$$

and from this, we get

$$5\cdot 2^a - \gamma^{2n} = \gamma^{2m} - 2((-1)^m + (-1)^n) + \delta^{2m} + \delta^{2n}.$$

Dividing both sides of the last equation by γ^{2n}, taking absolute values, and taking into account that $m < n$, we get

$$\begin{aligned}\left|5\cdot 2^a \cdot \gamma^{-2n} - 1\right| &= \left|\frac{1}{\gamma^{2n-2m}} - \frac{2((-1)^m + (-1)^n)}{\gamma^{2n}} + \frac{\delta^{2m}}{\gamma^{2n}} + \frac{\delta^{2n}}{\gamma^{2n}}\right| \\ &\le \frac{1}{\gamma^{2n-2m}} + \frac{2\left|(-1)^m + (-1)^n\right|}{\gamma^{2n}} + \frac{|\delta|^{2n}}{\gamma^{2n}} + \frac{|\delta|^{2m}}{\gamma^{2n}} < \frac{8}{\gamma^{n-m}}.\end{aligned}$$

As a result, we have

$$|\Lambda_1| < \frac{8}{\gamma^{n-m}},\ \Lambda_1 := 5\cdot 2^a \cdot \gamma^{-2n} - 1. \tag{14}$$

According to Theorem 2.1, we get

$$r := 3, \chi_1 := 5, \chi_2 := 2, \chi_3 := \gamma, t_1 := 1, t_2 := a, \text{ and } t_3 := -2n.$$

Because of $\chi_1, \chi_2, \chi_3 \in \mathbb{Q}(\sqrt{5})$, we should consider $\mathbb{T} := \mathbb{Q}(\sqrt{5})$, so we can take $d_{\mathbb{T}} := 2$. It is clear that $\Lambda_1 \ne 0$. Indeed, if $\Lambda_1 = 0$, we get $5\cdot 2^a = \gamma^{2n}$, which is impossible because $5\cdot 2^a \in \mathbb{Z}$, but γ^{2n} is not. So, $\Lambda_1 \ne 0$. From Eqs. (5) and (9), we can compute $h(\chi_1) = \log 5$, $h(\chi_2) = \log 2$, and $h(\chi_3) = \frac{1}{2}\log\gamma$. Then, we can take $A_1 := 2\log 5$,

$A_2 := 2\log 2$, and $A_3 := \log\gamma$, since $A_i \geq \max\{d_{\mathbb{T}} h(\chi_i), |\log\chi_i|, 0.16\}$, $i = 1,2,3$. Besides for $B := 2n$, $B \geq \max\{1, a, |-2n|\}$, since $a < 2n$. As a result, based on Theorem 2.1, we get

$$\log(|\Lambda_1|) > -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2) \times (1+\log 2n) \times 2\log 5 \times 2\log 2 \times \log\gamma$$

and with certain mathematical simplifications of the above inequality, we obtain,

$$\log(|\Lambda_1|) > -6.3 \times 10^{12} \times \log n, \tag{15}$$

where we used the fact that $1+\log 2n < 3\log n$, for $n > 2$. Further from inequality (14), we have

$$\log(|\Lambda_1|) < \log 8 - (n-m)\log\gamma. \tag{16}$$

Considering the inequalities (15) and (16) together, we get that

$$(n-m)\log\gamma < 6.4 \times 10^{12} \log n. \tag{17}$$

By the way, if rearranging Eq. (2) as

$$F_m^2 + F_n^2 = 2^a \Rightarrow \left(\frac{\gamma^m - \delta^m}{\sqrt{5}}\right)^2 + \left(\frac{\gamma^n - \delta^n}{\sqrt{5}}\right)^2 = 2^a$$
$$\Rightarrow \frac{\gamma^{2n}}{5} + \frac{\gamma^{2m}}{5} - 2^a = \frac{2(-1)^m + 2(-1)^n}{5} - \frac{\delta^{2m} + \delta^{2n}}{5}.$$

