ORIGINAL PAPER

THE RANDIĆ INDEX OF A POWER GRAPH

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Abstract. *In this study, the Randić index of the power graph of a finite cyclic group was redefined by considering its identity and generators. Then, bounds were obtained for the Randić index of the power graph in question by using the concepts of maximum degree, minimum degree, graph energy, and vertex energy.*
Keywords: *Power graph; Randić index; energy.*

1. INTRODUCTION

A power graph is a simple, connected graph defined on a finite group, which accepts the elements of the group as vertices and defines the adjacency relationship between two vertices as "if one vertex is a power of the other". The concept of power graph was first introduced to the mathematical literature by Kelarev and Quinn in 2000 [1]. Later, with the article of Chakrabarty et al. in 2009, studies in the field of power graphs gained momentum [2]. Another important study that shapes the power graph field is the one conducted by Chattopadhyay et al. in 2018 [3]. In their study, the adjacency matrix of the power graph was redefined by considering the block matrix structure.

On the other hand, the concept of the Randić index was introduced by Milan Randić in 1975 [4]. In his study, the Randić index concept was used to determine the degree of branching of the carbon atomic skeleton. Later, with the studies carried out 1990's the concept of Randić index entered the mathematical literature and began to be used frequently in this field. In this study, bounds were obtained for the Randić index of power graphs by considering the concepts of maximum degree, minimum degree, and vertex energy.

2. PRELIMINARIES

The power graph $P(C_n)$ of a finite cyclic group C_n with nth order, is the graph whose vertices are represented by the elements of C_n, and with adjacency relation between different two vertices i and j is defined as $i \sim j$ iff $i = j^k$ or $j = i^k, k \in \mathbb{Z}^+$. Let V_1, V_2 be the set of the identity and generators and the set of remaining elements of C_n, respectively. Due to the structure of V_1, $|V_1| = 1 + \varphi(n) = s$ (say), where $\varphi(n)$ is Euler's φ function. Also, adjacency matrix of the power graph $P(C_n)$ is a block matrix and is defined as

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$$A\big(P(C_n)\big) = \begin{pmatrix} (\mathrm{J}-\mathrm{I})_{\mathrm{s}\times\mathrm{s}} & J_{\mathrm{s}\times(\mathrm{n}-\mathrm{s})} \\ J_{(\mathrm{n}-\mathrm{s})\times\mathrm{s}} & A(P(V_2))_{(\mathrm{n}-\mathrm{s})\times(\mathrm{n}-\mathrm{s})} \end{pmatrix},$$

where I and J, are the identity matrix and the matrix with all elements being 1, respectively. Also $A(P(V_2))$ is the adjacency matrix characterized by the elements of V_2. Based on the concept of adjacency matrix, the energy of the power graph $P(C_n)$ is denoted by and defined as

$$E(P(C_n)) = \sum_{i=1}^{n} |\lambda_i(P(C_n))|,$$

where $\lambda_1(P(C_n)), \lambda_2(P(C_n)), \dots, \lambda_n(P(C_n))$ are the eigenvalues of the adjacency matrix. In addition, the energy of any vertex i of the power graph $P(C_n)$ is defined as

$$E_{P(C_n)}(i) = \left|A\big(P(C_n)\big)\right|_{ii},$$

where $\left|A\big(P(C_n)\big)\right| = \left[A\big(P(C_n)\big)A^*\big(P(C_n)\big)\right]^{\frac{1}{2}}$. Depending on the concept of vertex energy, the energy of the power graph $P(C_n)$ is given by

$$E(P(C_n)) = \textstyle\sum_{i=1}^{n} E_{P(C_n)}(i), \text{ for } 1,2,\dots,n \in C_n.$$

On the other hand, the Randić index of any simple graph G is defined as $R(G) = \sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}}$, where d_i, is the degree of a vertex i. This structure can be adapted for the power graph $P(C_n)$ by using the V_1 and V_2 sets. Accordingly, the Randić index of the power graph $P(C_n)$ is defined as

$$R\big(P(C_n)\big) = \frac{s(s-1)}{2(n-1)} + \frac{s}{\sqrt{n-1}} \sum_{j\in V_2} \frac{1}{\sqrt{d_j}} + \sum_{\substack{i\sim j \\ i,j\in V_2}} \frac{1}{\sqrt{d_i d_j}}.$$

The following theorems will help to obtain some results for the Randić index of the power graph $P(C_n)$. These theorems clearly reveal the relationship between any vertex energy and its degree.

Theorem 2.1. [5] For a graph G and a vertex i, $E_G(i) \le \sqrt{d_i}$. The equality holds iff $G \cong K_{1,n}$.

