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MULTIPLE SOLUTIONS FOR A LERAY-LIONS FOURTH-ORDER PROBLEM

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Abstract. *This work considers a Leray-Lions problem in nonstandard Sobolev spaces. We assume some assertions about the functions k and f to establish the existence and multiplicity of solutions for the problem*

$$\begin{cases} \Delta(k(x,\Delta w)) + b(x)|w|^{p(x)-2}w = \lambda f(x,w), & \text{in } Q \\ w = \Delta w = 0, & \text{on } \partial Q \end{cases}$$

Keywords: *Leray-Lions type operators; generalized Sobolev space; Fourth-order PDE.*

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1. INTRODUCTION

In recent years, there has been a lot of research about variational problems related to non-standard growth conditions. This is a consequence of their appearance in different fields of applied mathematics and physics. Readers can refer to the works [1-4] to fully understand the interest in this kind of problem. The fourth-order operator with nonstandard growth, $\Delta^2_{p(x)}w := \Delta\left(|\Delta w|^{p(x)-2}\Delta w\right)$, where p is a continuous function depending on the variable space is an extension of the classical p-biharmonic operator $\Delta^2_p w := \Delta(|\Delta w|^{p-2}\Delta w)$. However, the use of the $p(x)$-biharmonic operator is more difficult due to the non-homogeneity. Many authors have investigated this type of problem [5-20].

This manuscript focuses on a class of general operators introduced by Leray and Lions, which we refer to as the Leray-Lions type operators [23]. More precisely, we investigate the weak solvability of the problems using the operator $\Delta(k(x,\Delta w))$ under some additional conditions on the Carathéodory function k.

In this paper Q will be a regular, bounded domain in $\mathbb{R}^N (N \geq 2), \partial Q$ denotes the boundary and $b \in C(\overline{Q})$ where $\inf_{x\in Q} b(x) > 0$. We study the solvability and the multiplicity of solutions for the problem

$$\begin{cases} \Delta(k(x,\Delta w)) + b(x)|w|^{p(x)-2}w = \lambda f(x,w), & \text{in } Q \\ w = \Delta w = 0, & \text{on } \partial Q \end{cases} \# \tag{1.1}$$

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where $\lambda > 0, k, f$ are functions with some suitable conditions introduced later, and the continuous function p is log-Hölder satisfying

$$|p(x) - p(y)| \leq \frac{c}{-\log|x-y|} \quad \forall x, y \in Q, 0 < |x-y| \leq \frac{1}{2},$$

where c is a positive constant.

In the sequel, let us recall some studies related to the general operator in divergence form. In that context, Afrouzi et al. [22] studied the problem

$$\begin{cases} -\operatorname{div}(k(x, \Delta w)) = \lambda g(x) f(w), & \text{in } Q \\ w = 0, & \text{on } \partial Q \end{cases}$$

g and f are two continuous functions. Using a critical point theorem, the authors proved that their problems have at least two distinct nonnegative weak solutions in $W_0^{1,p}(Q)$ for any $\lambda \in]0, \lambda^*[$.

Moreover, Yücedağ [20] treated the following problem and established a result of existence and multiplicity.

$$\begin{cases} -\operatorname{div}(k(x, \Delta w)) = f(x, w), & \text{in } Q \\ w = 0, & \text{on } \partial Q \end{cases}$$

In fact, by the Mountain Pass theorem and the Fountain theorem, the author established the existence and multiplicity of a solution in $W_0^{1,p(x)}(Q)$, with p is a continuous function. Furthermore, in [10], the authors proved that there are at least two nontrivial solutions for problem (1.1) for every $\lambda > \lambda_0$, under some hypotheses on the carathéodory function f.

Motivated by the above papers, we deal with problem (1.1) in different cases, and the function k is assumed to verify the same conditions as the paper [6].

$(\mathbf{H}_1)$ $k: \overline{Q} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying $k(x, 0) = 0$, for a.e. $x \in Q$.

