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DIFFERENTIAL EQUATIONS CHARACTERIZATION FOR D_i -DARBOUX SLANT HELICES

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Abstract. *In this article, the differential equation characterizations for D_i -Darboux slant helices given with the help of three new orthogonal frames are introduced for the curves lying on a surface. These moving frames, obtained based on the Darboux frame, are called "Osculating Darboux Frame", "Normal Darboux Frame", and "Rectifying Darboux Frame", respectively.*

Keywords: *Darboux slant helix; Slant Helix; Darboux Frame; OD-Frame; ND-Frame; RD-Frame.*

1. INTRODUCTION

The concept of frame is important in the differential geometry of curves. One of the most important tools for analyzing a curve is a moving frame. The relationship of the vector fields forming the frame at the opposite points of two different curves reveals the special curve pairs [1,2]. Curvature functions are defined on the curve using moving frames [3,4]. These curvature functions are known as the differential invariants of the curve. Curves become special thanks to the relationships between the differential invariants of the curve [5-9]. Many different frames have been defined in different spaces [10-12]. The most commonly used moving frames are the Frenet frame and Bishop frame for the space curves, and the Darboux frame for the surface curves. The Darboux frame is known as the frame of the curve-surface pair [13-16]. Hananoi et al. describe three new vector fields associated with the Darboux frame along the curve on the surface [14]. In addition, Önder defines three new special curves on the surface, taking these three new vectors into account. In this definition, he names these curves as D_i -Darboux slant helices, where the indices $i \in \{o,n,r\}$ represent the osculating, normal, and rectifying planes of the curve on the surface, respectively [17,18]. Three new moving frames for surface curves are created by Akın et al., with the help of these vector fields and three special curves. These moving frames, obtained based on the Darboux frame, are called Osculating Darboux Frame (OD-Frame), Normal Darboux Frame (ND-Frame), and Rectifying Darboux Frame (RD-Frame), respectively [19,20].

In this study, the differential equation characterizations for D_i -Darboux slant helices are given with the help of the OD, ND, and RD frames. The relevant theorems are presented along with their proofs.

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1.1. PRELIMINARIES

Let M be an oriented surface in E^3, and $\alpha : I \subset IR \to M$ be a unit speed curve on surface M. The Darboux frame $\{T, V, U\}$ is well-defined along the curve α on M and derivative formulas of the Darboux frame are given by;

$$\begin{pmatrix} T' \\ V' \\ U' \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ V \\ U \end{pmatrix}, \tag{1}$$

where T is the unit tangent of α, U is the unit normal of M along the curve and $V = U \times T$. k_g, k_n and τ_g are the geodesic curvature, the normal curvature, and the geodesic torsion of the curve α, respectively.

Definition 1. Let $\alpha : I \to M$ be a unit speed curve on oriented surface M and $\{T, V, U\}$ be the Darboux frame along α. The vector fields

$$\begin{aligned} D_o(s) &= \tau_g(s)T(s) - k_n(s)V(s) \\ D_n(s) &= -k_n(s)V(s) + k_g(s)U(s) \\ D_r(s) &= \tau_g(s)T(s) + k_g(s)U(s) \end{aligned} \tag{2}$$

are called the osculating Darboux vector field, the normal Darboux vector field, and the rectifying Darboux vector field along the curve α on M, respectively [14].

Definition 2. Let α be a unit speed curve on an oriented surface M with Darboux vector fields $D_o(s)$, $D_n(s)$, $D_r(s)$. Then,

a) If the osculating Darboux vector field D_o (or equivalently, unit osculating Darboux vector field $\tilde{D}_o = D_o / \|D_o\|$) of α always makes a constant angle with a constant direction, then α is called a D_o-Darboux Slant helix on M.

b) If a normal Darboux vector field D_n (or equivalently, unit normal Darboux vector field $\tilde{D}_n = D_n / \|D_n\|$) of α always makes a constant angle with a constant direction, then α is called a D_n-Darboux Slant helix on M.

c) If rectifying Darboux vector D_r field (or equivalently, unit rectifying Darboux vector field $\tilde{D}_r = D_r / \|D_r\|$) of α always makes a constant angle with a constant direction, then α is called a D_r-Darboux Slant helix on M [17,18].