Theorem 2.2. [5] Let G be a connected graph with at least one edge. Then for all $i \in V$ $E_G(i) \ge \frac{d_i}{\Delta}$, where Δ is the maximum degree of G. Equality holds iff $G \cong K_{d,d}$.

In the next section, upper bounds for the Randić index of the power graph $P(C_n)$ were found. Then, results were obtained for Randić index using the relationship between the completeness of the power graph and the order of the group.

3. MAIN RESULTS

In this section, bounds for the Randić index $R(P(C_n))$ were obtained. First, the bound obtained by using the smallest degree of $P(C_n)$ is given.

Theorem 3.1. Let $R(P(C_n))$ be the Randić index of $P(C_n)$ and $n \geq 3$. Then

$$R(P(C_n)) \leq \frac{s(s-1)}{2(n-1)} + \frac{s(n-s)}{\sqrt{\delta(n-1)}} + \frac{(n-s-1)(n-s)}{2\delta},$$

where δ is the minimum degree of the power graph $P(C_n)$.

Proof: Since $\delta \leq d_i$, for all $i \in V(P(C_n))$,

$$\sum_{j \in V_2} \frac{1}{\sqrt{d_j}} \leq \sum_{j \in V_2} \frac{1}{\sqrt{\delta}} = \frac{n-s}{\sqrt{\delta}}. \tag{3.1}$$

On the other hand

$$\sum_{\substack{i \sim j \\ i,j \in V_2}} \frac{1}{\sqrt{d_i d_j}} \leq \sum_{\substack{i \sim j \\ i,j \in V_2}} \frac{1}{\delta} \leq \frac{(n-s-1)(n-s)}{2\delta}. \tag{3.2}$$

From (3.1), (3.2) and the definition of the Randić index of the power graph $P(C_n)$, we get

$$R(P(C_n)) \leq \frac{s(s-1)}{2(n-1)} + \frac{s(n-s)}{\sqrt{\delta(n-1)}} + \frac{(n-s-1)(n-s)}{2\delta}.$$

This completes the proof.

Theorem 3.2. Let $R(P(C_n))$ be the Randić index of $P(C_n)$ and $n \geq 3$. Then

$$R(P(C_n)) \leq \frac{s(s-1)}{2(n-1)} + \frac{n-s}{\sqrt{\delta}}\left[\frac{s}{\sqrt{n-1}} + \frac{\delta - s}{2\sqrt{\delta}}\right],$$

where δ is the minimum degree of the power graph $P(C_n)$.

Proof: Using the AM-GM inequality, we have

$$\begin{aligned} \sum_{\substack{i \sim j \\ i,j \in V_2}} \frac{1}{\sqrt{d_i d_j}} &\leq \frac{1}{2} \sum_{\substack{i \sim j \\ i,j \in V_2}} \left(\frac{1}{d_i} + \frac{1}{d_j}\right) \\ &= \frac{1}{2} \sum_{i \in V_2} \frac{d_i - s}{d_i} \end{aligned} \tag{3.3}$$

$$\leq \frac{1}{2}\left[n - s - s\sum_{i\in V_2}\frac{1}{\delta}\right]$$
$$= \frac{n-s}{2}\left[1 - \frac{s}{\delta}\right].$$

From (3.1) in Theorem 3.1, (3.3) and the concept of the Randić index of $P(C_n)$, we get

$$R\big(P(C_n)\big) \leq \frac{s(s-1)}{2(n-1)} + \frac{n-s}{\sqrt{\delta}}\left[\frac{s}{\sqrt{n-1}} + \frac{\delta - s}{2\sqrt{\delta}}\right].$$

Theorem 3.3. Let $R\big(P(C_n)\big)$ be the Randić index of $P(C_n)$ and $n \geq 3$. Then

$$R\big(P(C_n)\big) \leq \frac{s(s-1)}{2(n-1)} + \left(\frac{s}{\sqrt{n-1}} + n - s - 1\right)\sum_{j\in V_2}\frac{1}{E_{P(C_n)}(j)},$$

where $E_{P(C_n)}(j)$, is the energy of the vertex j.