$(\mathbf{H}_2)$ There exist $c_1 > 0$ and a nonnegative function $d \in L^{\frac{p(x)}{p(x)-1}}(Q)$, such that for any $t \in \mathbb{R}$, one has

$$|k(x,t)| \leq c_1\left(d(x) + |t|^{p(x)-1}\right), \quad \text{for a.e. } x \in Q$$

$(\mathbf{H}_3)$ For any $s, t \in \mathbb{R}$, we have

$$(k(x,t) - k(x,s))(t-s) \geq 0, \text{ for a.e. } x \in Q$$

$(\mathbf{H}_4)$ There exist $c_2 \geq 1$ with

$$c_2|t|^{p(x)} \leq k(x,t)t \leq p(x)K(x,t), \quad \text{for a.e. } x \in Q, \text{ and all } s, t \in \mathbb{R},$$

where $K: \overline{Q} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$K(x,t) = \int_0^t k(x,s)ds$$

$(\mathbf{H}_5)$ The mapping K is $p(x)$-uniformly convex and there exists a constant $k_0 > 0$ such that

$$K\left(x,\frac{\xi+\psi}{2}\right)\leq\frac{1}{2}K(x,\xi)+\frac{1}{2}K(x,\psi)-k_0|\xi-\psi|^{p(x)},\quad \text{for all } x\in Q \text{ and } \xi,\psi\in\mathbb{R}^N$$

$(\mathbf{H}_5)$ $K(x,-\zeta)=K(x,\zeta)$, for all $x\in Q$ and $\zeta\in\mathbb{R}^N$.

Our results are summarized as follows:

Theorem 1.1. Suppose that the conditions $(\mathbf{H}_1)-(\mathbf{H}_5)$ hold, moreover suppose that: $(\mathbf{f}_0) f: Q\times\mathbb{R}\to\mathbb{R}$ such that $f(x,0)\neq 0$ and

$$|f(x,s)|\leq a(x)+\alpha|s|^{\sigma(x)-1},\quad (x,s)\in Q\times\mathbb{R}$$

with $a(x)\geq 0, a\in L^{\frac{q(x)}{q(x)-1}}(Q), \alpha\geq 0, q\in C_+(\overline{Q}), \sigma(x)<p_2^*(x)$ and $1\leq\sigma<p^-$. Then problem (1.1) has at least one solution (in the weak sense).

Theorem 1.2. Suppose that the conditions $(\mathbf{H}_1)$-$(\mathbf{H}_5)$ hold, $f: Q\times\mathbb{R}\to\mathbb{R}$ satisfy $(\mathbf{f_1})$ $|f(x,t)|\leq a(x)+\alpha|t|^{q(x)-1}$ for all $x\in Q$, whith $a\in L^{\frac{q(x)}{q(x)-1}}(Q), \alpha\geq 0$ and $p(x)<q(x)<p_2^*(x)$.

$(\mathbf{f}_2)$ Suppose that there is a function $\theta(x)>p(x)$ and a positive constant $M>0$ satisfying the following inequality

$$0<\theta(x)F(x,t)\leq tf(x,t), \text{ for each } t\in\mathbb{R} \text{ and } |t|\geq M$$

So, one has a $\lambda^*>0$ such that for any $\lambda\in]0,\lambda^*[$, problem (1.1) has at least two distinct weak solutions.

Theorem 1.3. Assume that the conditions $(\mathbf{H}_1)-(\mathbf{H}_5)$ hold and that f satisfies $(\mathbf{f}_2)$ and $(\mathbf{f}_1')$ $\lim_{t\to 0}\frac{f(x,t)}{a(x)|t|^{p+1}}=0$, for $x\in Q$ uniformly, where $a\in L^{\frac{p(x)}{p(x)-1}}(Q)$.