If we denote the osculating Darboux frame (OD-frame) along the curve α on M by $\{\tilde{D}_o, U, Y_o\}$, derivative formulas of the OD-frame are given by;

$$\begin{aligned} \tilde{D}_o' &= -\delta_o \vec{Y}_o \\ \vec{U}' &= \mu_o \vec{Y}_o \\ \vec{Y}_o' &= \delta_o \tilde{D}_o - \mu_o \vec{U}, \end{aligned} \tag{3}$$

where $\tilde{D}_o = \dfrac{D_o}{\|D_o\|}$, U is the unit normal vector of M and $Y_o = \tilde{D}_o \times U$. $\mu_o = \sqrt{{k_n}^2 + {\tau_g}^2}$ and $\delta_o = \left(\dfrac{\tau_g}{k_n}\right)' \left(\dfrac{{k_n}^2}{{k_n}^2 + {\tau_g}^2}\right) + k_g$ are curvatures of α according to the OD-frame [19].

If we denote the normal Darboux frame (ND-frame) along the curve α on M by $\left\{\tilde{D}_n, T, Y_n\right\}$, derivative formulas of the ND-frame are given by;

$$\begin{pmatrix} \tilde{D}_n' \\ T' \\ Y_n' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\delta_n \\ 0 & 0 & \mu_n \\ \delta_n & -\mu_n & 0 \end{pmatrix} \begin{pmatrix} \tilde{D}_n \\ T \\ Y_n \end{pmatrix}, \tag{4}$$

where $\tilde{D}_n = \dfrac{D_n}{\|D_n\|}$, T is unit tangent vector of α and $Y_n = \tilde{D}_n \times T$. $\mu_n = \sqrt{{k_n}^2 + {k_g}^2}$ and $\delta_n = \left(\dfrac{k_n}{k_g}\right)' \left(\dfrac{{k_g}^2}{{k_n}^2 + {k_g}^2}\right) + \tau_g$ are curvatures of α according to the ND-frame [19].

If we denote the rectifying Darboux frame (RD-frame) along the curve α on M by $\left\{\tilde{D}_r, V, Y_r\right\}$, derivative formulas of the RD-frame are given by;

$$\begin{pmatrix} \tilde{D}_r' \\ V' \\ Y_r' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\delta_r \\ 0 & 0 & \mu_r \\ \delta_r & -\mu_r & 0 \end{pmatrix} \begin{pmatrix} \tilde{D}_r \\ V \\ Y_r \end{pmatrix},$$

where $\tilde{D}_r = \dfrac{D_r}{\|D_r\|}$, $V = U \times T$ and $Y_r = \tilde{D}_r \times V$. $\mu_r = \sqrt{k_g^2 + \tau_g^2}$ and $\delta_r = \left(\dfrac{\tau_g}{k_g}\right)' \left(\dfrac{k_g^2}{k_g^2 + \tau_g^2}\right) - k_n$ are curvatures of α according to the RD-frame [19].

Theorem 3.
a) Let α be a unit speed curve with OD-frame on an oriented surface M. Then α is a D_o-Darboux slant helix if and only if $\dfrac{\mu_o}{\delta_o}$ is a constant function.

b) Let α be a unit speed curve with an ND-frame on an oriented surface M. Then α is a D_n-Darboux slant helix if and only if $\frac{\mu_n}{\delta_n}$ is a constant function.

c) Let α be a unit speed curve with RD-frame on an oriented surface M. Then α is a D_r-Darboux slant helix if and only if $\frac{\mu_r}{\delta_r}$ is a constant function [18].

2. DIFFERENTIAL EQUATION CHARACTERIZATIONS FOR D_i -DARBOUX SLANT HELICES

2.1. DIFFERENTIAL EQUATION CHARACTERIZATIONS FOR D_o-DARBOUX SLANT HELICES

In this section, we give differential equation characterizations of curves and D_o-Darboux slant helices on an oriented surface M according to the OD-frame.

Theorem 4. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_o, U, Y_o\right\}$ be the osculating Darboux frame along α. The vector field $\tilde{D}_o$ satisfies the following differential equation

$$\tilde{D}_o''' + \left[\mu_o\delta_o\left(\frac{1}{\mu_o\delta_o}\right)' + \delta_o\left(\frac{1}{\delta_o}\right)'\right]\tilde{D}_o'' + \left\{\mu_o\delta_o\left[\frac{1}{\mu_o}\left(\frac{1}{\delta_o}\right)'\right]' + \mu_o^2 + \delta_o^2\right\}\tilde{D}_o' + \mu_o\delta_o\left(\frac{\delta_o}{\mu_o}\right)'\tilde{D}_o = 0, \tag{5}$$

where $\mu_o \neq 0$ and $\delta_o \neq 0$.

