ORIGINAL PAPER

ON THE LONG SEQUENCE AND THE HUREWICZ THEOREM FOR THE PERSISTENT HOMOTOPY GROUPS OF A PAIR

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Abstract. *In this paper, we show that the persistent homotopy groups fit into a long sequence which is exact of order 2. We prove a persistent version of the Hurewicz theorem for a pair of spaces* (X, A)*. We also show that this version implies the persistent version of the Hurewicz theorem for the space* X*.*

Keywords: *Filtration; homotopy; exact sequence; Hurewicz.*

1. INTRODUCTION

When one analyzes data obtained from experiments, some characteristic geometric structures can be observed in the data, but it is not true that every data set has such a well-organized geometric structure. Recent researches reveal that some methods of algebraic topology are effective in the characterization of complicated data. This kind of research is called Topological Data Analysis (TDA), and a common mathematical tool in TDA, capturing topological features, is the persistent homology. Persistent homology is an algebraic method to determine features of a topological space or a data set by using a filtration that comes from a suitable function on it.

Another algebraic method used to understand the topological properties of a space is the persistent homotopy groups. The persistent homotopy groups of a space X are defined in [1] through a filtration of the space and the corresponding homotopy groups of the spaces in the filtration with homomorphisms between them. Letsher [1] used the persistent homotopy groups to detect if a complex is knotted, and if so, if this knotting can be unknotted in a larger complex. He applied the persistent homotopy groups to proteins to detect types of knotting. Blumberg and Lesnick [2] stated a naive version of the Whitehead Conjecture for the persistent homotopy groups. Adams et al. [3] proved a persistent version of the Van Kampen theorem. They also showed that the excision and the Hurewicz theorems are valid for the persistent homotopy groups. Memoli and Zhou [4] studied the persistent homotopy groups of compact metric spaces and their stability properties. They gave the persistent version of the Hurewicz theorem for the persistent fundamental groups.

In this paper, we obtain a long sequence for the persistent homotopy groups that is exact of order 2. However, we give an example showing that the long sequence of the persistent homotopy groups is not exact. We prove that the Hurewicz theorem is valid for the persistent homotopy groups of a pair (X, A), and this theorem implies the Hurewicz theorem for the persistent homotopy groups of the space X.

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2. PRELIMINARIES

In this section, we give the basic notions that we need throughout the paper. Let X be a topological space with a fixed basepoint x_0. A filtration $\mathcal{X}$ of the space X, indexed by a subset I of $\mathbb{R}$, is a collection $\mathcal{X} = \{X_i, F_{i,j}\}_{i \leq j \in I}$ of topological spaces X_i with $X = \bigcup_{i \in I} X_i$ and inclusions $F_{i,j}: X_i \to X_j$ for each $i \leq j \in I$. The filtration $\mathcal{X}$ of X is called basepoint preserving if each space X_i contains the basepoint x_0.

For example, let $f: X \to \mathbb{R}$ be a continuous real-valued function on X. The set $X_i = \{x \in X \mid f(x) \leq i\}$ is a subspace of X. Clearly $X_i \subseteq X_j$ for $i \leq j$. So, $X = \bigcup_{i \in I} X_i$ and we have an inclusion map $F_{i,j}: X_i \to X_j$ for each $i \leq j \in \mathbb{R}$. Thus, $\mathcal{X} = \{X_i, F_{i,j}\}_{i \leq j \in \mathbb{R}}$ is a filtration of X induced by the function f and indexed by $\mathbb{R}$.

Let (X, A) be a pair of spaces with $A \subseteq X$ and a fixed basepoint $x_0 \in A$. Let $\mathcal{X} = \{X_i, F_{i,j}\}_{i \leq j \in I}$ be a basepoint preserving filtration of the space X and $A_i = X_i \cap A$ for each $i \in I$. Obviously, $\mathcal{A} = \{A_i, F_{A,i,j}\}_{i \leq j \in I}$ is a filtration for the subspace A, where $F_{A,i,j}: A_i \to A_j$ is the inclusion map induced by $F_{i,j}$. Notice that $x_0 \in A_i$ for each $i \in I$.

The collection $(\mathcal{X}, \mathcal{A}) = \{(X_i, A_i), \Phi_{i,j}\}_{i \leq j \in I}$ is called a basepoint preserving filtration for the pair (X, A), where $\Phi_{i,j}: (X_i, A_i) \to (X_j, A_j)$ is a map such that $\Phi_{i,j}(X_i) = F_{i,j}(X_i)$ and $\Phi_{i,j}(A_i) = F_{A,i,j}(A_i)$.