Taking absolute values after dividing both sides of the last equation by $\frac{\gamma^{2n}}{5}\left(1+\gamma^{2m-2n}\right)$, and taking into account that $m < n$, we get

$$\left|2^a \cdot \gamma^{-2n} \cdot 5\left(1+\gamma^{2m-2n}\right)^{-1} - 1\right| < \frac{5}{\gamma^n}$$

and

$$|\Lambda_2| < \frac{5}{\gamma^n},\ \Lambda_2 := 2^a \cdot \gamma^{-2n} \cdot 5\left(1+\gamma^{2m-2n}\right)^{-1} - 1.\ (n-m)\log\gamma < 6.4 \times 10^{12} \log n. \tag{18}$$

To apply Matveev's theorem to Eq. (18), we can consider the case where $r = 3$, $\chi_1 = 2$, $\chi_2 = \gamma$, $\chi_3 = 5 \cdot \left(1+\gamma^{2m-2n}\right)^{-1}$, $t_1 = a$, $t_2 = -2n$, and $t_3 = 1$. Since $\chi_1, \chi_2, \chi_3 \in \mathbb{Q}(\sqrt{5})$ we can take $\mathbb{T} = \mathbb{Q}(\sqrt{5})$ and $d_{\mathbb{T}} = 2$. As can be seen, since $\gamma^{2n} + \gamma^{2m} = 2^a \cdot 5$ is never satisfied $\Lambda_2 \neq 0$. Besides, if we take $B = 2n$, then $B \geq \max\{a, |-2n|, 1\}$, since $a < 2n$. In this case, we can compute the following:

$$h(\chi_1) = \log 2,\ h(\chi_2) = \frac{1}{2}\log\gamma,\ A_1 = 2\log 2,\ \text{and}\ A_2 = \log\gamma.$$

From (6), (7), (8), and (9), we get

$$h(\chi_3) \leq 3 + (n-m)\log\gamma.$$

Therefore, we can take

$$A_3 = 6 + 2(n-m)\log\gamma = d_{\mathbb{T}}(3 + (n-m)\log\gamma) \geq \max\{d_{\mathbb{T}} h(\chi_3), |\log\chi_3|, 0.16\}.$$

In this case, according to Matveev's theorem, we can write

$$\log(|\Lambda_2|) > -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2) \times (1+\log 2n) \times 2\log 2 \times \log\gamma \times (6 + 2(n-m)\log\gamma)$$

and with certain mathematical simplifications of the above inequality, we obtain,

$$\log(|\Lambda_2|) > -4 \times 10^{12} \times \log n \times (3 + (n-m)\log\gamma), \tag{19}$$

where we used the fact that $1 + \log 2n < 3\log n$, for $n > 2$. From the right-hand side of the inequality (18), we get

$$\log(|\Lambda_2|) < \log 5 - n\log\gamma. \tag{20}$$

Considering the inequalities (17), (19), and (20), we deduce that

$$n < 2.5 \times 10^{29}. \tag{21}$$

Thus, we can summarize the results mentioned above with a lemma as follows:

Lemma 3.1. If (m,n,a) is a solution in positive integers of Eq. (2) with $a < 2n$ and $1 \leq m \leq n$, then we obtain $n < 2.5 \times 10^{29}$.

3.1. REDUCING THE BOUNDS ON n FOR EQUATION (2)

As can be seen, we have determined a finite number of solutions to our problems, even though it has pretty extensive borders. The next step is to reduce the borders to a size that can be easily handled. We will use the Dujella-Pethő reduction lemma several times in this subsection to achieve this.

First, we consider the notation

$$\Gamma_1 := a\log 2 - 2n\log\gamma + \log 5. \tag{22}$$

Then, inequality (14) can be rewritten as

$$|\Lambda_1| = |e^{\Gamma_1} - 1| < \frac{8}{\gamma^{n-m}}. \tag{23}$$

Second, by using (1) and (2), we can write

$$\frac{\gamma^{2n}}{5} = F_n^2 + \frac{2(-1)^n}{5} - \frac{\delta^{2n}}{5} < F_n^2 + 1 \leq F_n^2 + F_m^2 = 2^a.$$