Proof: From the third equation of (3), we have

$$U = \frac{\delta_o}{\mu_0}\tilde{D}_o - \frac{1}{\mu_0}Y_o'. \tag{6}$$

From the first equation in (3), we get

$$\vec{Y}_o = -\frac{1}{\delta_o}\tilde{D}_o'. \tag{7}$$

Writing (7) in the second equation of (3), we obtain

$$U' = -\frac{\mu_o}{\delta_0}\tilde{D}_o'. \tag{8}$$

Differentiating (7), we have

$$Y_o' = -\left(\frac{1}{\delta_o}\right)' \tilde{D}_o' - \frac{1}{\delta_o}\tilde{D}_o'', \tag{9}$$

and writing this equation in (6), we obtain $U = \frac{1}{\mu_0\delta_o}\tilde{D}_o'' + \frac{1}{\mu_0}\left(\frac{1}{\delta_o}\right)'\tilde{D}_o' + \frac{\delta_o}{\mu_0}\tilde{D}_o$. Differentiating last equality and considering (8), we get (5) differential equation.

In Theorem 5, we give the differential equation of a curve α with respect to a vector field U.

Theorem 5. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_o, U, Y_o\right\}$ be the osculating Darboux frame along α. The vector field U satisfies the following differential equation

$$U''' + \left[\mu_o\delta_o\left(\frac{1}{\mu_o\delta_o}\right)' + \mu_o\left(\frac{1}{\mu_o}\right)'\right]U'' + \left\{\mu_o\delta_o\left[\frac{1}{\delta_o}\left(\frac{1}{\mu_o}\right)'\right]' + \mu_o^2 + \delta_o^2\right\}U' + \mu_o\delta_o\left(\frac{\mu_o}{\delta_o}\right)' U = 0, \tag{10}$$

where $\mu_o \neq 0$ and $\delta_o \neq 0$.

Proof: From the second equation of (3), we can write $Y_o = \frac{1}{\mu_o}\vec{U}'$. Differentiating $Y_o = \frac{1}{\mu_o}\vec{U}'$, we get

$$Y_o' = \left(\frac{1}{\mu_o}\right)'\vec{U}' + \frac{1}{\mu_o}\vec{U}''. \tag{11}$$

Writing (11) in the third equation of (3), we obtain

$$\tilde{D}_o = \frac{1}{\delta_o}\left(\frac{1}{\mu_o}\right)' U' + \frac{1}{\mu_o\delta_o}U'' + \frac{\mu_o}{\delta_o}U. \tag{12}$$

Differentiating (12), we get

$$\tilde{D}_o' = \frac{1}{\mu_o\delta_o}U''' + \left[\left(\frac{1}{\mu_o\delta_o}\right)' + \frac{1}{\delta_o}\left(\frac{1}{\mu_o}\right)'\right]U'' + \left\{\left[\frac{1}{\delta_o}\left(\frac{1}{\mu_o}\right)'\right]' + \frac{\mu_o}{\delta_o}\right\}U' + \left(\frac{\mu_o}{\delta_o}\right)' U. \tag{13}$$

Moreover, if Y_o is left alone in the second equation of (3) and written in the first equation of (3), we have $\tilde{D}_o' = -\frac{\delta_o}{\mu_o}U'$. Writing last equation in (13), we obtain the differantial equation (10).

Theorem 6. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_o, U, Y_o\right\}$ be the OD-frame along α. The vector field Y_o satisfies the following differential equation

$$Y_o''' + \frac{1}{\lambda_1}\left(\lambda_1' - \lambda_2\right)Y_o'' + \frac{1}{\lambda_1}\left(-\lambda_2' + \lambda_3\right)Y_o' + \frac{1}{\lambda_1}\left(-\delta_o + \lambda_3'\right)Y_o = 0, \tag{14}$$

where $\lambda_1 = \frac{\mu_0}{\mu_o'\delta_o - \mu_o\delta_o'}$, $\lambda_2 = \frac{1}{\delta_o}\left(1 + \delta_o'\lambda_1\right)$ and $\lambda_3 = \lambda_1\left(\mu_o^2 + \delta_o^2\right)$.