The k-th homotopy group of X, denoted by $\pi_k(X, x_0)$, is the set of homotopy classes of maps $\gamma: S^k \to X$ such that $\gamma(s_0) = x_0$, where S^k is the k-sphere and s_0 is a basepoint of S^k. For $k = 0$, $\pi_k(X, x_0)$ is not a group. It is just a set of path-connected components in X. The k-th relative homotopy group $\pi_k(X, A, x_0)$ for a pair (X, A) is the set of homotopy classes of maps $\rho: (D^k, S^{k-1}) \to (X, A)$ such that $\rho(s_0) = x_0$, where D^k is the k-disk and s_0 is a basepoint of S^{k-1}. Notice that $\pi_k(X, A, x_0)$ is a group for $k \geq 2$. We say the space X with the basepoint x_0 is n-connected if $\pi_k(X, x_0) = 0$ for all $k \leq n$ and the pair (X, A) with the base point x_0 is called n-connected if $\pi_k(X, A, x_0) = 0$ for all $k \leq n$. We say that a space X is simply connected if it is 1-connected (see [5] for more details).

Let $\mathcal{X}$ be a basepoint preserving filtration of X. Suppose that each space in $\mathcal{X}$ is path-connected. Then the k-th $\{i, j\}$-persistent homotopy group of X with respect to the filtration $\mathcal{X}$ is defined as the image of the homomorphism $\pi_k F_{i,j}: \pi_k(X_i, x_0) \to \pi_k(X_j, x_0)$ between the k-th homotopy groups $\pi_k(X_i, x_0)$ and $\pi_k(X_j, x_0)$ induced by the inclusion $F_{i,j}$, and it is denoted by $\pi_k^{i,j}(X)$. The group $\pi_k^{i,j}(X)$ detects homotopy classes of maps $\gamma: S^k \to X_i$ which are completely contained in X_j (see [1] for more details).

Let $(\mathcal{X}, \mathcal{A}) = \{(X_i, A_i), \Phi_{i,j}\}_{i \leq j \in I}$ be a basepoint preserving filtration for the pair (X, A) with basepoint $x_0 \in A$ such that each pair (X_i, A_i) is 1-connected. The k-th $\{i, j\}$-persistent homotopy group of the pair (X, A) with respect to the filtration $(\mathcal{X}, \mathcal{A})$ is defined as the image of the homomorphism $\pi_k \Phi_{i,j}: \pi_k(X_i, A_i, x_0) \to \pi_k(X_j, A_j, x_0)$ induced by $\Phi_{i,j}$ and denoted by $\pi_k^{i,j}(X, A)$.

3. A LONG SEQUENCE OF THE PERSISTENT HOMOTOPY GROUPS

In [5], Hatcher showed that the relative homotopy groups (homotopy groups of a pair) fit into a long exact sequence, which is stated in Theorem 3.1. In this section, we obtain a long sequence for the persistent homotopy groups of a pair. We show that this sequence is not exact by an example. However, we prove that it is exact of order 2, that is, the image of one map is a subset of the kernel of the next.

Theorem 3.1. The sequence

$$\cdots \to \pi_k(A, x_0) \xrightarrow{\alpha_k} \pi_k(X, x_0) \xrightarrow{\beta_k} \pi_k(X, A, x_0) \xrightarrow{\partial_k} \pi_{k-1}(A, x_0) \to \cdots \to \pi_0(X, x_0)$$

of the pair (X, A) is exact, where the maps α_k and β_k are induced by the inclusions $(A, x_0) \to (X, x_0)$ and $(X, x_0, x_0) \to (X, A, x_0)$, respectively, and ∂_k is obtained by restricting maps $(D^k, S^{k-1}, s_0) \to (X, A, x_0)$ to S^{k-1} with $s_0 \in S^{k-1}$.

Notice that the maps $\beta_1 : \pi_1(X, x_0) \to \pi_1(X, A, x_0)$, $\partial_1 : \pi_1(X, A, x_0) \to \pi_0(A, x_0)$ and $\alpha_0 : \pi_0(A, x_0) \to \pi_0(X, x_0)$ given in Theorem 3.1 are not homomorphisms. Since $\pi_1(X, A, x_0)$, $\pi_0(X, x_0)$ and $\pi_0(A, x_0)$ are not groups when we talk about the kernel of the maps β_1, ∂_1 and α_0, we mean the elements that are mapped to the homotopy class of the constant map at x_0 under these maps .