$(\mathbf{f}_2')$ $\lim_{t\to\infty}\frac{f(x,t)}{|t|^{q(x)-1}}=0$, for $x\in Q$ uniformly. If $q^->p^+$, problem (1.1) admits a nontrivial weak solution.

Theorem 1.4. Assume that conditions $(\mathbf{H}_4),(\mathbf{H}_5),(\mathbf{f}_2)$ are satisfied and $q^->p^+$. Moreover, assume
that
$(\mathbf{f}_3)$ $f(x,-t)=-f(x,t)$, for $(x,t)\in Q\times\mathbb{R}$.

Then, problem (1.1) admits infinite pairs of solutions. Finally, by using the critical point theory in the calculus of variations, a result of existence and multiplicity is developed, and precisely for the case when $(p()-1.)-sublinearatinfinity$.

Theorem 1.5. Assume that conditions $(\mathbf{H}_1)$ to $(\mathbf{H}_4)$ are satisfied. Also, assume the following
$(\mathbf{f}_4)$ $f\in L^\infty(Q\times[-t_1,t_1])$, for any $t_1\in\mathbb{R}_+$.
$(\mathbf{f_5})$ $\lim_{|t|\to\infty}\frac{f(x,t)}{|t|^{p^--1}}=0$, uniformly for $x\in Q$.
$(\mathbf{f_6})$ There exists a constant $t_0>0$ and a ball B such that $\bar{B}\subset Q$ and $0<\int_B F(x,t_0)dx$.

Then there is $\lambda_* > 0$ such that problem (1.1) admits at least one nontrivial weak solution for any $\lambda > \lambda_*$. In addition, if $(\mathbf{f}_7)$ $\lim_{|t|\to 0} \frac{f(x,t)}{|t|^{p^+-1}} = 0$, uniformly for $x \in Q$, with $p^+ < p_2^*(x)$, for all $x \in \overline{Q}$.

The problem (1.1) has at least two nontrivial weak solutions for every $\lambda > \lambda_*$.

2. PRELIMINARIES

In what follows, we recall some properties and definitions of variable-exponent Sobolev spaces. The reader can refer to the works in [23-26] for more understanding.

Let

$$C_+(\overline{Q}) := \{r : r \in C(\overline{Q}), r(x) > 1, \text{ for all } x \in \overline{Q}\}$$

Let $p \in C_+(\overline{Q})$, with

$$1 < p^- := \min_{x\in\overline{Q}} p(x) \le p(x) \le p^+ := \max_{x\in\overline{Q}} p(x) < +\infty\,. \tag{2.1}$$

We define the Lebesgue space with non-standard exponents as

$$L^{p(x)}(Q) = \left\{w : w: Q \to \mathbb{R}\,, \text{measurable} : \int_Q |w(x)|^{p(x)} dx < \infty\right\}.$$

We equip this space with the Luxembourg norm

$$|w|_{p(x)} = \inf\left\{\tau > 0 : \int_Q \left|\frac{w(x)}{\tau}\right|^{p(x)} dx \le 1\right\}$$

Note that the Lebesgue spaces with variable exponent are Banach spaces and reflexive if and only if $1 < q^- \le q^+ < \infty$. Besides, if q_1, q_2 verify $q_1(x) \le q_2(x)$ a.e. $x \in Q$, then the injection

$$L^{q_2(x)}(Q) \hookrightarrow L^{q_1(x)}(Q)$$

is compact and continuous.

Besides, for $w \in L^{p(x)}(Q)$ and $w_0 \in L^{p'(x)}(Q)$, one has

$$\left|\int_Q w w_0 dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |w|_{p(x)} |w_0|_{p'(x)}, \tag{2.2}$$

where $p'(x)$ is the conjugate function of $p(x)$.

Next, we define on the space $L^{p(x)}(Q)$ the so-called modular which is the function $\rho_{p(x)} : L^{p(x)}(Q) \to \mathbb{R}$ defined as follows

$$\rho_{p(x)}(w) := \int_Q |w|^{p(x)} dx$$

and satisfying some interesting properties needed later.