Hence, $1<2^a\cdot\gamma^{-2n}\cdot 5$ and so $\Gamma_1>0$. Considering this inequality with (23), we obtain

$$0<\Gamma_1\le e^{\Gamma_1}-1<\frac{8}{\gamma^{n-m}}.$$

Substituting Γ_1 in the above inequality by its formula (22), we get

$$\Gamma_1=a\log 2-2n\log\gamma+\log 5<\frac{8}{\gamma^{n-m}}, \tag{24}$$

and dividing both sides of the above inequality by $2\log\gamma$, we have

$$0<a\frac{\log 2}{2\log\gamma}-n+\frac{\log 5}{2\log\gamma}<\frac{8}{\gamma^{n-m}\cdot 2\log\gamma}<\frac{9}{\gamma^{n-m}}.$$

From Lemma 3.1, we put $M:=5\times 10^{29}, M>2n>a$. We also put

$$\tau:=\frac{\log 2}{2\log\gamma},\ \mu:=\frac{\log 5}{2\log\gamma},\ X:=9,\ Y:=\gamma,\ \text{and}\ \nu:=n-m.$$

Obviously τ is an irrational number and

$$\frac{p_{68}}{q_{68}}=\frac{20721505928824926197089563175427}{28771475858671197523902386653746}$$

is the 68th convergent of the continued fraction expansion of τ. Applying Lemma 2.2, we obtain

$$n-m\le\frac{\log\left(Xq_{68}/\varepsilon\right)}{\log Y}<160.2,$$

where $6M<q_{68}$ and $\varepsilon:=\|\mu q_{68}\|-M\|\tau q_{68}\|, \varepsilon>0.09.$ If (m,n,a) is a possible solution of the Eq. (2), then $n-m\le 160$. When we insert this upper bound for $n-m$ into (19), the result is $n<2.6\times 10^{16}$.

Now we consider the notation

$$\Gamma_2:=a\log 2-2n\log\gamma+\log\left(\frac{5}{1+\gamma^{2m-2n}}\right) \tag{25}$$

and

$$\left|\Lambda_2\right|=\left|e^{\Gamma_2}-1\right|<\frac{5}{\gamma^n}. \tag{26}$$

Since $\Lambda_2\neq 0$, we observe that $\Gamma_2\neq 0$. So, we differentiate the following cases. If $\Gamma_2>0$, then $e^{\Gamma_2}-1>0$, hence from (26) and used the fact that $x\le e^x-1$ for all $x\in\mathbb{R}$, we get

$$0<\Gamma_2<e^{\Gamma_2}-1<\frac{5}{\gamma^n}.$$

Assume now that $\Gamma_2 < 0$. It is easy to see that $\frac{5}{\gamma^n} < \frac{1}{2}$ for all $n > 4$. So, from (26), we have that $\left|e^{\Gamma_2} - 1\right| < \frac{1}{2}$. Hence, $e^{|\Gamma_2|} < 2$. Because of $\Gamma_2 < 0$, we can deduce that

$$0 < \left|\Gamma_2\right| < e^{|\Gamma_2|} - 1 = e^{|\Gamma_2|}\left|e^{\Gamma_2} - 1\right| < \frac{10}{\gamma^n}.$$

In any case, we obtain that the inequality

$$0 < \left|\Gamma_2\right| < \frac{10}{\gamma^n}$$

holds for all $n > 4$. By replacing Γ_2 in the previous inequality with its formula and following the argument in (25) and dividing both sides of the inequality by $2\log\gamma$, we have

$$0 < \left| a\frac{\log 2}{2\log\gamma} - n + \frac{\log\left(5/\left(1+\gamma^{2m-2n}\right)\right)}{2\log\gamma} \right| < \frac{10}{\gamma^n \cdot 2\log\gamma} < \frac{11}{\gamma^n}.$$

Again, we apply Lemma 2.2, setting $\tau = \frac{\log 2}{2\log\gamma}$. Also, from the new result above, we take $M := 5.2\times 10^{16}$ with $a < 2n < M$ and $6M < q_{41}$, where

$$\frac{p_{41}}{q_{41}} = \frac{1622342818532681326}{2252596765806035975}$$

is the 41st convergent of the continued fraction expansion of τ. As a result, we have

$$n \leq \frac{\log\left(Xq_{41}/\varepsilon\right)}{\log Y} < 94.3,$$

where

$$\varepsilon := \left\|\mu q_{41}\right\| - M\left\|\tau q_{41}\right\| > 0.4860,\ \mu = \frac{\log\left(5/\left(1+\gamma^{2m-2n}\right)\right)}{2\log\gamma}, X := 11,\ Y := \gamma, \text{ and } \nu := n.$$

So, we obtain that $n \leq 94$. Thus, $a < 2n < 188$. Structuring an iterative algorithm in Mathematica© for Eq. (2) over the range $n \leq 94$, and $a < 188$ shall prove the validity of Theorem 3.1.