Proof: From the third equation of (3), we can write

$$\tilde{D}_o = \frac{1}{\delta_o}Y_o' + \frac{\mu_o}{\delta_o}U \tag{15}$$

Differentiating (15) and using OD-frame derivative formulas, we get

$$\frac{1}{\delta_o}Y_o'' + \left(\frac{1}{\delta_o}\right)'Y_o' + \frac{\mu_o^2 + \delta_o^2}{\delta_o}Y_o + \left(\frac{\mu_o}{\delta_o}\right)'U = 0. \tag{16}$$

From the last equation, it follows

$$U = -\frac{\delta_o}{\mu_o'\delta_o - \mu_o\delta_o'}Y_o'' + \frac{\delta_o'}{\mu_o'\delta_o - \mu_o\delta_o'}Y_o' - \frac{\delta_o\left(\mu_o^2 + \delta_o^2\right)}{\mu_o'\delta_o - \mu_o\delta'}Y_o. \tag{17}$$

Writing (17) in (15), we have

$$\tilde{D}_o = -\lambda_1 Y_o'' + \lambda_2 Y_o' - \lambda_3 Y_o \tag{18}$$

where $\lambda_1 = \frac{\mu_0}{\mu_o'\delta_o - \mu_o\delta_o'}$, $\lambda_2 = \frac{1}{\delta_o}\left(1 + \delta_o'\lambda_1\right)$ and $\lambda_3 = \lambda_1\left(\mu_o^2 + \delta_o^2\right)$. Differentiating (18) and using OD-frame derivative formulas, we obtain the differential equation (14).

Theorem 7. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_o, U, Y_o\right\}$ be the OD-frame along α. Then α is a D_o-Darboux slant helix if and only if the vector field Y_o satisfies the following differential equation

$$Y_o'' - \frac{\delta_o'}{\delta_o}Y_o' + \left(\mu_o^2 + \delta_o^2\right)Y_o = 0, \tag{19}$$

where $\delta_o \neq 0$.

Proof: Considering α is a D_o-Darboux slant helix in (16), we obtain the differential equation (19).

2.2. DIFFERENTIAL EQUATION CHARACTERIZATIONS FOR D_n-DARBOUX SLANT HELICES

In this section, we give differential equation characterizations of curves and D_n-Darboux slant helices on an oriented surface M according to the ND-frame. Since the theorems in this section will be proven in a similar way to the theorems in section 2.1, they will be presented without proof.

Theorem 8. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_n, T, Y_n\right\}$ be the ND-frame along α . The vector field $\tilde{D}_n$ satisfies the following differential equation

$$\tilde{D}_n''' + \left[\mu_n\delta_n\left(\frac{1}{\mu_n\delta_n}\right)' + \delta_n\left(\frac{1}{\delta_n}\right)'\right]\tilde{D}_n'' + \left\{\mu_n\delta_n\left[\frac{1}{\mu_n}\left(\frac{1}{\delta_n}\right)'\right]' + \mu_n^2 + \delta_n^2\right\}\tilde{D}_n' + \mu_n\delta_n\left(\frac{\delta_n}{\mu_n}\right)'\tilde{D}_n = 0 ,$$

where $\mu_n \neq 0$ and $\delta_n \neq 0$.

Theorem 9. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_n, T, Y_n\right\}$ be the ND-frame along α . The vector field T satisfies the following differential equation

$$T''' + \left[\mu_n\delta_n\left(\frac{1}{\mu_n\delta_n}\right)' + \mu_n\left(\frac{1}{\mu_n}\right)'\right]T'' + \left\{\mu_n\delta_n\left[\frac{1}{\delta_n}\left(\frac{1}{\mu_n}\right)'\right]' + \mu_n^2 + \delta_n^2\right\}T' + \mu_n\delta_n\left(\frac{\mu_n}{\delta_n}\right)'T = 0,,$$

where $\mu_n \neq 0$ and $\delta_n \neq 0$.

Theorem 10. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_n, T, Y_n\right\}$ be the ND-frame along α . The vector field Y_n satisfies the following differential equation

$$Y_n''' + \frac{1}{a_1}\left(a_1' - a_2\right)Y_n'' + \frac{1}{a_1}\left(-a_2' + a_3\right)Y_n' + \frac{1}{a_1}\left(-\delta_n + a_3'\right)Y_n = 0$$

where $a_1 = \dfrac{\mu_n}{\mu_n'\delta_n - \mu_n\delta_n'}$, $a_2 = \dfrac{1}{\delta_n}\left(1 + \delta_n' a_1\right)$ and $a_3 = a_1\left(\mu_n^2 + \delta_n^2\right)$.

