ORIGINAL PAPER

A NOTE ON EDOUARD SEQUENCE

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Abstract. *In this study, we examine the k-Edouard, k-Edouard-Lucas, and Modified k-Edouard sequences, and some terms of these sequences, are given. Then, we give the generating functions, summation formulas, etc. Also, we obtain the Binet formulas in three different ways. The first is in the known classical way, the second is with the help of the sequence's generating functions, and the third is with the help of the matrices. In addition, we examine the relations among the terms of the k-Edouard, k-Edouard-Lucas, Modified k-Edouard, Edouard, Edouard-Lucas, Modified Edouard, Balancing, Balancing-Lucas, and Modified Balancing sequences. Finally, we associate the terms of these sequences with matrices.*

Keywords: *Edouard sequences; Lucas number; Binet formula; generating function; balancing numbers.*

1. INTRODUCTION

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci sequences have been applied in various fields such as Engineering [1], Chemistry [2], Biomathematics [3], Economy [4], Liberal Arts [5], etc. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the k-Fibonacci [6], Balancing [7], Narayana [8], Copper Fibonacci [9], k-Jacobsthal-Lucas [10], Padovan [11], k-Oresme [12], Fibonacci link [13], Jacobsthal hybrid [14], Lacunary [15], Bronze Fibonacci [16], Copper Lucas [17], k-Mersenne [18], k-Pell [19], Olivier [20], Bronze Leonardo [21], Dickson k-Fibonacci [22], k-Chebyshev [23] and so on.

For $n \in \mathbb{N}$, Fibonacci numbers F_n, Lucas numbers L_n, Balancing numbers B_n, Balancing-Lucas numbers C_n, and Modified Balancing numbers H_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n, \text{ with } F_0 = 0 \text{ and } F_1 = 1,$$
$$L_{n+2} = L_{n+1} + L_n, \text{ with } L_0 = 2 \text{ and } L_1 = 1,$$
$$B_{n+2} = 6B_{n+1} - B_n, \text{ with } B_0 = 0 \text{ and } B_1 = 1,$$
$$C_{n+2} = 6C_{n+1} - C_n, \text{ with } C_0 = 1 \text{ and } C_1 = 3,$$
$$H_{n+2} = 6H_{n+1} - H_n, \text{ with } H_0 = 2 \text{ and } H_1 = 6.$$

For F_n, L_n, B_n, C_n, and H_n the Binet formulas are given by the following relations, respectively,

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$$F_n = \frac{\varphi^n - \omega^n}{\varphi - \omega},\ L_n = \varphi^n + \omega^n,\ B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ C_n = \frac{\alpha^n + \beta^n}{2},\ \text{and } H_n = \alpha^n + \beta^n$$

where $\varphi = \frac{1+\sqrt{5}}{2}$, $\omega = \frac{1-\sqrt{5}}{2}$, $\alpha = 3 + 2\sqrt{2}$, and $\beta = 3 - 2\sqrt{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$ and $r^2 - 6r + 1 = 0$, respectively. Here the number φ is the golden ratio and the number α is the square of the silver ratio.

For $n \in \mathbb{N}$, the Edouard numbers E_n, Edouard-Lucas numbers K_n, and Modified Edouard numbers G_n are defined by the recurrence relations, respectively,

$$E_{n+2} = 6E_{n+1} - E_n + 1, \text{ with } E_0 = 0 \text{ and } E_1 = 1,$$
$$K_{n+2} = 6K_{n+1} - K_n - 4, \text{ with } K_0 = 3 \text{ and } K_1 = 7,$$
$$G_{n+2} = 6G_{n+1} - G_n + 1, \text{ with } G_0 = 2 \text{ and } G_1 = 6.$$

For E_n, K_n, and G_n the Binet formulas are given by the following relations, respectively,

$$E_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4},\ K_n = \alpha^n + \beta^n + 1,$$

and

$$G_n = \frac{(41-4\beta)}{(\alpha-\beta)(\alpha-1)}\alpha^{n+1} + \frac{(41-4\alpha)}{(\beta-\alpha)(\beta-1)}\beta^{n+1} + \frac{1}{(\alpha-1)(\beta-1)}.$$

Hence, $\alpha = 3 + 2\sqrt{2}$ and $\beta = 3 - 2\sqrt{2}$ are the roots of the characteristic equation $x^2 - 6x + 1 = 0$. In [24-28], Soykan generalized these sequences and found many features about these sequences.

With the help of the recurrence relation of the Fibonacci sequence, k-Fibonacci and k-Lucas sequences have been revealed, and these sequences have an important place in number theory. In [29], Falcon and Plaza introduced the k-Fibonacci sequence and obtained many properties related to this sequence. In addition, Falcon defined the k-Lucas sequences [30]. Moreover, Falcon applied the Hankel transform to the k-Fibonacci sequence and obtained the terms of the Fibonacci sequence differently [31].

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far. In this study, we give new generalizations inspired by the k-Fibonacci and Edouard sequences. We call these sequences the k-Edouard, k-Edouard-Lucas, and Modified k-Edouard sequences and denote them as $\mathcal{E}_{k,n}$, $\mathcal{S}_{k,n}$ and $\mathcal{G}_{k,n}$, respectively.

In section 2, we define the k-Edouard, k-Edouard-Lucas, and Modified k-Edouard sequences. Then, we give the characteristic equation, Binet formulas, the summation formulas, generating functions, and some properties for these sequences. In addition, we obtain Binet formulas with the help of generating functions of these sequences.

In section 3, we examine the relations among the terms of the k-Edouard, k-Edouard-Lucas, Modified k-Edouard, Edouard, Edouard-Lucas, and Modified Edouard sequences. Then, we associate the k-Edouard, k-Edouard-Lucas, and Modified k-Edouard sequences with B_n, C_n, and H_n sequences. Finally, we associate the terms of these sequences with matrices.

2. k-EDOUARD, k-EDOUARD-LUCAS, AND MODIFIED k-EDOUARD SEQUENCES

In this section, a new generalization of Edouard sequences is studied. Then, we obtain many properties of this generalization, such as special summation formulas and generating functions.