Proposition 2.1. ([24]) For all $w \in L^{p(x)}(Q)$, one has
1. $|w|_{p(x)} < 1(\text{ resp.} = 1, > 1) \Leftrightarrow \rho_{p(x)}(w) < 1(\text{ resp.} = 1, > 1)$.
2. If $|w|_{p(x)} > 1$, then we have

$$|w|_{p(x)}^{p^-} \leq \rho_{p(x)}(w) \leq |w|_{p(x)}^{p^+}$$

3. If $|w|_{p(x)} < 1$, then we have

$$|w|_{p(x)}^{p^+} \leq \rho_{p(x)}(w) \leq |w|_{p(x)}^{p^-}.$$

For any positive integer m, we define the Sobolev space with variable exponents as:

$$W^{m,p(x)}(Q) = \left\{w \in L^{p(x)}(Q), D^\alpha w \in L^{p(x)}(Q); |\alpha| \leq m\right\},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$ and $D^\alpha w = \frac{\partial^{|\alpha|} w}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_N} x_N}$. So $W^{m,p(x)}(Q)$ is a separable and reflexive Banach space equipped with the norm

$$\|w\|_{m,p(x)} = \sum_{|\alpha| \leq m} |D^\alpha w|_{p(x)}$$

The space $W_0^{m,p(x)}(Q)$ is the closure of $C_0^\infty(Q)$ in $W^{m,p(x)}(Q)$. Now since $W^{2,p(x)}(Q)$ and $W_0^{1,p(x)}(Q)$ are reflexive, separable Banach spaces, then $X = W^{2,p(x)}(Q) \cap W_0^{1,p(x)}(Q)$, satisfy the same characteristic, equipped with the norm

$$\|w\|_X = \|w\|_{W^{2,p(x)}(Q)} + \|w\|_{W_0^{1,p(x)}(Q)}$$

we also mention due to the fact (b) $b \in L^\infty(Q)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for a.e. $x \in Q$, then

$$\|w\|_b = \inf\left\{\mu > 0: \int_Q \left(\left|\frac{\Delta w}{\mu}\right|^{p(x)} + b(x)\left|\frac{w}{\mu}\right|^{p(x)}\right) dx \leq 1\right\}$$

is an equivalent norm to $\|\cdot\|_X$ on X, (see Remark 2.1 in [14]). So, in the rest of the paper, we will consider $(X, \|\cdot\|_b)$. We also define on X the function $\rho_{p(x)}: X \to \mathbb{R}$ which is called modular and defined by

$$\rho_{p(x)}^b(w) := \int_Q \left(|\Delta w|^{p(x)} + b(x)|w|^{p(x)}\right) dx.$$

and one has (see [13]).

Lemma 2.1. For $w, w_n \in X$ we have
1. $\|w\|_b < 1(\text{ resp.} = 1, > 1) \Leftrightarrow \rho_{p(x)}^b(w) < 1(\text{ resp .} = 1, > 1)$.
2. $\min\left(\|w\|_b^{p^-}, \|w\|_b^{p^+}\right) \leq \rho_{p(x)}^b(w) \leq \max\left(\|w\|_b^{p^-}, \|w\|_b^{p^+}\right)$.
3. $\|w_n\|_b \to 0(\text{ resp.} \to \infty) \Leftrightarrow \rho_{p(x)}^b(w_n) \to 0(\text{ resp.} \to \infty)$.