Let (X, A) be a pair of spaces with a basepoint $x_0 \in A$, $\mathcal{X} = \{X_i, F_{i,j}\}_{i \le j \in I}$ be a basepoint preserving filtration for the space X and $(\mathcal{X}, \mathcal{A}) = \{(X_i, A_i), \Phi_{i,j}\}_{i \le j \in I}$ be the basepoint preserving filtration for the pair (X, A) induced by $\mathcal{X}$. In the remaining part of this section, we suppose that X_i and A_i are 0-connected (i.e., path-connected) and the pair (X_i, A_i) is 1-connected for each $i \in I$.

Lemma 3.2. Each horizontal line in the diagram

$$\ldots \to \pi_k(A_i, x_0) \xrightarrow{\alpha_k^i} \pi_k(X_i, x_0) \xrightarrow{\beta_k^i} \pi_k(X_i, A_i, x_0) \xrightarrow{\partial_k^i} \pi_{k-1}(A_i, x_0) \to \ldots \xrightarrow{\alpha_1^i} \pi_1(X_i, x_0) \to 0$$

$$\downarrow \pi_k F_{A,i,j} \qquad \downarrow \pi_k F_{i,j} \qquad \downarrow \pi_k \Phi_{i,j} \qquad \downarrow \pi_{k-1} F_{A,i,j} \qquad \downarrow \pi_1 F_{i,j}$$

$$\ldots \to \pi_k(A_j, x_0) \xrightarrow{\alpha_k^j} \pi_k(X_j, x_0) \xrightarrow{\beta_k^j} \pi_k(X_j, A_j, x_0) \xrightarrow{\partial_k^j} \pi_{k-1}(Aj, x_0) \to \ldots \xrightarrow{\alpha_1^j} \pi_1(X_j, x_0) \to 0$$

is exact. Moreover, the diagram is commutative. Here, for $l = i, j$, the maps α_k^l and β_k^l are induced by the inclusions $(A_l, x_0) \to (X_l, x_0)$ and $(X_l, x_0, x_0) \to (X_l, A_l, x_0)$, respectively, and ∂_k^l is obtained by restricting maps $(D^k, S^{k-1}, s_0) \to (X_l, A_l, x_0)$ to S^{k-1} with $s_0 \in S^{k-1}$.

Proof: Consider the long exact sequences for the pairs (X_i, A_i) and (X_j, A_j) given in Theorem 3.1. By the assumption given for X_l, A_l and (X_l, A_l), for $l = i, j$, we have $\pi_0(X_l, x_0) = \pi_0(A_l, x_0) = 0$ and $\pi_1(X_l, A_l, x_0) = 0$. So the long exact sequences for the pairs (X_i, A_i) and (X_j, A_j) imply the exactness of the horizontal lines given in the diagram. Moreover, by naturality, we see that the given diagram is commutative, which completes the proof.

By using Lemma 3.2, we obtain the long sequence

$$\cdots \to \pi_k^{i,j}(A) \xrightarrow{\alpha_{\mathrm{k}}^{i,j}} \pi_k^{i,j}(X) \xrightarrow{\beta_{\mathrm{k}}^{i,j}} \pi_k^{i,j}(X, A) \xrightarrow{\partial_{\mathrm{k}}^{i,j}} \pi_{k-1}^{i,j}(A) \to \cdots \to \pi_1^{i,j}(A) \xrightarrow{\alpha_1^{i,j}} \pi_1^{i,j}(X) \to 0 \qquad (1)$$

containing $\{\mathrm{i}, \mathrm{j}\}$-persistent homotopy groups of A, X and the pair (X, A). Here, $\alpha_{\mathrm{k}}^{i,j}$ is the restriction of the map α_{k}^{j} to $im\, \pi_k F_{A,i,j}$, $\beta_{\mathrm{k}}^{i,j}$ is the restriction of the map β_{k}^{j} to $im\, \pi_k F_{i,j}$ and

$\partial_k^{i,j}$ is the restriction of the map ∂_k^j to $im\pi_k\Phi_{i,j}$. Notice that the maps α_k^j, β_k^j and ∂_k^j are the ones given in Lemma 3.2, and the maps $\alpha_k^{i,j}$, $\beta_k^{i,j}$ and $\partial_k^{i,j}$ are all homomorphisms.

Now, we will show that the sequence given in (1) is not exact by an example.

Example 3.3. Let X be the upside down 2-disk given in Figure 1, A be the boundary circle denoted by red, x_0 be any point on A and $f: X \to \mathbb{R}$ be the height function on X.