Definition 2.1. For $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$, the k-Edouard $\mathcal{E}_{k,n}$, k-Edouard-Lucas $\mathcal{S}_{k,n}$, and modified k-Edouard $\mathcal{G}_{k,n}$ sequences are defined by, respectively,

$$\mathcal{E}_{k,n+2} = 6\mathcal{E}_{k,n+1} - \mathcal{E}_{k,n} + k, \text{ with } \mathcal{E}_{k,0} = 0 \text{ and } \mathcal{E}_{k,1} = 1,$$
$$\mathcal{S}_{k,n+2} = 6\mathcal{S}_{k,n+1} - \mathcal{S}_{k,n} - 4k, \text{ with } \mathcal{S}_{k,0} = 3 \text{ and } \mathcal{S}_{k,1} = 7,$$
$$\mathcal{G}_{k,n+2} = 6\mathcal{G}_{k,n+1} - \mathcal{G}_{k,n} + k, \text{ with } \mathcal{G}_{k,0} = 2 \text{ and } \mathcal{G}_{k,1} = 6.$$

Also, the third-order recurrence relations of the sequences $\mathcal{E}_{k,n}$, $\mathcal{S}_{k,n}$, and $\mathcal{G}_{k,n}$ are as follows,

$$\mathcal{E}_{k,n+3} = 7\mathcal{E}_{k,n+2} - 7\mathcal{E}_{k,n+1} + \mathcal{E}_{k,n}, \text{ with } \mathcal{E}_{k,0} = 0, \mathcal{E}_{k,1} = 1, \text{ and } \mathcal{E}_{k,2} = k + 6,$$
$$\mathcal{S}_{k,n+3} = 7\mathcal{S}_{k,n+2} - 7\mathcal{S}_{k,n+1} + \mathcal{S}_{k,n}, \text{ with } \mathcal{S}_{k,0} = 3, \mathcal{S}_{k,1} = 2, \text{ and } \mathcal{S}_{k,2} = 39 - 4k,$$
$$\mathcal{G}_{k,n+3} = 7\mathcal{G}_{k,n+2} - 7\mathcal{G}_{k,n+1} + \mathcal{G}_{k,n}, \text{ with } \mathcal{G}_{k,0} = 2, \mathcal{G}_{k,1} = 6, \text{ and } \mathcal{G}_{k,2} = k + 34,$$

respectively.

Then, let's give some information about the equations of these sequences. The characteristic equation of the k-Edouard, k-Edouard-Lucas, and Modified k-Edouard sequences is

$$t^3 - 7t^2 + 7t - 1 = (t^2 - 6t + 1)(t - 1) = 0. \quad (1)$$

The roots of this equation are

$$\alpha = 3 + 2\sqrt{2}, \beta = 3 - 2\sqrt{2}, \text{ and } \delta = 1. \quad (2)$$

The relationship among these roots is given below

$$\alpha + \beta = 6, \alpha\beta = 1, \alpha + \beta + \delta = 7, \alpha\beta + \alpha\delta + \beta\delta = 7, \text{ and } \alpha\beta\delta = 1.$$

The $\mathcal{E}_{k,n}$, $\mathcal{S}_{k,n}$, and $\mathcal{G}_{k,n}$ values for the first eight n natural numbers are given below

- $\mathcal{E}_{k,0} = 0$ $\mathcal{S}_{k,0} = 3$ $\mathcal{G}_{k,0} = 2$,
- $\mathcal{E}_{k,1} = 1$ $\mathcal{S}_{k,1} = 7$ $\mathcal{G}_{k,1} = 6$,
- $\mathcal{E}_{k,2} = k + 6$ $\mathcal{S}_{k,2} = 39 - 4k$ $\mathcal{G}_{k,2} = k + 34$,
- $\mathcal{E}_{k,3} = 7k + 35$ $\mathcal{S}_{k,3} = 227 - 28k$ $\mathcal{G}_{k,3} = 7k + 198$,
- $\mathcal{E}_{k,4} = 42k + 204$ $\mathcal{S}_{k,4} = 1323 - 168k$ $\mathcal{G}_{k,4} = 42k + 1154$,
- $\mathcal{E}_{k,5} = 246k + 1189$ $\mathcal{S}_{k,5} = 7711 - 984k$ $\mathcal{G}_{k,5} = 246k + 6726$,

In the following theorem, the Binet formulas of the k-Edouard $\mathcal{E}_{k,n}$, k-Edouard-Lucas $\mathcal{S}_{k,n}$, and modified k-Edouard $\mathcal{G}_{k,n}$ sequences are expressed.

Theorem 2.1. Let $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. We obtain
i. $\mathcal{E}_{k,n} = \frac{5+k-\beta}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{5+k-\alpha}{(\beta-\alpha)(\beta-1)}\beta^n + \frac{k}{(\alpha-1)(\beta-1)}$,

ii. $\mathcal{S}_{k,n}=\frac{32-4k-4\beta}{(\alpha-\beta)(\alpha-1)}\alpha^n+\frac{32-4k-4\alpha}{(\beta-\alpha)(\beta-1)}\beta^n-\frac{4k}{(\alpha-\delta)(\beta-1)}$,
iii. $\mathcal{G}_{k,n}=\frac{28+k-4\beta}{(\alpha-\beta)(\alpha-1)}\alpha^n+\frac{28+k-4\alpha}{(\beta-\alpha)(\beta-1)}\beta^n+\frac{k}{(\alpha-1)(\beta-1)}$.

Proof: i. The Binet form of a sequence is as follows

$$\mathcal{E}_{k,n}=x\alpha^n+y\beta^n+z\delta^n. \tag{3}$$

For these n values, we obtain

$$\mathcal{E}_{k,0}=x+y+z,$$
$$\mathcal{E}_{k,1}=x\alpha+y\beta+z\delta,$$
$$\mathcal{E}_{k,2}=x\alpha^2+y\beta^2+z\delta^2.$$

We find

$$x=\frac{5+k-\beta}{(\alpha-\beta)(\alpha-1)},\ y=\frac{5+k-\alpha}{(\beta-\alpha)(\beta-1)},\ \text{and } z=\frac{k}{(\alpha-1)(\beta-1)}.$$

Thus, we obtain

$$\mathcal{E}_{k,n}=\frac{5+k-\beta}{(\alpha-\beta)(\alpha-1)}\alpha^n+\frac{5+k-\alpha}{(\beta-\alpha)(\beta-1)}\beta^n+\frac{k}{(\alpha-1)(\beta-1)}.$$

The proofs of the others are shown similarly. □

In the following theorems, we give the generating functions of the k-Edouard $\mathcal{E}_{k,n}$, k-Edouard-Lucas $\mathcal{S}_{k,n}$, and modified k-Edouard $\mathcal{G}_{k,n}$ sequences. In addition, we obtain the Binet formulas of $\mathcal{E}_{k,n}$, $\mathcal{S}_{k,n}$, and $\mathcal{G}_{k,n}$ sequences with the help of generating functions.