In what follows, we remind the definition of the critical Sobolev exponent:

$$p_2^*(x) = \begin{cases} \dfrac{Np(x)}{N-2p(x)}, & p(x) < \dfrac{N}{2} \\ +\infty, & p(x) \geq \dfrac{N}{2} \end{cases}$$

Theorem 2.1. (See Theorem 3.2 in [5]) Let $p \in C_+(\overline{Q})$ verifying

$$p(x) < \frac{N}{2}, \text{ for all } x \in \overline{Q}. \tag{2.3}$$

Let $s: \overline{Q} \longrightarrow (1, \infty)$ be a continuous function such that

$$p_2^*(x) = \frac{Np(x)}{N-2p(x)} > s(x) \geq s^- > 1, x \in \overline{Q}\,. \tag{2.4}$$

If (2.3) and (2.4) hold, there exists a constant $C = C(N, q, r, Q)$ such that

$$|g|_{s(x)} \leq C\|g\|_b, \text{ for all } g \in X$$

So, for any $s \in (1, p^*)$, the injection $X \hookrightarrow L^{s(x)}(Q)$ is compact and continuous. Proposition 2.2. (See [12]) Let p be a measurable function in $L^\infty(Q)$ and q be a measurable function such that $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in Q$. If w is a nontrivial function in $L^{q(x)}(Q)$, then

$$\min\left(|w|^{p^+}_{p(x)q(x)}, |w|^{p^-}_{p(x)q(x)}\right) \leq \left||w|^{p(x)}\right|_{q(x)} \leq \max\left(|w|^{p^+}_{p(x)q(x)}, |w|^{p^-}_{p(x)q(x)}\right)$$

In the following, we shall designate by $c, c_i, i = 1,2, \ldots$, a positive constant that may vary from line to line.

3 PREREQUISITE RESULTS

Definition 3.1. $w \in X \setminus \{0\}$ is a weak solution of (1.1) if $\Delta w = 0$ on ∂Q and

$$\int_Q \left(k(x, \Delta w)\Delta v + b(x)|w|^{p(x)-2}wv\right)dx - \lambda \int_Q f(x, w)v dx = 0, \quad \forall v \in X$$

First, let us denote by

$$J(w) = \int_Q \left(K(x, \Delta w) + \frac{b(x)}{p(x)}|w|^{p(x)}\right)dx \text{ and } \phi(w) = \int_Q F(x, w)dx$$

with $F(x,t) = \int_0^t f(x,s)ds$. The energy of problem (1.1) is defined by $\Psi_\lambda: X \to \mathbb{R}$, where

$$\Psi_\lambda(w) = J(w) - \lambda\phi(w), \forall w \in X.$$

In what follows, we recall an important result.

Theorem 3.1. (See [8]) The energy functional $J: X \to \mathbb{R}$ is sequentially weakly lower semi-continuous and of class C^1. Moreover, the mapping $J': X \to X^*$ is a strictly monotone bounded homeomorphism and is of type (S_+), that is,

$$\text{if } w_n \rightharpoonup w \text{ and } \limsup_{n\to\infty} J'(w_n)(w_n - w) \leq 0, \text{ then } w_n \to w \in X$$

Due to Theorem 3.1, we see that $J \in C^1(X, \mathbb{R})$. Besides, under assertions $(\mathbf{H}_1)$ and Proposition 2 in [7], one has $\phi \in C^1(X, \mathbb{R})$. Thus, $\Psi_\lambda \in C^1(X, \mathbb{R})$, moreover

$$< d\Psi_\lambda(w), v >= \int_Q \left(k(x, \Delta w)\Delta v + b(x)|w|^{p(x)-2} wv\right)dx - \lambda \int_Q f(x, w)v dx$$

for every $w, v \in X$. Therefore, the critical points of Ψ_λ are exactly the weak solutions of the problem (1.1).

Theorem 3.2. (Critical point theorem,[3]) Let X be a real Banach space and let $J, \phi: X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that J is bounded from below and $J(0) = \phi(0) = 0$. Fix $r > 0$ such that $\sup_{J(w)<r} \phi(w) < +\infty$ and assume that, for each $\lambda \in]0, \frac{r}{\sup_{J(w)<r}\phi(w)}[$, the functional $\Psi_\lambda := J - \lambda\phi$ satisfies (PS)-condition and it is unbounded from below. Then, for each $\lambda \in]0, \frac{r}{\sup_{J(w)<r}\phi(w)}[$, the functional Ψ_λ admits two distinct critical points.