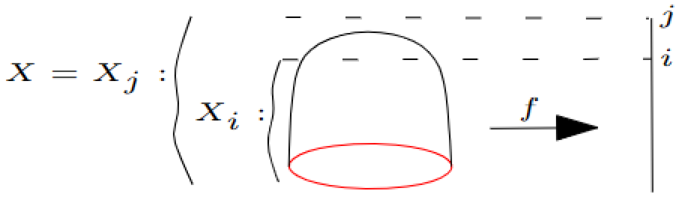

Figure 1. A filtration of X induced by the height function f.

Clearly, the function f induces a basepoint preserving filtration $\mathcal{X}$ for the spaces X such that each space in the filtration contains the point x_0. Notice that, the filtration $\mathcal{X}$ induces a basepoint preserving filtration for A and for the pair (X, A). Let i, j be the two fixed filtration levels such that $i < j$ and $X_i = \{x \in X \mid f(x) \leq i\}$, $X_j = \{x \in X \mid f(x) \leq j\}$, $A_i = \{x \in A \mid f(x) \leq i\}$ and $A_j = \{x \in A \mid f(x) \leq j\}$, which are given in Figure 1. Notice that $A_i = A_j = A$ and all the spaces in Figure 1 are path-connected. Thus, $\pi_0(A_i, x_0) = \pi_0(X_i, x_0) = \pi_0(A_j, x_0) = \pi_0(X_j, x_0) = 0$ for any $x_0 \in A$, i.e., A_i, A_j, X_i and X_j are 0-connected. Moreover, $\pi_1(A_i, x_0) = \pi_1(A_j, x_0) = \mathbb{Z}$. Now consider the following long exact sequences of homotopy groups for the pairs (X_i, A_i) and (X_j, A_j), respectively:

$$\cdots \to \pi_2(A_i, x_0) \xrightarrow{\alpha_2^i} \pi_2(X_i, x_0) \xrightarrow{\beta_2^i} \pi_2(X_i, A_i, x_0) \xrightarrow{\partial_2^i} \pi_1(A_i, x_0) \xrightarrow{\alpha_1^i} \pi_1(X_i, x_0) \xrightarrow{\beta_1^i} \pi_1(X_i, A_i, x_0) \xrightarrow{\partial_1^i} 0 \qquad (2)$$

and

$$\cdots \to \pi_2(A_j, x_0) \xrightarrow{\alpha_2^j} \pi_2(X_j, x_0) \xrightarrow{\beta_2^j} \pi_2(X_j, A_j, x_0) \xrightarrow{\partial_2^j} \pi_1(A_j, x_0) \xrightarrow{\alpha_1^j} \pi_1(X_j, x_0) \xrightarrow{\beta_1^j} \pi_1(X_j, A_j, x_0) \xrightarrow{\partial_1^j} 0 \qquad (3)$$

Since X_i is a cylinder, which is homotopy equivalent to A, then $\pi_k(A_i, x_0)$ and $\pi_k(X_i, x_0)$ are isomorphic for each k. So, the homomorphisms α_2^i and α_1^i are isomorphisms. Then, by the exactness of the above sequences, we obtain $im\, \alpha_1^i = \pi_1(X_i, x_0) = ker\beta_1^i$ which implies $im\beta_1^i = 0$. Hence, we obtain $im\beta_1^i = 0 = ker\partial_1^i$. Moreover, we have $ker\partial_1^i = \pi_1(X_i, A_i, x_0)$. So, we obtain $\pi_1(X_i, A_i, x_0) = 0$. Since $X_j = X$ is a disk, $\pi_k(X_j, x_0) = 0$ for each $k \geq 1$. So, $im\beta_1^j = 0$. Then, by the same argument given for the level i, we see that $\pi_1(X_j, A_j, x_0) = 0$. Notice that $\pi_0(X_i, A_i, x_0)$ and $\pi_0(X_j, A_j, x_0)$ can be considered as the quotient sets $\pi_0(X_i, x_0)/\pi_0(A_i, x_0)$ and $\pi_0(X_j, x_0)/\pi_0(A_j, x_0)$, respectively. Then we obtain $\pi_0(X_i, A_i, x_0) = \pi_0(X_j, A_j, x_0) = 0$. Thus, the pairs (X_i, A_i) and (X_j, A_j) are 1-connected. Hence, we can talk about the persistent homotopy groups of the spaces X_i, X_j, A_i, A_j and the pairs (X_i, A_i), (X_j, A_j). Moreover, we obtain the sequence

$$\cdots \to \pi_k^{i,j}(A) \xrightarrow{\alpha_k^{i,j}} \pi_k^{i,j}(X) \xrightarrow{\beta_k^{i,j}} \pi_k^{i,j}(X, A) \xrightarrow{\partial_k^{i,j}} \cdots \to \pi_2^{i,j}(A) \xrightarrow{\alpha_2^{i,j}} \pi_2^{i,j}(X) \xrightarrow{\beta_2^{i,j}} \pi_2^{i,j}(X, A) \xrightarrow{\partial_2^{i,j}} \pi_1^{i,j}(A) \xrightarrow{\alpha_1^{i,j}} \pi_1^{i,j}(X) \to 0$$

given in (1).