Theorem 2.2. The generating functions for k-Edouard $\mathcal{E}_{k,n}$, k-Edouard-Lucas $\mathcal{S}_{k,n}$, and modified k-Edouard $\mathcal{G}_{k,n}$ sequences are given as follows, respectively,
i. $e(t)=\sum_{n=0}^{\infty}\mathcal{E}_{k,n}t^n=\frac{t+(k-1)t^2}{-t^3+7t^2-7t+1}$,
ii. $s(t)=\sum_{n=0}^{\infty}\mathcal{S}_{k,n}t^n=\frac{3-14t+(11-4k)t^2}{-t^3+7t^2-7t+1}$,
iii. $g(t)=\sum_{n=0}^{\infty}\mathcal{G}_{k,n}t^n=\frac{2-8t+(k+6)t^2}{-t^3+7t^2-7t+1}$.

Proof: i. For the k-Edouard sequence, we have

$$e(t)=\sum_{n=0}^{\infty}\mathcal{E}_{k,n}t^n=t+(k+6)t^2+\sum_{n=3}^{\infty}\mathcal{E}_{k,n}t^n$$
$$=t+(k+6)t^2+7\sum_{n=3}^{\infty}\mathcal{E}_{k,n-1}t^n-7\sum_{n=3}^{\infty}\mathcal{E}_{k,n-2}t^n+\sum_{n=3}^{\infty}\mathcal{E}_{k,n-3}t^n$$
$$=t+(k+6)t^2+7t\sum_{n=2}^{\infty}\mathcal{E}_{k,n}t^n-7t^2\sum_{n=1}^{\infty}\mathcal{E}_{k,n}t^n+t^3\sum_{n=0}^{\infty}\mathcal{E}_{k,n}t^n.$$

Thus, we obtain

$$e(t)(-t^3+7t^2-7t+1)=t+(k+6)t^2-7t^2.$$

So,

$$e(t)=\sum_{n=0}^{\infty}\mathcal{E}_{k,n}t^n=\frac{t+(k-1)t^2}{-t^3+7t^2-7t+1}.$$

The proofs of the others are shown similarly. □

Theorem 2.3. For $\mathcal{E}_{k,n}$, $\mathcal{S}_{k,n}$, and $\mathcal{G}_{k,n}$ sequences, the Binet formulas can be obtained with the help of the generating functions.

Proof: With the help of the roots of the characteristic equation of these sequences, the roots of the $-t^3+7t^2-7t+1$ equation become $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\delta}$. For $\mathcal{E}_{k,n}$, we have

$$\frac{t+(k-1)t^2}{-t^3+7t^2-7t+1} = \frac{5+k-\beta}{(\alpha-\beta)(\alpha-1)}\frac{1}{1-\alpha t}+\frac{5+k-\alpha}{(\beta-\alpha)(\beta-1)}\frac{1}{1-\beta t}+\frac{k}{(\alpha-1)(\beta-1)}\frac{1}{1-\delta t}$$
$$=\frac{5+k-\beta}{(\alpha-\beta)(\alpha-1)}\sum_{n=0}^{\infty}\alpha^n t^n+\frac{5+k-\alpha}{(\beta-\alpha)(\beta-1)}\sum_{n=0}^{\infty}\beta^n t^n+\frac{k}{(\alpha-1)(\beta-1)}\sum_{n=0}^{\infty}\delta^n t^n.$$
$$=\sum_{n=0}^{\infty}\mathcal{E}_{k,n}t^n.$$

Similarly, the Binet formulas of the sequences $\mathcal{S}_{k,n}$ and $\mathcal{G}_{k,n}$ are found. □

Next, we give special sum formulas of the k-Edouard $\mathcal{E}_{k,n}$, k-Edouard-Lucas $\mathcal{S}_{k,n}$, and modified k-Edouard $\mathcal{G}_{k,n}$ sequences.

Theorem 2.4. Let $k\in\mathbb{R}^+$ and $n\in\mathbb{N}$. We obtain
i. $\sum_{s=0}^{n}\mathcal{E}_{k,s}=\frac{5\mathcal{E}_{k,n}-\mathcal{E}_{k,n-1}-(n-1)k-1}{4}$,
ii. $\sum_{s=0}^{n}\mathcal{S}_{k,s}=\frac{5\mathcal{S}_{k,n}-\mathcal{S}_{k,n-1}+(n-1)4k+8}{4}$,
iii. $\sum_{s=0}^{n}\mathcal{G}_{k,s}=\frac{5\mathcal{G}_{k,n}-\mathcal{G}_{k,n-1}-(n-1)k+4}{4}$.

Proof: ii. From the definition of the k-Edouard-Lucas sequence, we obtain

$$\mathcal{S}_{k,2}=6\mathcal{S}_{k,1}-\mathcal{S}_{k,0}-4k,$$
$$\mathcal{S}_{k,3}=6\mathcal{S}_{k,2}-\mathcal{S}_{k,1}-4k,$$
$$\vdots$$
$$\mathcal{S}_{k,n}=6\mathcal{S}_{k,n-1}-\mathcal{S}_{k,n-2}-4k.$$

So, we have

$$-10+\sum_{s=0}^{n}\mathcal{S}_{k,s}=6\sum_{s=1}^{n-1}\mathcal{S}_{k,s}-\sum_{s=0}^{n-2}\mathcal{S}_{k,s}-(n-1)4k,$$
$$-10+\sum_{s=0}^{n}\mathcal{S}_{k,s}=6\left(-\mathcal{S}_{k,n}-\mathcal{S}_{k,0}+\sum_{s=0}^{n}\mathcal{S}_{k,s}\right)-\mathcal{S}_{k,n}-\mathcal{S}_{k,n-1}+\sum_{s=0}^{n}\mathcal{S}_{k,s}-(n-1)4k.$$

Thus, we obtain

$$\sum_{s=0}^{n}\mathcal{S}_{k,s}=\frac{5\mathcal{S}_{k,n}-\mathcal{S}_{k,n-1}+(n-1)4k+8}{4}.$$