Theorem 3.3. (Mountain-pass theorem) Let X be a Banach space and let $\Psi_\lambda \in C^1(Q, \mathbb{R})$ which fulfills the Palais-Smale condition. Suppose that $\Psi_\lambda(0) = 0$ and
1. There exist two positive real numbers η and r such that $\Psi_\lambda(w) \geq r$ with $\|w\| = \eta$,
2. There exists $w_1 \in X$ such that $\|w_1\| > \rho$ and $\Psi_\lambda(w_1) < 0$.

Let

$$\Gamma = \{g \in C([0,1], X): g(0) = 0, g(1) = w_1\}$$

and

$$\beta = \inf\{\max \Psi_\lambda(g([0,1])): g \in \Gamma\}$$

Then $\beta \geq r$ and β is a critical value of Ψ_λ. Now, let X be a reflexive and separable Banach space, then there are $\{e_j\} \in X$ and $\{e_j^*\} \in X^*$ such that

$$X = \overline{\text{span}\{e_j \mid j = 1,2,3 \ldots\}}, \quad X^* = \overline{\text{span}\{e_j^*, j = 1,2,3 \ldots\}}$$

and

$$< e_i^*, e_j >= \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

For convenience, we write $X_j = \text{span}\{e_j\}, Y_k = \oplus_{j=1}^k X_j, Z_k = \oplus_{j=k}^{+\infty} X_j$. So we have Theorem 3.4. (See Lemma 4.9 [18]) If $q(x) \in C_+(\overline{Q}), q(x) < p_2^*(x)$ for any $x \in \overline{Q}$, denote

$$\beta_k = \sup\{|w|_{q(x)}: \|w\| = 1, w \in Z_k\}.$$

Then $\lim_{k\to\infty}\beta_k = 0$.

Theorem 3.5. (Fountain theorem [3]) Assume that X is a Banach space, $\Psi_\lambda \in C^1(X,\mathbb{R})$ is an even function and X_k, Y_k and Z_k are defined as above. If there exists $\rho_k > \gamma_k > 0$ for each $k = 1,2,\ldots$ such that:
1. $\inf_{w\in Z_k,\|w\|=\gamma_k}\Psi_\lambda(w) \to \infty$ as $k \to \infty$,
2. $\max_{w\in Y_k,\|w\|=\rho_k}\Psi_\lambda(w) \le 0$,
3. Ψ_λ fulfill the condition (PS) for any $c > 0$, then Ψ_λ admits a sequence of critical values tending to $+\infty$.

4. PROOF OF THEOREMS

Proof of Theorem 1.1. Since $|f(x,s)| \le a(x) + \alpha|s|^{\sigma(x)-1}$ then

$$|F(x,s)| \le a(x)|s| + \frac{\alpha}{\sigma(x)}|s|^{\sigma(x)} \le \beta(x) + c_1|s|^{\sigma(x)}$$

where $\beta \ge 0$ and $\beta \in L^1(Q)$, it follows

$$\Psi_\lambda(w) = \int_Q \left(\Delta(k(x,\Delta w)) + \frac{b(x)}{p(x)}|w|^{p(x)}\right)dx - \lambda\int_Q F(x,w)dx$$

Using the fact that $\Delta(k(x,\Delta w)) \ge c|\Delta w|^{p(x)}$, one has for $\|w\|_b > 1$ and since $X \hookrightarrow L^{\sigma(x)}(Q)$

$$\Psi_\lambda(w) \ge \frac{c}{p^+}\|w\|_b^{p^-} - |\beta|_{L^1} - c_1\|w\|_b^{\sigma(x)}$$

So $\Psi_\lambda \to +\infty$ as $\|w\|_b \to +\infty$. Since Ψ_λ is weakly lower semi-continuous, then Ψ_λ admits a minimum point $w \in X$, so w is a weak solution of problem (1.1). The fact that $f(x,0) \ne 0$, ends the proof.