Observe that, by the definition of the persistent homotopy group of spaces, $\pi_1^{i,j}(A)$ and $\pi_1^{i,j}(X)$ are the images of the homomorphisms $\pi_1(A_i,x_0)\to\pi_1(A_j,x_0)$ and $\pi_1(X_i,x_0)\to\pi_1(X_j,x_0)$ induced by the inclusions, respectively. Since $A_i=A_j=A$, $\pi_1(A_i,x_0)\to\pi_1(A_j,x_0)$ is an isomorphism and $\pi_1^{i,j}(A)=\mathbb{Z}$. Since $\pi_1(X_j,x_0)=0$, $\pi_1^{i,j}(X)=0$.

Notice that by the exactness of the sequence given in (2), we obtain $im\,\alpha_2^i=\pi_2(X_i,x_0)=ker\beta_2^i$ which implies $im\beta_2^i=0$. Hence, we obtain $im\beta_2^i=0=ker\partial_2^i$ and $im\partial_2^i=ker\alpha_1^i=0$. By the first isomorphism theorem, we see that $\pi_2(X_i,A_i,x_0)=0$. Since $\pi_2^{i,j}(X,A)$ is the image of the homomorphsim $\pi_2(X_i,A_i,x_0)\to\pi_2(X_j,A_j,x_0)$, then $\pi_2^{i,j}(X,A)=0$. Then the above sequence is derived as

$$\cdots\to\pi_k^{i,j}(A)\xrightarrow{\alpha_k^{i,j}}\pi_k^{i,j}(X)\xrightarrow{\beta_k^{i,j}}\pi_k^{i,j}(X,A)\xrightarrow{\partial_k^{i,j}}\cdots\to\pi_2^{i,j}(A)\xrightarrow{\alpha_2^{i,j}}\pi_2^{i,j}(X)\xrightarrow{\beta_2^{i,j}}0\xrightarrow{\partial_2^{i,j}}\mathbb{Z}\xrightarrow{\alpha_1^{i,j}}0\to 0$$

such that $im\partial_2^{i,j}=0$ and $ker\alpha_1^{i,j}=\mathbb{Z}$. Thus, $im\partial_2^{i,j}\neq ker\alpha_1^{i,j}$ which means that the sequence given in (1) is not exact.

In the following theorem, we show that the long sequence given in (1) is exact of order 2.

Theorem 3.4 The sequence

$$\cdots\to\pi_k^{i,j}(A)\xrightarrow{\alpha_k^{i,j}}\pi_k^{i,j}(X)\xrightarrow{\beta_k^{i,j}}\pi_k^{i,j}(X,A)\xrightarrow{\partial_k^{i,j}}\pi_{k-1}^{i,j}(A)\to\cdots\to\pi_1^{i,j}(A)\xrightarrow{\alpha_1^{i,j}}\pi_1^{i,j}(X)\to 0$$

of homomorphisms of $\{i,j\}$-persistent homotopy groups of A, X and the pair (X,A) is exact of order 2. That is,

1. $im\,\alpha_k^{i,j}\subseteq ker\,\beta_k^{i,j}$,
2. $im\,\beta_k^{i,j}\subseteq ker\,\partial_k^{i,j}$,
3. $im\,\partial_k^{i,j}\subseteq ker\,\alpha_{k-1}^{i,j}$.

Proof: By the definition of persistent homotopy groups for the space X and the pair (X,A), $\pi_k^{i,j}(A)=im\pi_kF_{A,i,j}$, $\pi_k^{i,j}(X)=im\pi_kF_{i,j}$ and $\pi_k^{i,j}(X,A)=im\pi_k\Phi_{i,j}$. Clearly, the maps $\alpha_k^{i,j},\beta_k^{i,j}$ and $\partial_k^{i,j}$ are all homomorphisms for any $k\geq 2$. We only show item 1. Items 2 and 3 can be shown similarly.