The proofs of the others are shown similarly. □

Theorem 2.5. Let $k\in\mathbb{R}^+$ and $n\in\mathbb{N}$. We obtain
i. $\sum_{s=0}^{n}\mathcal{E}_{k,2s}=\frac{3\mathcal{E}_{k,2n+1}-\mathcal{E}_{k,2n}-4nk-3}{16}$,
ii. $\sum_{s=0}^{n}\mathcal{E}_{k,2s+1}=\frac{17\mathcal{E}_{k,2n+1}-3\mathcal{E}_{k,2n}-4nk-1}{9}$,
iii. $\sum_{s=0}^{n}\mathcal{S}_{k,2s}=\frac{3\mathcal{S}_{k,2n+1}-\mathcal{S}_{k,2n}+16nk+30}{16}$,

iv. $\sum_{s=0}^{n} \mathcal{S}_{k,2s+1} = \frac{17\mathcal{S}_{k,2n+1}-3\mathcal{S}_{k,2n}+16nk+2}{16}$,
v. $\sum_{s=0}^{n} \mathcal{G}_{k,2s} = \frac{3\mathcal{G}_{k,2n+1}-\mathcal{G}_{k,2n}-4nk+16}{9}$,
vi. $\sum_{s=0}^{n} \mathcal{G}_{k,2s+1} = \frac{17\mathcal{G}_{k,2n+1}-3\mathcal{G}_{k,2n}-4nk}{9}$.

Proof: The proofs are shown similarly to Theorem 2.4. □

3. RELATIONS AMONG SPECIAL SEQUENCES

In this chapter, we examine the relationships among the k-Edouard, k-Edouard-Lucas, Modified k-Edouard, Edouard, Edouard-Lucas, Modified Edouard, Balancing, Balancing-Lucas, and Modified Balancing sequences. Finally, we associate the terms of these sequences with matrices.

In the following theorems, we examine the relations among the k-Edouard $\mathcal{E}_{k,n}$, k-Edouard-Lucas $\mathcal{S}_{k,n}$, Modified k-Edouard $\mathcal{G}_{k,n}$, Edouard E_n, Edouard-Lucas K_n, and Modified Edouard G_n sequences.

Theorem 3.1. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:
i. $\mathcal{E}_{k,n} = \frac{-3k+13}{16k^2-160k+272}\mathcal{S}_{k,n+2} + \frac{5k-33}{8k^2-80k+136}\mathcal{S}_{k,n+1} + \frac{-4k^2+33k-15}{16k^2-160k+272}\mathcal{S}_{k,n}$,
ii. $\mathcal{E}_{k,n} = -\frac{2k+36}{k^2+32k+128}\mathcal{G}_{k,n+2} + \frac{9k+224}{k^2+32k+128}\mathcal{G}_{k,n+1} + \frac{k^2+25k-60}{k^2+32k+128}\mathcal{G}_{k,n}$,
iii. $\mathcal{E}_{k,n} = (k-1)E_{n+2} + (-7k+7)E_{n+1} + (7k-6)E_n$,
iv. $\mathcal{E}_{k,n} = \frac{7k-2}{64}K_{n+2} + \frac{-11k+4}{16}K_{n+1} + \frac{21k-4}{64}K_n$,
v. $\mathcal{E}_{k,n} = -\frac{34k+4}{161}G_{n+2} + \frac{200k+33}{161}G_{n+1} - \frac{5k+29}{161}G_n$.

Proof: i. The following relation is used for proofs

$$\mathcal{E}_{k,n} = a \times \mathcal{S}_{k,n+5} + b \times \mathcal{S}_{k,n+4} + c \times \mathcal{S}_{k,n+3}. \quad (4)$$

For these n values, we obtain

$$\mathcal{E}_{k,0} = a \times \mathcal{S}_{k,5} + b \times \mathcal{S}_{k,4} + c \times \mathcal{S}_{k,3},$$
$$\mathcal{E}_{k,1} = a \times \mathcal{S}_{k,6} + b \times \mathcal{S}_{k,5} + c \times \mathcal{S}_{k,4},$$
$$\mathcal{E}_{k,2} = a \times \mathcal{S}_{k,7} + b \times \mathcal{S}_{k,6} + c \times \mathcal{S}_{k,5}.$$

We find

$$a = \frac{-3k+13}{16k^2-160k+272},\ b = \frac{5k-33}{8k^2-80k+136},\ \text{and } c = \frac{-4k^2+33k-15}{16k^2-160k+272}.$$

Thus, we have

$$\mathcal{E}_{k,n} = \frac{-3k+13}{16k^2-160k+272}\mathcal{S}_{k,n+2} + \frac{5k-33}{8k^2-80k+136}\mathcal{S}_{k,n+1} + \frac{-4k^2+33k-15}{16k^2-160k+272}\mathcal{S}_{k,n}.$$

The proofs of the others are shown similarly. □

Theorem 3.2. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:
i. $\mathcal{S}_{k,n} = \frac{3k+4}{k^2+4k-4}\mathcal{E}_{k,n+2} - \frac{10k+36}{k^2+4k-4}\mathcal{E}_{k,n+1} + \frac{-4k^2-9k+48}{k^2+4k-4}\mathcal{E}_{k,n}$,
ii. $\mathcal{S}_{k,n} = \frac{11k+184}{k^2+32k+128}\mathcal{G}_{k,n+2} - \frac{46k+1120}{k^2+32k+128}\mathcal{G}_{k,n+1} + \frac{-4k^2-93k+424}{k^2+32k+128}\mathcal{G}_{k,n}$,

iii. $\mathcal{S}_{k,n} = (11-4k)E_{n+2} + (28k-74)E_{n+1} + (-28k+63)E_n$,
iv. $\mathcal{S}_{k,n} = \frac{-7k+7}{16}K_{n+2} + \frac{11k-11}{4}K_{n+1} + \frac{-21k+37}{16}K_n$,
v. $\mathcal{S}_{k,n} = \frac{136k+59}{161}G_{n+2} - \frac{800k+366}{161}G_{n+1} + \frac{20k+307}{161}G_n$.