Proof of Eq. (3): Now, we assume that the second equation, namely Eq. (3), holds. By Wiles' solution of Fermat's last theorem in [21], it is shown that there is no solution to the equation $F_m^3 + F_n^3 = 2^3$, where there does not exist a triple $(m,n,3)$. If $n = m$, then Eq. (3) becomes $F_n^3 = 2^{a-1}$. Using Theorem 1 in [18] and the fact that $F_3 = 2$, we find that the solutions to this equation are $(n,a) \in \{(1,1),(2,1),(3,4),(6,10)\}$, which correspond to the solutions stated in (11) of Theorem 3.1. Therefore, we assume that $m < n$.

Now we get a relation between n and a. Considering Lemma 2.1. and Eq. (3), we can write

$$2^a = F_m^3 + F_n^3 \le \gamma^{3(m-1)} + \gamma^{3(n-1)} < \gamma^{3(n-1)} + \gamma^{3(n-1)} = 2\cdot\gamma^{3(n-1)} < 2^{3n}. \tag{27}$$

Hence, $a < 3n$, which will be used many times later in the proof. Applying the Binet's formula in Eq. (1) to Eq. (3) yields

$$F_m^3 + F_n^3 = 2^a \Rightarrow \left(\frac{\gamma^m - \delta^m}{\sqrt{5}}\right)^3 + \left(\frac{\gamma^n - \delta^n}{\sqrt{5}}\right)^3 = 2^a \tag{28}$$

and from this, we get

$$5\sqrt{5}\cdot 2^a - \gamma^{3n} = \gamma^{3m} - 3(-1)^m(\gamma^m - \delta^m) - 3(-1)^n(\gamma^n - \delta^n) - \delta^{3m} - \delta^{3n}.$$

Dividing both sides of the last equation by γ^{3n}, taking absolute values, and taking into account that $m < n$, we get

$$\begin{aligned}\left|5\sqrt{5}\cdot 2^a\cdot\gamma^{-3n} - 1\right| &= \left|\frac{\gamma^{3m}}{\gamma^{3n}} - \frac{3(-1)^m(\gamma^m - \delta^m) + 3(-1)^n(\gamma^n - \delta^n)}{\gamma^{3n}} - \frac{\delta^{3n}}{\gamma^{3n}} - \frac{\delta^{3m}}{\gamma^{3n}}\right| \\ &\le \frac{1}{\gamma^{3n-3m}} + \frac{3\gamma^m + 3|\delta|^m}{\gamma^{3n}} + \frac{3\gamma^n + 3|\delta|^n}{\gamma^{3n}} + \frac{|\delta|^{3n}}{\gamma^{3n}} + \frac{|\delta|^{3m}}{\gamma^{3n}} < \frac{15}{\gamma^{n-m}}.\end{aligned}$$

As a result, we have

$$|\Lambda_3| < \frac{15}{\gamma^{n-m}},\ \Lambda_3 := 5\sqrt{5}\cdot 2^a\cdot\gamma^{-3n} - 1. \tag{29}$$