Let $\eta\in im\,\alpha_k^{i,j}$ be is a non-zero element. Then, there exists a non-zero element $\omega\in\pi_k(A_i,x_0)$ such that $\eta=\alpha_k^j(\pi_kF_{A,i,j}(\omega))$. By Lemma 3.2, $im\alpha_k^j=ker\beta_k^j$ and $\mathrm{im}\,\alpha_k^{i,j}\subseteq im\pi_kF_{i,j}$. Thus, $\beta_k^j(\eta)=0$ and $\eta\in im\pi_kF_{i,j}$. By the definition of the map $\beta_k^{i,j}$, $\eta\in ker\,\beta_k^{i,j}$.

4. THE HUREWICZ THEOREM FOR THE PERSISTENT HOMOTOPY GROUPS OF A PAIR

In this section, we prove that the Hurewicz theorem is valid for the persistent homotopy groups of a pair under some connectivity assumptions. Now, we recall the Hurewicz theorem given for both a space and a pair in the following theorem (see [5] for details).

Theorem 4.1. If a space X is $(n-1)$-connected for $n \geq 2$, then $\widetilde{H}_k(X) = 0$ for $k < n$ and $\pi_n(X) \cong H_n(X)$. If the pair of spaces (X, A) is $(n-1)$-connected for $n \geq 2$ and A is simply connected and non-empty, then $H_k(X, A) = 0$ for $k < n$ and $\pi_n(X, A) \cong H_n(X, A)$.

Let (X, A) be a pair of spaces with a basepoint $x_0 \in A$. Let $\mathcal{X} = \{X_i, F_{i,j}\}_{i \leq j \in I}$ be a basepoint preserving filtration for X and $(\mathcal{X}, \mathcal{A}) = \{(X_i, A_i), \Phi_{i,j}\}_{i \leq j \in I}$ be the basepoint preserving filtration for the pair (X, A).

Let us recall the definition of the persistent homology group. The k-th $\{i, j\}$-persistent homology group of a space X with respect to the filtration $\mathcal{X} = \{X_i, F_{i,j}\}_{i \leq j \in I}$ is defined as the image of the homomorphism $H_k F_{i,j}: H_k(X_i) \to H_k(X_j)$ and denoted by $H_k^{i,j}(X)$. Similarly, the k-th $\{i, j\}$-persistent homology group of a pair (X, A) with respect to the filtration $(\mathcal{X}, \mathcal{A}) = \{(X_i, A_i), \Phi_{i,j}\}_{i \leq j \in I}$ is defined as the image of the homomorphism $H_k \Phi_{i,j}: H_k(X_i, A_i) \to H_k(X_j, A_j)$ and denoted by $H_k^{i,j}(X, A)$. In [3], the authors showed a persistent version of the Hurewicz theorem given for a space X with a filtration $\mathcal{X}$, which is stated in the following theorem.

Theorem 4.2. For the fixed levels i and j, if the space X_i is $(m-1)$-connected and the space X_j is $(n-1)$-connected, for $m, n \geq 2$, then $H_k^{i,j}(X) = 0$ for $0 < k < min\{m, n\}$ and $\pi_k^{i,j}(X) \cong H_k^{i,j}(X)$ for $k = min\{m, n\}$.

Now, we show that a persistent version of the Hurewicz theorem for a pair (X, A) is valid by considering the filtration $(\mathcal{X}, \mathcal{A}) = \{(X_i, A_i), \Phi_{i,j}\}_{i \leq j \in I}$.

Theorem 4.3. For the fixed levels i and j of the filtration $(\mathcal{X}, \mathcal{A}) = \{(X_i, A_i), \Phi_{i,j}\}_{i \leq j \in I}$ of the pair (X, A), if (X_i, A_i) is $(m-1)$-connected, (X_j, A_j) is $(n-1)$-connected for $m, n \geq 2$, and A_i, A_j are simply connected and non-empty, then $H_k^{i,j}(X, A) = 0$ for $k < min\{m, n\}$ and $\pi_k^{i,j}(X, A) \cong H_k^{i,j}(X, A)$ for $k = min\{m, n\}$.

Proof: By Theorem 4.1, $H_k(X_i, A_i) = 0$ for $k < m$ and $H_l(X_j, A_j) = 0$ for $l < n$. So $H_k(X_i, A_i) = 0$ for $k < min\{m, n\}$, which implies $H_k^{i,j}(X, A) = 0$ by the definition of the persistent homology groups. Moreover, (X_i, A_i) and (X_j, A_j) are both $(k-1)$-connected for $k = min\{m, n\}$. Therefore, by Theorem 4.1, we have the isomorphisms

$$p_i: \pi_k(X_i, A_i) \to H_k(X_i, A_i)$$

and

$$p_j: \pi_k(X_j, A_j) \to H_k(X_j, A_j).$$

Thus, by naturality, we have the following commutative diagram:

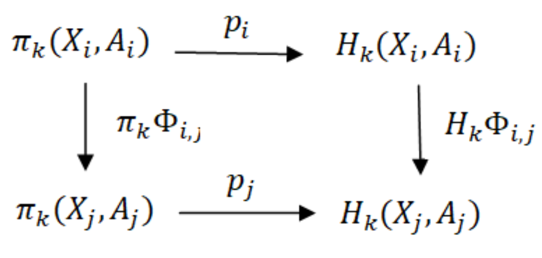

Figure 2. Commutative diagram between levels i and j.