Proof: Since $d_i \geq 1$ for every vertex i,

$$\sum_{\substack{i\sim j\\ i,j\in V_2}}\frac{1}{\sqrt{d_i d_j}} \leq \sum_{\substack{i\sim j\\ i,j\in V_2}}\left(\frac{\sqrt{d_i}+\sqrt{d_j}}{\sqrt{d_i d_j}}\right)$$
$$= \sum_{\substack{i\sim j\\ i,j\in V_2}}\left(\frac{1}{\sqrt{d_i}} + \frac{1}{\sqrt{d_j}}\right).$$

From Theorem 2.1, we have

$$\leq \sum_{\substack{i\sim j\\ i,j\in V_2}}\left(\frac{1}{E_{P(C_n)}(i)} + \frac{1}{E_{P(C_n)}(j)}\right)$$
$$\leq \sum_{j\in V_2}\frac{n-s-1}{E_{P(C_n)}(j)}\ . \tag{3.4}$$

On the other hand, using Theorem 2.1, we get

$$\sum_{j\in V_2}\frac{1}{\sqrt{d_j}} \leq \sum_{j\in V_2}\frac{1}{E_{P(C_n)}(j)}. \tag{3.5}$$

From (3.4), (3.5) and the definition of the Randić index of the power graph $P(C_n)$, we have

$$R\big(P(C_n)\big) \leq \frac{s(s-1)}{2(n-1)} + \left(\frac{s}{\sqrt{n-1}} + n - s - 1\right)\sum_{j\in V_2}\frac{1}{E_{P(C_n)}(j)}.$$

Theorem 3.4. Let $R\big(P(C_n)\big)$ and $E\big(P(C_n)\big)$ be the Randić index and energy of $P(C_n)$, respectively and $n \geq 3$. Then

$$R\big(P(C_n)\big) \leq \frac{\sqrt{2\Delta}}{\delta}(n-t-1)E\big(P(C_n)\big),$$

where Δ and δ be the maximum degree and minimum degree of the power graph $P(C_n)$, respectively.

Proof: Since $d_i \geq 1$ for every vertex i,

$$\sum_{\substack{i\sim j \\ i,j\in V_2}} \frac{1}{\sqrt{d_i d_j}} \leq \sum_{\substack{i\sim j \\ i,j\in V_2}} \left(\frac{\sqrt{d_i}}{d_i} + \frac{\sqrt{d_j}}{d_j}\right).$$

From Theorem 2.2 and the fact $\delta \leq d_i$, for all $i \in V\big(P(C_n)\big)$, we get

$$\begin{aligned}
&\leq \sum_{\substack{i\sim j \\ i,j\in V_2}} \left(\frac{\sqrt{\Delta E_{P(C_n)}(i)}}{\delta} + \frac{\sqrt{\Delta E_{P(C_n)}(j)}}{\delta}\right) \\
&= \frac{\sqrt{\Delta}}{\delta} \sum_{\substack{i\sim j \\ i,j\in V_2}} \left(\sqrt{E_{P(C_n)}(i)} + \sqrt{E_{P(C_n)}(j)}\right) \\
&\leq \frac{\sqrt{\Delta}}{\delta}(n-s-1) \sum_{j\in V_2} E_{P(C_n)}(j) \\
&\leq \frac{\sqrt{\Delta}}{\delta}(n-s-1)E\big(P(C_n)\big).
\end{aligned}$$

Corollary 3.5. Let $R\big(P(C_n)\big)$ be the Randić index of $P(C_n)$ and $n = a^t \geq 3$. Then

$$R\big(P(C_n)\big) = \frac{s(s-1)}{2(n-1)},$$

where a, t are a prime number and a positive integer, respectively.

Proof: Since a is a prime number and t is a positive integer, $P(C_n)$ is a complete graph. In this case $V_1 = V\big(P(C_n)\big)$ and $V_2 = \emptyset$. From the definition of the Randić index of the power graph $P(C_n)$, we get

$$R\big(P(C_n)\big) = \frac{s(s-1)}{2(n-1)} + \frac{s}{\sqrt{n-1}} \sum_{j\in V_2} \frac{1}{\sqrt{d_j}} + \sum_{\substack{i\sim j \\ i,j\in V_2}} \frac{1}{\sqrt{d_i d_j}} = \frac{s(s-1)}{2(n-1)}.$$

4. CONCLUSIONS

In this study, the concept of the Randić index for a power graph is introduced. Then, by considering the concepts of vertex energy, degree, maximum degree, and minimum degree, bounds for the Randic index are obtained. In the following studies, other types of indices on a power graph will be examined, and boundary studies will be conducted for these concepts.

REFERENCES

[1] Kelarev, A. V., Quinn, S. J., *Contributions to General Algebra*, **12**, 229, 2000.
[2] Chakrabarty, I., Ghosh, S., Sen, M. K., *Semigroup Forum,* **78**, 410, 2009.
[3] Chattopadhyay, S., Panigrahi, P., Atik, F., *Indagationes Mathematicae,* **29**(2), 730, 2018.
[4] Randić, M., *Journal of the American Chemical Society,* **97**, 6609, 1975.
[5] Arizmendi, O., Hidolgo, J. F., Romero, O. J., *Linear Algebra and its Applications*, **557**, 464, 2018.