According to Theorem 2.1, we get $r := 3,\ \chi_1 := 5\sqrt{5},\ \chi_2 := 2,\ \chi_3 := \gamma,\ t_1 := 1,\ t_2 := a$, and $t_3 := -3n$. Because of $\chi_1, \chi_2, \chi_3 \in \mathbb{Q}(\sqrt{5})$, we should consider $\mathbb{T} := \mathbb{Q}(\sqrt{5})$, so we can take $d_{\mathbb{T}} := 2$. It is clear that $\Lambda_3 \ne 0$. Indeed, if $\Lambda_3 = 0$, we get $125\cdot 4^a = \gamma^{6n}$, which is impossible because $125\cdot 4^a \in \mathbb{Z}$, but γ^{6n} is not. So, $\Lambda_3 \ne 0$. From Eqs. (5) and (9), we can compute $h(\chi_1) = \log 5\sqrt{5},\ h(\chi_2) = \log 2$, and $h(\chi_3) = \frac{1}{2}\log\gamma$. Then, we can take $A_1 := 2\log 5\sqrt{5}$, $A_2 := 2\log 2$, and $A_3 := \log\gamma$, since $A_i \ge max\{d_{\mathbb{T}} h(\chi_i), |\log\chi_i|, 0.16\}$, i=1, 2, 3. Besides, for $B := 3n$, $B \ge \max\{1, a, |-3n|\}$, since $a < 3n$. As a result, based on Theorem 2.1, we get

$$\log(|\Lambda_3|) > -1.4\times 30^6\times 3^{4.5}\times 2^2\times(1+\log 2)\times(1+\log 3n)\times 2\log 5\sqrt{5}\times 2\log 2\times\log\gamma$$

and with certain mathematical simplifications of the above inequality, we obtain,

$$\log(|\Lambda_3|) > -1.6\times 10^{13}\times\log n, \tag{30}$$

where we used the fact that $1+\log 3n < 5\log n$, for $n \ge 2$. Further from inequality (29), we have

$$\log(|\Lambda_3|) < \log 15 - (n-m)\log\gamma. \tag{31}$$

Considering the inequalities (30) and (31) together, we get that

$$(n-m)\log\gamma < 1.7\times 10^{13}\log n. \tag{32}$$

By the way, if rearranging Eq. (3) as

$$F_m^3 + F_n^3 = 2^a \Rightarrow \left(\frac{\gamma^m-\delta^m}{\sqrt{5}}\right)^3 + \left(\frac{\gamma^n-\delta^n}{\sqrt{5}}\right)^3 = 2^a$$

$$\Rightarrow \frac{\gamma^{3n}}{5\sqrt{5}} + \frac{\gamma^{3m}}{5\sqrt{5}} - 2^a = \frac{3(-1)^m\left(\gamma^m-\delta^m\right)+3(-1)^n\left(\gamma^n-\delta^n\right)}{5\sqrt{5}} + \frac{\delta^{3m}+\delta^{3n}}{5\sqrt{5}}.$$

Taking absolute values after dividing both sides of the last equation by $\frac{\gamma^{3n}}{5\sqrt{5}}\left(1+\gamma^{3m-3n}\right)$, and taking into account that $m<n$, we get

$$\left|2^a\cdot\gamma^{-3n}\cdot 5\sqrt{5}\left(1+\gamma^{3m-3n}\right)^{-1}-1\right| < \frac{15}{\gamma^n}$$

and

$$|\Lambda_4| < \frac{15}{\gamma^n},\ \Lambda_4 := 2^a\cdot\gamma^{-3n}\cdot 5\sqrt{5}\left(1+\gamma^{3m-3n}\right)^{-1}-1. \tag{33}$$

To apply Matveev's theorem to Eq. (33), we consider the case where $r=3$, $\chi_1=2$, $\chi_2=\gamma$, $\chi_3=5\sqrt{5}\cdot\left(1+\gamma^{3m-3n}\right)^{-1}$, $t_1=a$, $t_2=-3n$, and $t_3=1$. Since $\chi_1,\chi_2,\chi_3\in\mathbb{Q}(\sqrt{5})$, we can take $\mathbb{T}=\mathbb{Q}(\sqrt{5})$ and $d_{\mathbb{T}}=2$. It is clear that $\Lambda_4\neq 0$. If $\Lambda_4=0$, then we get the relation

$$5\sqrt{5}\cdot 2^a = \gamma^{3n}+\gamma^{3m}. \tag{34}$$

Conjugating Eq. (34) in $\mathbb{Q}(\sqrt{5})$, we have

$$-5\sqrt{5}\cdot 2^a = \delta^{3n}+\delta^{3m}. \tag{35}$$

Considering Eqs. (34) and (35) together, we deduce that

$$\gamma^{3n} < \gamma^{3n}+\gamma^{3m} = \left|\delta^{3n}+\delta^{3m}\right| \le |\delta|^{3n}+|\delta|^{3m} < 1,$$

which is never satisfied for $m,n\in\mathbb{Z}^+$. So, $\Lambda_4\neq 0$. Besides, if we take $B=3n$, then $B\ge\max\{a,|-3n|,1\}$, since $a<3n$. In this case, we can compute the followings:

$$h(\chi_1)=\log 2, h(\chi_2)=\frac{1}{2}\log\gamma, A_1 := 2\log 2,\text{ and } A_2 := \log\gamma.$$