Now we show that $\pi_k^{i,j}(X,A) \cong H_k^{i,j}(X,A)$. Observe that if $m \neq n$, $H_k^{i,j}(X,A) = 0$ and $\pi_k^{i,j}(X,A) = 0$ by the definitions of persistent homology and homotopy groups. For $m = n$, we have a well-defined homomorphism $f: \pi_k^{i,j}(X,A) \to H_k^{i,j}(X,A)$ defined as $f(\alpha) = H_k\Phi_{i,j}(p_i(\beta))$ such that $\alpha = \pi_k\Phi_{i,j}(\beta)$ for some $\beta \in \pi_k(X_i, A_i)$.

Now we will show that f is one-to-one and onto, which implies that f is an isomorphism. Let $\alpha, \alpha' \in \pi_k^{i,j}(X,A)$ be two non-zero elements such that $f(\alpha) = f(\alpha')$. Then there exist $\beta, \beta' \in \pi_k(X_i, A_i)$ such that $\alpha = \pi_k\Phi_{i,j}(\beta)$, $\alpha' = \pi_k\Phi_{i,j}(\beta')$ and $H_k\Phi_{i,j}(p_i(\beta)) = H_k\Phi_{i,j}(p_i(\beta'))$. By commutativity of the given diagram, we have $p_j(\pi_k\Phi_{i,j}(\beta)) = p_j(\pi_k\Phi_{i,j}(\beta'))$. Since the map p_j is an isomorphism, we have $\pi_k\Phi_{i,j}(\beta) = \pi_k\Phi_{i,j}(\beta')$, which implies that $\alpha = \alpha'$. Hence f is one-to-one.

Let $\gamma \in H_k^{i,j}(X,A)$ be a non-zero element. By the definition of the persistent homology group, there exists an element $\delta \in H_k(X_i, A_i)$ such that $H_k\Phi_{i,j}(\delta) = \gamma$. Since p_i is an isomorphism, there exists a non-zero element $\delta' \in \pi_k(X_i, A_i)$ such that $p_i(\delta') = \delta$. Let $\alpha = \pi_k\Phi_{i,j}(\delta')$. Then $\alpha \in \pi_k^{i,j}(X,A)$. Thus we obtain $f(\alpha) = H_k\Phi_{i,j}(p_i(\delta')) = H_k\Phi_{i,j}(\delta) = \gamma$, which implies f is onto.Therefore, we obtain that $\pi_k^{i,j}(X,A) \cong H_k^{i,j}(X,A)$.

Now we show that the Hurewicz Theorem for the persistent homotopy groups of the pair (X,A) implies Theorem 4.2 by considering the filtration $\mathcal{X} = \{X_i, F_{i,j}\}_{i \leq j \in I}$ for X and the filtration $(\mathcal{X}, \mathcal{A}) = \{(X_i, A_i), \Phi_{i,j}\}_{i \leq j \in I}$ for (X,A).

Theorem 4.4. For fixed levels i and j, if the space X_i is $(m-1)$-connected and the space X_j is $(n-1)$-connected for $m, n \geq 2$, then the Hurewicz theorem for the persistent homotopy groups of a pair in dimension $k = min\{m, n\}$ implies the Hurewicz theorem for the persistent homotopy groups of X in dimension $(k-1)$.

In order to prove above theorem, we need the following lemma. Let CX be the cone of the space X. Clearly, $CX_i \subseteq CX_j$ for $i \leq j$ and we have the inclusion $\phi_{i,j}: CX_i \to CX_j$. Thus we have a filtration $\mathcal{CX} = \{CX_i, \phi_{i,j}\}_{i \leq j \in I}$ for CX obtained by the filtration $\mathcal{X}$ and a filtration $(\mathcal{CX}, \mathcal{X}) = \{(CX_i, X_i), \Phi_{i,j}\}_{i \leq j \in I}$ obtained by the filtration $\mathcal{CX}$.