Proof: iii. If Binet formulas are used for proofs, we get

$$(11-4k)E_{n+2} + (28k-74)E_{n+1} + (-28k+63)E_n$$
$$= (11-4k)\left(\frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}\right)$$
$$+ (28k-74)\left(\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}\right)$$
$$+(-28k+63)\left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4}\right)$$
$$= \frac{32-4k-4\beta}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{32-4k-4\alpha}{(\beta-\alpha)(\beta-1)}\beta^n - \frac{4k}{(\alpha-1)(\beta-1)} = \mathcal{S}_{k,n}.$$

The proofs of the others are shown similarly. □

Theorem 3.3. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:
i. $\mathcal{G}_{k,n} = \frac{2k+5}{k^2+4k-4}\mathcal{E}_{k,n+2} - \frac{9k+38}{k^2+4k-4}\mathcal{E}_{k,n+1} + \frac{k^2+11k+29}{k^2+4k-4}\mathcal{E}_{k,n}$,
ii. $\mathcal{G}_{k,n} = \frac{-11k+57}{16k^2-160k+272}\mathcal{S}_{k,n+2} + \frac{23k-163}{8k^2-80k+136}\mathcal{S}_{k,n+1} + \frac{-4k^2+5k+201}{16k^2-160k+272}\mathcal{S}_{k,n}$,
iii. $\mathcal{G}_{k,n} = (k+6)E_{n+2} - (7k+40)E_{n+1} + (7k+34)E_n$,
iv. $\mathcal{G}_{k,n} = \frac{7k+16}{64}K_{n+2} - \frac{11k+24}{16}K_{n+1} + \frac{21k+80}{64}K_n$.
v. $\mathcal{G}_{k,n} = \frac{-34k+34}{161}G_{n+2} + \frac{200k-200}{161}G_{n+1} + \frac{-5k+166}{161}G_n$.

Proof: **v.** If Binet formulas are used for proofs, we get

$$\frac{7k+16}{64}K_{n+2} - \frac{11k+24}{16}K_{n+1} + \frac{21k+80}{64}K_n = \frac{7k+16}{64}(\alpha^{n+2}+\beta^{n+2}+1)$$
$$-\frac{11k+24}{16}(\alpha^{n+1}+\beta^{n+1}+1) + \frac{21k+80}{64}(\alpha^n+\beta^n+1)$$
$$= \frac{28+k-4\beta}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{28+k-4\alpha}{(\beta-\alpha)(\beta-1)}\beta^n + \frac{k}{(\alpha-1)(\beta-1)} = \mathcal{G}_{k,n}.$$

The proofs of the others are shown similarly. □

Theorem 3.4. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:
i. $G_n = \frac{-8k^2+37k+17}{16k^3-160k^2+272k}\mathcal{S}_{k,n+2} + \frac{16k^2-105k-51}{8k^3-80k^2+136k}\mathcal{S}_{k,n+1} + \frac{-28k^2+213k+17}{16k^3-160k^2+272k}\mathcal{S}_{k,n}$,
$G_n = \frac{2k^2+4k+1}{k^3+4k^2-4k}\mathcal{E}_{k,n+2} - \frac{8k^2+33k+6}{k^3+4k^2-4k}\mathcal{E}_{k,n+1} + \frac{7k^2+33k+1}{k^3+4k^2-4k}\mathcal{E}_{k,n}$,
$G_n = \frac{2k^2+30k-32}{k^3+32k^2+128k}\mathcal{G}_{k,n+2} + \frac{-8k^2-184k+192}{k^3+32k^2+128k}\mathcal{G}_{k,n+1} + \frac{7k^2+186k-32}{k^3+32k^2+128k}\mathcal{G}_{k,n}$.
ii. $K_n = \frac{-3k^2+20k-17}{4k^3-40k^2+68k}\mathcal{S}_{k,n+2} + \frac{7k^2-58k+51}{2k^3-20k^2+34k}\mathcal{S}_{k,n+1} + \frac{-7k^2+56k-17}{2k^3-13k^2+20k}\mathcal{S}_{k,n}$,
$K_n = \frac{3k^2+8k-4}{k^3+4k^2-4k}\mathcal{E}_{k,n+2} + \frac{-14k^2-56k+24}{k^3+4k^2-4k}\mathcal{E}_{k,n+1} + \frac{7k^2+32k-4}{k^3+4k^2-4k}\mathcal{E}_{k,n}$,
$K_n = \frac{3k^2+64k+128}{k^3+32k^2+128k}\mathcal{G}_{k,n+2} - \frac{14k^2+384k+768}{k^3+32k^2+128k}\mathcal{G}_{k,n+1} + \frac{7k^2+192k+128}{k^3+32k^2+128k}\mathcal{G}_{k,n}$.
iii. $E_n = \frac{-k+1}{k^3+4k^2-4k}\mathcal{E}_{k,n+2} + \frac{k^2+5k-6}{k^3+4k^2-4k}\mathcal{E}_{k,n+1} + \frac{1}{k^3+4k^2-4k}\mathcal{E}_{k,n}$,

$$E_n = \frac{-7k+17}{16k^3-160k^2+272k}\mathcal{S}_{k,n+2} + \frac{-2k^2+25k-51}{16k^3-160k^2+272k}\mathcal{S}_{k,n+1} + \frac{-3k+17}{16k^3-160k^2+272k}\mathcal{S}_{k,n},$$
$$E_n = -\frac{6k+32}{k^3+32k^2+128k}\mathcal{G}_{k,n+2} + \frac{k^2+40k+192}{k^3+32k^2+128k}\mathcal{G}_{k,n+1} - \frac{2k+32}{k^3+32k^2+128k}\mathcal{G}_{k,n}.$$

Proof: i. The following relation is used for proofs

$$G_n = K \times \mathcal{S}_{k,n+2} + M \times \mathcal{S}_{k,n+1} + N \times \mathcal{S}_{k,n}. \quad (5)$$

For these n values, we obtain

$$G_0 = K \times \mathcal{S}_{k,2} + M \times \mathcal{S}_{k,1} + N \times \mathcal{S}_{k,0},$$
$$G_1 = K \times \mathcal{S}_{k,3} + M \times \mathcal{S}_{k,2} + N \times \mathcal{S}_{k,1},$$
$$G_2 = K \times \mathcal{S}_{k,4} + M \times \mathcal{S}_{k,3} + N \times \mathcal{S}_{k,2}.$$

We find

$$K = \frac{-8k^2+37k+17}{16k^3-160k^2+272k},\ M = \frac{16k^2-105k-51}{8k^3-80k^2+136k},\text{ and } N = \frac{-28k^2+213k+17}{16k^3-160k^2+272k}.$$

Thus, we have

$$G_n = \frac{-8k^2+37k+17}{16k^3-160k^2+272k}\mathcal{S}_{k,n+2} + \frac{16k^2-105k-51}{8k^3-80k^2+136k}\mathcal{S}_{k,n+1} + \frac{-28k^2+213k+17}{16k^3-160k^2+272k}\mathcal{S}_{k,n}.$$

The proofs of the others are shown similarly. □

In the following theorem, we examine the relationships among the k-Edouard, k-Edouard-Lucas, Modified k-Edouard sequences, and special number sequences.