Theorem 11. Let α be a unit speed curve on an oriented surface M and $\{\tilde{D}_n, T, Y_n\}$ be the ND- frame along α. Then α is a D_n-Darboux slant helix if and only if the vector field Y_n satisfies the following differential equation

$$Y_n'' - \frac{\delta_n'}{\delta_n} Y_n' + \left(\mu_n^2 + \delta_n^2\right) Y_n = 0$$

where $\delta_n \neq 0$.

2.3. DIFFERENTIAL EQUATION CHARACTERIZATIONS FOR D_r-DARBOUX SLANT HELICES

In this section, we give differential equation characterizations of curves and D_r-Darboux slant helices on an oriented surface M according to the RD-frame. Since the theorems in this section will be proven in a similar way to the theorems in section 2.1, they will be presented without proof.

Theorem 12. Let α be a unit speed curve on an oriented surface M and $\{\tilde{D}_r, V, Y_r\}$ be the RD-frame along α. The vector field $\tilde{D}_r$ satisfies the following differential equation

$$\tilde{D}_r''' + \left[\mu_r\delta_r\left(\frac{1}{\mu_r\delta_r}\right)' + \delta_r\left(\frac{1}{\delta_r}\right)'\right]\tilde{D}_r'' + \left\{\mu_r\delta_r\left[\frac{1}{\mu_r}\left(\frac{1}{\delta_r}\right)'\right]' + \mu_r^2 + \delta_r^2\right\}\tilde{D}_r' + \mu_r\delta_r\left(\frac{\delta_r}{\mu_r}\right)'\tilde{D}_r = 0,$$

where $\mu_r \neq 0$ and $\delta_r \neq 0$.

Theorem 13. Let α be a unit speed curve on an oriented surface M and $\{\tilde{D}_r, V, Y_r\}$ be the RD-frame along α. The vector field V satisfies the following differential equation

$$V''' + \left[\mu_r\delta_r\left(\frac{1}{\mu_r\delta_r}\right)' + \mu_r\left(\frac{1}{\mu_r}\right)'\right]V'' + \left\{\mu_r\delta_r\left[\frac{1}{\delta_r}\left(\frac{1}{\mu_r}\right)'\right]' + \mu_r^2 + \delta_r^2\right\}V' + \mu_r\delta_r\left(\frac{\mu_r}{\delta_r}\right)'V = 0,$$

where $\mu_r \neq 0$ and $\delta_r \neq 0$.

Theorem 14. Let α be a unit speed curve on an oriented surface M and $\{\tilde{D}_r, V, Y_r\}$ be the RD-frame along α. The vector field Y_r satisfies the following differential equation

$$Y_r''' + \frac{1}{b_1}\left(b_1' - b_2\right)Y_r'' + \frac{1}{b_1}\left(-b_2' + b_3\right)Y_r' + \frac{1}{b_1}\left(-\delta_r + b_3'\right)Y_r = 0,$$

where $b_1 = \dfrac{\mu_r}{\mu_r'\delta_r - \mu_r\delta_r'}$, $b_2 = \dfrac{1}{\delta_r}\left(1 + \delta_r' b_1\right)$ and $b_3 = b_1\left(\mu_r^2 + \delta_r^2\right)$.

Theorem 15. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_r, V, Y_r\right\}$ be the RD-frame along α. Then α is a D_r-Darboux slant helix if and only if the vector field V satisfies the following differential equation

$$Y_r'' - \frac{\delta_r'}{\delta_r}Y_r' + \left(\mu_r^2 + \delta_r^2\right)Y_r = 0,$$

where $\delta_r \neq 0$.

3. RESULTS

Corollary 16. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_o, U, Y_o\right\}$ be the osculating Darboux frame along α. Then α is a D_o-Darboux slant helix if and only if the vector field $\tilde{D}_o$ satisfies the following differential equation

$$\tilde{D}_o''' + \left[\mu_o\delta_o\left(\frac{1}{\mu_o\delta_o}\right)' + \delta_o\left(\frac{1}{\delta_o}\right)'\right]\tilde{D}_o'' + \left\{\mu_o\delta_o\left[\frac{1}{\mu_o}\left(\frac{1}{\delta_o}\right)'\right]' + \mu_o^2 + \delta_o^2\right\}\tilde{D}_o' = 0,$$

where $\mu_o \neq 0$ and $\delta_o \neq 0$.

Proof: The proof is clear from Theorem 3 and Theoem 4.