From (6), (7), (8), and (9) we get

$$h(\chi_3) \leq 4 + 2(n-m)\log\gamma.$$

Therefore, we can take

$$A_3 := 8 + 4(n-m)\log\gamma = d_{\mathbb{T}}(4 + 2(n-m)\log\gamma) \geq \max\{d_{\mathbb{T}}h(\chi_3), |\log\chi_3|, 0.16\}.$$

In this case, according to Matveev's theorem, we can write

$$\log(|\Lambda_4|) > -1.4\times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2)\times(1+\log 3n)\times 2\log 2\times\log\gamma\times(8+4(n-m)\log\gamma)$$

and with certain mathematical simplifications of the above inequality, we obtain,

$$\log(|\Lambda_4|) > -1.3\times 10^{13} \times \log n \times (2+(n-m)\log\gamma), \tag{36}$$

where we used the fact that $1+\log 3n < 5\log n$, for $n \geq 2$. From the right-hand side of the inequality (33), we get

$$\log(|\Lambda_4|) < \log 15 - n\log\gamma. \tag{37}$$

Considering the inequalities (32), (36), and (37), we deduce that

$$n < 2.25\times 10^{30}. \tag{38}$$

Thus, we can summarize the results mentioned above with a lemma as follows:

Lemma 3.1.1. If (m,n,a) is a solution in positive integers of Eq. (3) with $a < 3n$ and $1 \leq m \leq n$, then we obtain $n < 2.25\times 10^{30}$.

3.2. REDUCING THE BOUNDS ON n FOR EQUATION (3)

As can be seen, we have determined a finite number of solutions to our problems, even though it has pretty extensive borders. The next step is to reduce the borders to a size that can be easily handled. We will use the Dujella-Pethő reduction lemma several times in this subsection to achieve this.

First, we consider the notation

$$\Gamma_3 := a\log 2 - 3n\log\gamma + \log 5\sqrt{5}. \tag{39}$$

Then, inequality (29) can be rewritten as

$$|\Lambda_3| = |e^{\Gamma_3} - 1| < \frac{15}{\gamma^{n-m}}. \tag{40}$$

Since $\Lambda_3 \neq 0$, we can observe that $\Gamma_3 \neq 0$. So, we differentiate the following cases. If $\Gamma_3 > 0$, then $e^{\Gamma_3} - 1 > 0$, hence from (40), and use the fact that $x \leq e^x - 1$ for all $x \in \mathbb{R}$, we get

$$0 < \Gamma_3 < e^{\Gamma_3} - 1 < \frac{15}{\gamma^{n-m}}. \tag{41}$$

Assume now that $\Gamma_3 < 0$. It is easy to see that $\frac{15}{\gamma^{n-m}} < \frac{1}{2}$ for all $n-m>7$. So, from (40), we have that $\left|e^{\Gamma_3}-1\right| < \frac{1}{2}$. So, $\left|e^{\Gamma_3}\right| < 2$. Since $\Gamma_3 < 0$, we can deduce that

$$0 < \left|\Gamma_3\right| < e^{\left|\Gamma_3\right|} - 1 = e^{\left|\Gamma_3\right|}\left|e^{\Gamma_3}-1\right| < \frac{30}{\gamma^{n-m}}. \tag{42}$$

In any case, we obtain that the inequality

$$0 < \left|\Gamma_3\right| < \frac{30}{\gamma^{n-m}} \tag{43}$$

satisfy all $n-m>7$. By replacing Γ_3 in the previous inequality with its formula (39) and dividing both sides of the inequality by $3\log\gamma$, we obtain

$$0 < \left|a\frac{\log 2}{3\log\gamma} - n + \frac{\log 5\sqrt{5}}{3\log\gamma}\right| < \frac{30}{\gamma^{n-m}\cdot 3\log\gamma} < \frac{21}{\gamma^{n-m}}.$$