Theorem 3.5. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The relationships of the k-Edouard $\mathcal{E}_{k,n}$, k-Edouard-Lucas $\mathcal{H}_{k,n}$, and modified k-Edouard $\mathcal{G}_{k,n}$, sequences with the Balancing sequence B_n, Balancing-Lucas sequence C_n, and Modified Balancing sequence H_n are as follows:
i. For the k-Edouard sequence

- $\mathcal{E}_{k,n} = \frac{k}{4}B_{n+1} + \frac{4-5k}{4}B_n - \frac{k}{4}$,
- $\mathcal{E}_{k,n} = \frac{-k+2}{16}C_{n+1} + \frac{7k-6}{16}C_n - \frac{k}{4}$,
- $\mathcal{E}_{k,n} = \frac{-k+2}{32}H_{n+1} + \frac{7k-6}{32}H_n - \frac{k}{4}$,
- $B_n = \frac{1}{k^2+4k-4}(k\mathcal{E}_{k,n+1} - (k+4)\mathcal{E}_{k,n} - k)$,
- $B_n = \frac{1}{k^2+4k-4}(-\mathcal{E}_{k,n+2} + (k+6)\mathcal{E}_{k,n+1} - (k+5)\mathcal{E}_{k,n})$
- $C_n = \frac{1}{k^2+4k-4}((2k-4)\mathcal{E}_{k,n+1} + (2k+12)\mathcal{E}_{k,n} + k^2 + 2k)$,
- $C_n = \frac{1}{k^2+4k-4}((k+2)\mathcal{E}_{k,n+2} - (4k+16)\mathcal{E}_{k,n+1} + (3k+14)\mathcal{E}_{k,n})$,
- $H_n = \frac{1}{k^2+4k-4}((4k-8)\mathcal{E}_{k,n+1} + (4k+24)\mathcal{E}_{k,n} + 2k^2 + 4k)$,
- $H_n = \frac{1}{k^2+4k-4}((2k+4)\mathcal{E}_{k,n+2} - (8k+32)\mathcal{E}_{k,n+1} + (6k+28)\mathcal{E}_{k,n})$.

ii. For the k-Edouard-Lucas sequence

- $\mathcal{S}_{k,n} = (3-k)B_{n+1} + (5k-11)B_n + k$,
- $\mathcal{S}_{k,n} = \frac{k-1}{4}C_{n+1} + \frac{15-7k}{4}C_n + k$,
- $\mathcal{S}_{k,n} = \frac{k-1}{8}H_{n+1} + \frac{15-7k}{8}H_n + k$,
- $B_n = \frac{1}{4k^2-40k+68}((3-k)\mathcal{S}_{k,n+1} + (k-7)\mathcal{S}_{k,n} + 4k)$,

- $B_n = \frac{1}{4k^2-40k+68}(-\mathcal{S}_{k,n+2} + (-k+9)\mathcal{S}_{k,n+1} + (k-8)\mathcal{S}_{k,n})$,
- $C_n = \frac{1}{2k^2-20k+34}((1-k)\mathcal{S}_{k,n+1} + (9-k)\mathcal{S}_{k,n} + 2k^2 - 10k)$,
- $C_n = \frac{1}{4k^2-40k+68}((5-k)\mathcal{S}_{k,n+2} + (4k-28)\mathcal{S}_{k,n+1} + (-3k+23)\mathcal{S}_{k,n})$,
- $H_n = \frac{1}{k^2-10k+17}((1-k)\mathcal{S}_{k,n+1} + (9-k)\mathcal{S}_{k,n} + 2k^2 - 10k)$,
- $H_n = \frac{1}{2k^2-20k+34}((5-k)\mathcal{S}_{k,n+2} + (4k-28)\mathcal{S}_{k,n+1} + (-3k+23)\mathcal{S}_{k,n})$.

iii. For the Modified k-Edouard sequence

- $\mathcal{G}_{k,n} = \frac{k+8}{4}B_{n+1} - \frac{5k+24}{4}B_n - \frac{k}{4}$,
- $\mathcal{G}_{k,n} = \frac{-k}{16}C_{n+1} + \frac{7k+32}{16}C_n - \frac{k}{4}$,
- $\mathcal{G}_{k,n} = \frac{-k}{32}H_{n+1} + \frac{7k+32}{32}H_n - \frac{k}{4}$,
- $B_n = \frac{1}{k^2+32k+128}((k+8)\mathcal{G}_{k,n+1} - (k+24)\mathcal{G}_{k,n} - 4k$,
- $B_n = \frac{1}{k^2+32k+128}(-4\mathcal{G}_{k,n+2} + (k+32)\mathcal{G}_{k,n+1} - (k+28)\mathcal{G}_{k,n}$,
- $C_n = \frac{1}{k^2+32k+128}(2k\mathcal{G}_{k,n+1} + (2k+64)\mathcal{G}_{k,n} + k^2 + 16k)$,
- $C_n = \frac{1}{k^2+32k+128}((k+16)\mathcal{G}_{k,n+2} - (4k+96)\mathcal{G}_{k,n+1} + (3k+80)\mathcal{G}_{k,n})$,
- $H_n = \frac{1}{k^2+32k+128}(4k\mathcal{G}_{k,n+1} + (4k+128)\mathcal{G}_{k,n} + 2k^2 + 32k)$,
- $H_n = \frac{1}{k^2+32k+128}((2k+32)\mathcal{G}_{k,n+2} - (8k+192)\mathcal{G}_{k,n+1} + (6k+160)\mathcal{G}_{k,n})$.