Lemma 4.5. For any $k > 0$ and $i, j \in I$ with $i \leq j$, $\mathrm{H}_{\mathrm{k+1}}^{\mathrm{i,j}}(\mathrm{CX}, \mathrm{X}) \cong \mathrm{H}_{\mathrm{k}}^{\mathrm{i,j}}(\mathrm{X})$.

Proof: Let us consider the following long exact sequence for relative pairs:

$$\cdots \to H_{k+1}(CX_i) \to H_{k+1}(CX_i, X_i) \to H_k(X_i) \to H_k(CX_i) \to H_k(CX_i, X_i) \to \cdots$$

Since CX_i is contractible for any i, $H_k(CX_i) = 0$ for $k > 0$. Thus, we have $H_{k+1}(CX_i, X_i) \cong H_k(X_i)$ for any $k > 0$. Let $h_i: H_{k+1}(CX_i, X_i) \to H_k(X_i)$ and $h_j: H_{k+1}(CX_j, X_j) \to H_k(X_j)$ be the corresponding isomorphisms. Now, let us consider the following diagram, which is commutative by naturality:

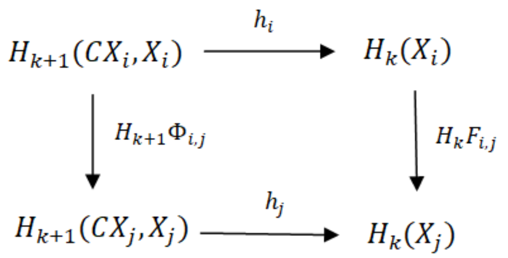

Figure 3. Commutative diagram between levels i and j.

Then we get a well-defined homomorphism $f: H_{k+1}^{i,j}(CX, X) \to H_k^{i,j}(X)$ defined as $f(\alpha) = H_k F_{i,j}(h_i(\beta))$ such that $\alpha = H_{k+1}\Phi_{i,j}(\beta)$ for some $\beta \in H_{k+1}(CX_i, X_i)$. Using the commutativity of the above diagram and the isomorphisms h_i and h_j, we can conclude that f is an isomorphism.

Proof of Theorem 4.4: Let us consider the following long exact sequence given for the pair (CX_i, X_i) for any $i \in I$;

$$\cdots \to \pi_k(X_i) \to \pi_k(CX_i) \to \pi_k(CX_i, X_i) \to \pi_{k-1}(X_i) \to \cdots$$

Since CX_i is contractible, $\pi_k(CX_i) = 0$ for $k > 0$. Since X_i is $(m-1)$-connected, $\pi_k(X_i) = 0$ for $k < m$. We then obtain $\pi_k(CX_i, X_i) = 0$ for $k \leq m$. Similarly, we see that $\pi_k(CX_j, X_j) = 0$ for $k \leq n$, which shows that (CX_i, X_i) is m-connected and (CX_j, X_j) is n-connected. Since X_i and X_j are simply connected by assumptions, we have $H_k^{i,j}(CX, X) = 0$ for $k < min\{m, n\}$ and $\pi_k^{i,j}(CX, X) \cong H_k^{i,j}(CX, X)$ for $k = min\{m, n\}$ by Theorem 4.3.

By Lemma 4.5, we have $H_{k-1}^{i,j}(X) \cong H_k^{i,j}(CX, X)$ for any $k > 1$, which implies that $H_{k-1}^{i,j}(X) = 0$ for $1 < k < min\{m, n\}$. By [3, Lemma 4.5] we have $\pi_{k-1}^{i,j}(X) \cong \pi_k^{i,j}(CX, X)$ for $k > 1$. Thus, by the composition of the isomorphisms $\pi_{k-1}^{i,j}(X) \cong \pi_k^{i,j}(CX, X)$, $\pi_k^{i,j}(CX, X) \cong H_k^{i,j}(CX, X)$ and $H_k^{i,j}(CX, X) \cong H_{k-1}^{i,j}(X)$, we obtain

$$\pi_{k-1}^{i,j}(X) \cong H_{k-1}^{i,j}(X)$$

for $k = min\{m, n\}$.

5. CONCLUSIONS

In this study, we worked on the persistent homotopy groups of a pair. We investigated the relation between persistent (absolute) homotopy groups and persistent relative homotopy groups, which is useful for calculations. We obtained a long sequence containing persistent homotopy groups and persistent relative homotopy groups. We observed that this sequence is not exact but exact of order 2. We also gave the relation between the persistent homology and persistent homotopy groups of a pair by proving a persistent version of the Hurewicz Theorem for a pair given in [5]. We observed that the persistent version of the Hurewicz Theorem for a space given in [3] can be obtained by the persistent version of the relative Hurewicz theorem.

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