Corollary 17. Let α be a unit speed curve on an oriented surface M and $\left\{\tilde{D}_o, U, Y_o\right\}$ be the OD-frame along α. Then α is a D_o-Darboux slant helix if and only if the vector field U satisfies the following differential equation

$$U''' + \left[\mu_o\delta_o\left(\frac{1}{\mu_o\delta_o}\right)' + \mu_o\left(\frac{1}{\mu_o}\right)'\right]U'' + \left\{\mu_o\delta_o\left[\frac{1}{\delta_o}\left(\frac{1}{\mu_o}\right)'\right]' + \mu_o^2 + \delta_o^2\right\}U' = 0$$

where $\mu_o \neq 0$ and $\delta_o \neq 0$.

Proof: The proof is clear from Theorem 3 and Theorem 5.

Corollary 18. Let α be a unit speed curve on an oriented surface M and $\{\tilde{D}_n, T, Y_n\}$ be the ND- frame along α. Then α is a D_n-Darboux slant helix if and only if the vector field $\tilde{D}_n$ satisfies the following differential equation

$$\tilde{D}_n''' + \left[\mu_n\delta_n\left(\frac{1}{\mu_n\delta_n}\right)' + \delta_n\left(\frac{1}{\delta_n}\right)'\right]\tilde{D}_n'' + \left[\mu_n^{\ 2} + \delta_n^{\ 2} + \mu_n\delta_n\left[\frac{1}{\mu_n}\left(\frac{1}{\delta_n}\right)'\right]'\right]\tilde{D}_n' = 0,$$

where $\mu_n \neq 0$ and $\delta_n \neq 0$.

Proof: The proof is clear from Theorem 3 and Theorem 8.

Corollary 19. Let α be a unit speed curve on an oriented surface M and $\{\tilde{D}_n, T, Y_n\}$ be the ND- frame along α. Then α is a D_n-Darboux slant helix if and only if the vector field $\tilde{D}_n$ satisfies the following differential equation

$$T''' + \left[\mu_n\delta_n\left(\frac{1}{\mu_n\delta_n}\right)' + \mu_n\left(\frac{1}{\mu_n}\right)'\right]T'' + \left\{\mu_n\delta_n\left[\frac{1}{\delta_n}\left(\frac{1}{\mu_n}\right)'\right]' + \mu_n^2 + \delta_n^2\right\}T' = 0,$$

where $\mu_n \neq 0$ and $\delta_n \neq 0$.

Proof: The proof is clear from Theorem 3 and Theorem 9.

Corollary 20. Let α be a unit speed curve on an oriented surface M and $\{\tilde{D}_r, V, Y_r\}$ be the RD-frame along α. Then α is a D_r-Darboux slant helix if and only if the vector field $\tilde{D}_r$ satisfies the following differential equation

$$\tilde{D}_r''' + \left[\mu_r\delta_r\left(\frac{1}{\mu_r\delta_r}\right)' + \delta_r\left(\frac{1}{\delta_r}\right)'\right]\tilde{D}_r'' + \left[\mu_r^{\ 2} + \delta_r^{\ 2} + \mu_r\delta_r\left[\frac{1}{\mu_r}\left(\frac{1}{\delta_r}\right)'\right]'\right]\tilde{D}_r' = 0,$$

where $\mu_r \neq 0$ and $\delta_r \neq 0$.

Proof: The proof is clear from Theorem 3 and Theorem 12.

Corollary 21. Let α be a unit speed curve on an oriented surface M and $\{\tilde{D}_r, V, Y_r\}$ be the RD-frame along α. Then α is a D_r-Darboux slant helix if and only if the vector field V satisfies the following differential equation

$$V''' + \left[\mu_r \delta_r \left(\frac{1}{\mu_r \delta_r} \right)' + \mu_r \left(\frac{1}{\mu_r} \right)' \right] V'' + \left\{ \mu_r \delta_r \left[\frac{1}{\delta_r} \left(\frac{1}{\mu_r} \right)' \right]' + \mu_r^2 + \delta_r^2 \right\} V' = 0,$$

where $\mu_r \neq 0$ and $\delta_r \neq 0$.

Proof: The proof is clear from Theorem 3 and Theorem 13.

4. CONCLUSIONS

In this study, the differential equation characterizations for D_i -Darboux slant helices are given with the help of the OD, ND, and RD frames. The relevant theorems are presented along with their proofs.

This study has a unique value in terms of both the frames studied and the type of curves considered, and in this respect, the study contributes to the literature. It can also form the basis for many new studies.

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