From Lemma 3.1.1, we put $M := 6.75\times 10^{30}$, $M > 3n > a$. We also put

$$\tau := \frac{\log 2}{3\log\gamma},\ \mu := \frac{\log 5\sqrt{5}}{3\log\gamma},\ X := 21,\ Y := \gamma, \text{ and } \nu := n-m.$$

Obviously, τ is an irrational number and

$$\frac{p_{53}}{q_{53}} = \frac{624077771882059188165772565863 53}{1299782839817197865252560039396 78}$$

is 53rd convergent of the continued fraction expansion of τ. Applying Lemma 2.2, we obtain

$$n-m \le \frac{\log\left(Xq_{53}/\varepsilon\right)}{\log Y} < 165.84,$$

where $6M < q_{53}$ and $\varepsilon := \left\|\mu q_{53}\right\| - M\left\|\tau q_{53}\right\|$, $\varepsilon > 0.06$. If (m,n,a) is a possible solution of Eq. (3), then $n-m \le 165$. When we insert this upper bound for $n-m$ into (36) and (37), the result is $n < 8.6\times 10^{16}$.

Now we consider the notation

$$\Gamma_4 := a\log 2 - 3n\log\gamma + \log\left(\frac{5\sqrt{5}}{1+\gamma^{3m-3n}}\right) \tag{44}$$

and

$$\left|\Lambda_4\right| = \left|e^{\Gamma_4}-1\right| < \frac{15}{\gamma^{n}}. \tag{45}$$

Because of $\Lambda_4 \ne 0$, we observe that $\Gamma_4 \ne 0$. As previously shown, for all values of Γ_4, either $\Gamma_4 > 0$ or $\Gamma_4 < 0$, it can be demonstrated that

$$0<|\Gamma_4|<\frac{30}{\gamma^n} \tag{46}$$

holds for all $n>7$. By replacing Γ_4 in the previous inequality with its formula (44) and dividing both sides of the inequality by $3\log\gamma$, we get

$$0<\left|a\frac{\log 2}{3\log\gamma}-n+\frac{\log\left(5\sqrt{5}/\left(1+\gamma^{3m-3n}\right)\right)}{3\log\gamma}\right|<\frac{30}{\gamma^n\cdot 3\log\gamma}<\frac{21}{\gamma^n}.$$

Again, we apply Lemma 2.2, taking $\tau=\frac{\log 2}{3\log\gamma}$. Also, from the above new result, we take $M:=2.6\times 10^{17}$, with $a<3n<M$ and $6M<q_{29}$, where

$$\frac{p_{29}}{q_{29}}=\frac{1565498096762651489}{3260503183444777387}$$

is the 29th convergent of the continued fraction expansion of τ. As a result, we have

$$n\leq\frac{\log\left(Xq_{29}/\varepsilon\right)}{\log Y}<96.93,$$

where

$$\varepsilon:=\|\mu q_{29}\|-M\|\tau q_{29}\|>0.38,\ \mu:=\frac{\log\left(5\sqrt{5}/\left(1+\gamma^{3m-3n}\right)\right)}{3\log\gamma},\ X:=21,\ Y:=\gamma,\text{ and }\nu:=n.$$

So, we obtain that $n\leq 96$. Thus, $a<3n\leq 288$. Structuring an iterative algorithm in Mathematica© for Eq. (3) over the range $n\leq 96$ and $a\leq 288$ shall prove the validity of Theorem 3.1.

4. CONCLUSIONS

In this study, we explored numbers that are powers of two and can be expressed as the sum of the squares or cubes of two Fibonacci numbers. By applying Matveev's theorem and the Dujella-Pethő reduction lemma, we established a comprehensive framework for solving the Diophantine equation $F_m^x+F_n^x=2^a$ with $x\in\{2,3\}$, where m, n, and a are positive integers. Our findings reveal all possible solutions to this equation, contributing to the broader understanding of the interplay between Fibonacci numbers and powers of two in the context of Diophantine problems. This work underscores the power of advanced number-theoretic techniques in addressing such equations and provides a pathway for further exploration in related areas.

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