Proof: i. If Binet formulas are used for proofs, we get

$$\begin{aligned}\frac{k}{4}B_{n+1} + \frac{4-5k}{4}B_n - \frac{k}{4} &= \frac{k}{4}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} + \frac{4-5k}{4}\frac{\alpha^n-\beta^n}{\alpha-\beta} - \frac{k}{4}\\ &= \frac{5+k-\beta}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{5+k-\alpha}{(\beta-\alpha)(\beta-1)}\beta^n + \frac{k}{(\alpha-1)(\beta-1)} = \mathcal{E}_{k,n}.\end{aligned}$$

The proofs of the others are shown similarly. □

In the following theorems, we associate the terms of the k-Edouard $\mathcal{E}_{k,n}$, k-Edouard-Lucas $\mathcal{S}_{k,n}$, and modified k-Edouard $\mathcal{G}_{k,n}$ sequences with matrices. In addition, we obtain the Binet formulas of these sequences with the help of the matrices.

Theorem 3.6. Let $k \in \mathbb{R}^+$, and $n \in \mathbb{N}$. The following equations are true:

i. For the k-Edouard sequence,

1. $\begin{bmatrix}\mathcal{E}_{k,n+2}\\ \mathcal{E}_{k,n+1}\\ \mathcal{E}_{k,n}\end{bmatrix} = \begin{bmatrix}7 & -7 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}^n \begin{bmatrix}k+6\\ 1\\ 0\end{bmatrix}$,
2. $\begin{bmatrix}\mathcal{E}_{k,n}\\ \mathcal{E}_{k,n+1}\\ \mathcal{E}_{k,n+2}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 7 & -7 & 1\end{bmatrix}^n \begin{bmatrix}0\\ 1\\ k+6\end{bmatrix}$,
3. $\begin{bmatrix}\mathcal{E}_{k,n+2}\\ \mathcal{E}_{k,n+1}\\ \mathcal{E}_{k,n}\end{bmatrix} = \begin{bmatrix}7 & -7 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \begin{bmatrix}\mathcal{E}_{k,n+1}\\ \mathcal{E}_{k,n}\\ \mathcal{E}_{k,n-1}\end{bmatrix}$.

ii. For the k-Edouard-Lucas sequence,

1. $\begin{bmatrix} \mathcal{S}_{k,n+2} \\ \mathcal{S}_{k,n+1} \\ \mathcal{S}_{k,n} \end{bmatrix} = \begin{bmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 39-4k \\ 7 \\ 3 \end{bmatrix}$,

2. $\begin{bmatrix} \mathcal{S}_{k,n} \\ \mathcal{S}_{k,n+1} \\ \mathcal{S}_{k,n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & -7 & 1 \end{bmatrix}^n \begin{bmatrix} 3 \\ 7 \\ 39-4k \end{bmatrix}$,

3. $\begin{bmatrix} \mathcal{S}_{k,n+2} \\ \mathcal{S}_{k,n+1} \\ \mathcal{S}_{k,n} \end{bmatrix} = \begin{bmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S}_{k,n+1} \\ \mathcal{S}_{k,n} \\ \mathcal{S}_{k,n-1} \end{bmatrix}$.

iii. For the Modified k-Edouard sequence,

1. $\begin{bmatrix} \mathcal{G}_{k,n+2} \\ \mathcal{G}_{k,n+1} \\ \mathcal{G}_{k,n} \end{bmatrix} = \begin{bmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} k+34 \\ 6 \\ 2 \end{bmatrix}$,

2. $\begin{bmatrix} \mathcal{G}_{k,n} \\ \mathcal{G}_{k,n+1} \\ \mathcal{G}_{k,n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & -7 & 1 \end{bmatrix}^n \begin{bmatrix} 2 \\ 6 \\ k+34 \end{bmatrix}$,

3. $\begin{bmatrix} \mathcal{G}_{k,n+2} \\ \mathcal{G}_{k,n+1} \\ \mathcal{G}_{k,n} \end{bmatrix} = \begin{bmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{G}_{k,n+1} \\ \mathcal{G}_{k,n} \\ \mathcal{G}_{k,n-1} \end{bmatrix}$.

Proof: i. 1. Let show the proof by induction over n. For $n = 1$, the equality is true. For $n - 1$, assume the equality is true. We obtain

$$\begin{bmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} k+6 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} k+6 \\ 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_{k,n+1} \\ \mathcal{E}_{k,n} \\ \mathcal{E}_{k,n-1} \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{E}_{k,n+2} \\ \mathcal{E}_{k,n+1} \\ \mathcal{E}_{k,n} \end{bmatrix}.$$

From the last equation, for n, it can be seen that the equality is true.

The proofs of the others may be found similarly. □

In the following theorem, we calculate the Simson formulas of these sequences. The Simson formula is the most general form of the Cassini identity.

Theorem 3.7. (Simson Formulas) Let $k \in \mathbb{R}^+$ and $t \in \mathbb{N}$. The following equations are satisfied.

i. $det \begin{bmatrix} \mathcal{E}_{k,t+2} & \mathcal{E}_{k,t+1} & \mathcal{E}_{k,t} \\ \mathcal{E}_{k,t+1} & \mathcal{E}_{k,t} & \mathcal{E}_{k,t-1} \\ \mathcal{E}_{k,t} & \mathcal{E}_{k,t-1} & \mathcal{E}_{k,t-2} \end{bmatrix} = k(4 - 4k - k^2)$,

ii. $det \begin{bmatrix} \mathcal{S}_{k,t+2} & \mathcal{S}_{k,t+1} & \mathcal{S}_{k,t} \\ \mathcal{S}_{k,t+1} & \mathcal{S}_{k,t} & \mathcal{S}_{k,t-1} \\ \mathcal{S}_{k,t} & \mathcal{S}_{k,t-1} & \mathcal{S}_{k,t-2} \end{bmatrix} = 64k(17 - 10k + k^2)$,

iii. $det\begin{bmatrix} \mathcal{G}_{k,t+2} & \mathcal{G}_{k,t+1} & \mathcal{G}_{k,t} \\ \mathcal{G}_{k,t+1} & \mathcal{G}_{k,t} & \mathcal{G}_{k,t-1} \\ \mathcal{G}_{k,t} & \mathcal{G}_{k,t-1} & \mathcal{G}_{k,t-2} \end{bmatrix} = -k(128+32k+k^2).$

Proof: The proofs are shown by the induction method using the definition and determinant properties. □

Theorem 3.8. For $\mathcal{E}_{k,n}$, $\mathcal{S}_{k,n}$ and $\mathcal{G}_{k,n}$ sequences, the Binet formulas can be obtained with the help of the matrices.

Proof: The following relation is used for proof (see for details Corollary 3.1 in [32]).

$$t_n = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} r_{m+1-j} \det(\Lambda_j). \tag{6}$$

Thus,

$$\mathcal{E}_{k,n} = \frac{1}{\det(\Lambda)} \sum_{j=1}^{i} \mathcal{E}_{k,m+1-j} \det(\Lambda_j).$$

Let $m = i = 3$,

$$\Lambda = \begin{bmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ 1 & 1 & 1 \end{bmatrix}, \Lambda_1 = \begin{bmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ 1 & 1 & 1 \end{bmatrix}, \Lambda_2 = \begin{bmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{and } \Lambda_3 = \begin{bmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ 1 & 1 & 1 \end{bmatrix}.$$

So, we obtain

$$\mathcal{E}_{k,n} = \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} \mathcal{S}_{k,4-j} \det(\Lambda_j)$$

$$= \frac{1}{\det(\Lambda)} (\mathcal{E}_{k,3} \det(\Lambda_1) + \mathcal{E}_{k,2} \det(\Lambda_2) + \mathcal{E}_{k,1} \det(\Lambda_3))$$

$$= \frac{5+k-\beta}{(\alpha-\beta)(\alpha-1)} \alpha^n + \frac{5+k-\alpha}{(\beta-\alpha)(\beta-1)} \beta^n + \frac{k}{(\alpha-1)(\beta-1)}.$$

Similarly, the Binet formulas of the $\mathcal{S}_{k,n}$, and $\mathcal{G}_{k,n}$ sequences are found. □

4. CONCLUSIONS

In this study, we defined the k-Edouard, k-Edouard-Lucas, and Modified k-Edouard sequences. Then, we found the features of these sequences, such as generating function and special summation formulas. In addition, we examined the relationships among the terms of these sequences. Moreover, we associated k-Edouard, k-Edouard-Lucas, and Modified k-Edouard sequences with Balancing, Balancing-Lucas, and Modified Balancing numbers. Ultimately, we linked the elements of these sequences to matrices and calculated the Simson formulas of these sequences. If this study is examined, such features can be found in other sequences, such as the Fermat and Mersenne sequences. Also, research can be conducted on the application area of the sequences defined in these articles [33-34].

REFERENCES

[1] de Oliveira, R. R., Alves, F. R. V., *Acta Scientiae*, **21**(3), 170, 2019.
[2] Battaloglu, R., Simsek, Y., *Symmetry*, **13**(08), 1334, 2021.
[3] Avazzadeh, Z., Hassani, H., Agarwal, P., Mehrabi, S., Javad Ebadi, M., Hosseini Asl, M. K., *Mathematical Methods in the Applied Sciences*, **46**(8), 9332, 2021.
[4] Baishya, K., *International Journal of Economic Perspectives*, **17**(6), 344, 2021.
[5] Sekido, H., Iwasaki, M., *Research in Mathematics*, **9**(1), 2069647, 2022.
[6] Bolat, C., Köse, H., *International Journal of Contemporary Mathematical Sciences*, **5**(22), 1097, 2010.
[7] Panda, G. K., *Fibonacci Quarterly*, **45**(3), 265, 2007.
[8] Soykan, Y., *International Journal of Advances in Applied Mathematics and Mechanics*, **7**(3), 43, 2020.
[9] Akkuş, H., Özkan, E., *Proceedings of the Indian National Science Academy*, 1, 2025.
[10] Akkuş, H., Üregen, R. N., Özkan, E., *Sakarya University Journal of Science*, **25**(4), 969, 2021.
[11] Vieira, R. P. M., Alves, F. R. V., Catarino, P. M. M. C., *International Journal of Trends in Mathematics Education Research*, **3**(1), 8, 2020.
[12] Özkan, E., Akkuş, H., *Symmetry*, **16**, 1, 2024.
[13] Shannon, A. G., Akkuş, H., Aküzüm, Y., Deveci, Ö., Özkan, E., *Notes on Number Theory and Discrete Mathematics*, **30**(3), 530, 2024.
[14] Szynal-Liana, A., Wloch, I., *Annales Mathematicae Silesianae*, **33**, 276, 2019.
[15] Akkuş, H., Deveci, Ö., Özkan, E., Shannon, A. G., *Notes on Number Theory and Discrete Mathematics*, **30**(1), 8, 2024.
[16] Özkan, E., Akkuş, H., Özkan, A., *Axioms*, **14**(1), 14, 2025.
[17] Özkan, E., Akkuş, H., *AIP Advances*, **14**(7), 1, 2024.
[18] Özkan, E., Şen, B., Akkuş, H., Uysal, M., *Universal Journal of Mathematics and Applications*, **8**(1), 1, 2025.
[19] Kilic, E., *Journal of Computational and Applied Mathematics*, **209**(2), 133, 2007.
[20] Akkuş, H., Özkan, E., *Mathematica Montisnigri*, **61**, 14, 2024.
[21] Özkan, E., Akkuş, H., *Notes on Number Theory and Discrete Mathematics*, **30**(4), 811, 2024.
[22] Özkan, E., Akkus, H., *WSEAS Transactions on Mathematics*, **23**, 856, 2024.
[23] Özkan, E., Akkus, H., *WSEAS Transactions on Mathematics*, **22**, 503, 2023.
[24] Soykan, Y., *Journal of Progressive Research in Mathematics*, **18**(4), 58, 2021.
[25] Soykan, Y., *International Journal of Advances in Applied Mathematics and Mechanics*, **3**(9), 41, 2022.
[26] Soykan, Y., *Universal Journal of Mathematics and Applications*, **3**(1), 1, 2020.
[27] Soykan, Y., *Earthline Journal of Mathematical Sciences*, **13**(1), 1, 2023.
[28] Soykan, Y., *MathLAB Journal*, **7**(1), 101, 2020.
[29] Falcon, S., Plaza, A., *Chaos Solitons Fractals*, **32**(5), 1615, 2007.
[30] Falcon, S., *International Journal of Contemporary Mathematical Sciences*, **6**(21), 1039, 2011.
[31] Falcon, S., *Communications of the Korean Mathematical Society*, **28**(4), 827, 2013.
[32] Kiliç, E., Stanica, P., *Bulletin of the Malaysian Mathematical Sciences Society*, **34**(1), 51, 2011.
[33] Marin, M., *Rendiconti di Matematica e delle sue Applicazioni*, **17**(7), 103, 1997.
[34] Marin, M., *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, **60**(3), 3, 1998.