Proof of Theorem 1.2. Our objective is to apply Theorem 3.2 to problem (1.1). First, we show that Ψ_λ fulfill (PS)-condition for any $\lambda > 0$. For that, we will prove that any sequence $\{w_n\} \subset X$ such that

$$|\Psi_\lambda(w_n)| \le c, \quad \text{and} \quad \Psi'_\lambda(w_n) \to 0, \quad \text{as} \quad n \to \infty\# \tag{4.1}$$

has a convergent subsequence. We assume that $\|w_n\|_b > 1$, by using (H_4) and (f_1) we can write

$$
\begin{aligned}
c+\|w_n\|_b &\geq \Psi_\lambda(w_n)-\frac{1}{\theta}<\Psi_\lambda'(w_n),w_n> \\
&=\int_Q\left(K(x,\Delta w_n)+\frac{b(x)}{p(x)}|w_n|^{p(x)}\right)dx-\lambda\int_Q F(x,w_n)dx \\
&-\frac{1}{\theta}\left(\int_Q\left(k(x,\Delta w_n)\Delta w_n+b(x)|w_n|^{p(x)}\right)dx-\lambda\int_Q f(x,w_n)w_n dx\right) \\
&\geq \int_Q\left(c_2\frac{|\Delta w_n|^{p(x)}}{p(x)}+\frac{b(x)}{p(x)}|w_n|^{p(x)}\right)dx-\lambda\int_Q F(x,w_n)dx \\
&-\frac{1}{\theta}\left(\int_Q\left(c_2|\Delta w_n|^{p(x)}+b(x)|w_n|^{p(x)}\right)dx-\lambda\int_Q f(x,w_n)w_n dx\right) \\
&\geq c_2\left(\frac{1}{p^+}-\frac{1}{\theta}\right)\int_Q\left(|\Delta w_n|^{p(x)}+b(x)|w_n|^{p(x)}\right)dx \\
&+\lambda\left(\int_Q\left(\frac{1}{\theta}f(x,w_n)w_n-F(x,w_n)\right)dx\right. \\
&\geq c_2\left(\frac{1}{p^+}-\frac{1}{\theta}\right)\|w_n\|_b^{p^-}.
\end{aligned}
$$

Since $p^- > 1$, we obtain a contradiction. So $\{w_n\}$ is bounded in X. Thus, up to a subsequence $w_n \rightharpoonup w$ in X and due to the compact embedding $X \to L^{q(x)}(Q)$, we have

$$w_n \to w \text{ in } L^{q(x)}(Q)\# \tag{4.2}$$

From (4.1), we have

$$<\Psi_\lambda'(w_n), w_n - w> \to 0$$

so,

$$<\Psi_\lambda'(w_n),w_n-w>=\int_Q\left(k(x,\Delta w_n)(\Delta w_n-\Delta w)+b(x)|w_n|^{p(x)}(w_n-w)\right)dx-\lambda\int_Q f(x,w_n)(w_n-w)dx \to 0.$$

By (f_1), we get

$$\left|\int_Q f(x,w_n)(w_n-w)dx\right| \leq |a(x)|_{\frac{q(x)}{q(x)-1}}|w_n-w|_{q(x)}+\alpha\left||w_n|^{q(x)-1}\right|_{\frac{q(x)}{q(x)-1}}|w_n-w|_{q(x)}.$$

By using (4.2), we obtain

$$\int_Q f(x,w_n)(w_n-w)dx \to 0.$$

Then, we have

$$\int_Q\left(k(x,\Delta w_n)(\Delta w_n-\Delta w)+b(x)|w_n|^{p(x)-2}(w_n-w)\right)dx \to 0$$

so,

$$\limsup_{n\to\infty} J'(w_n)(w_n - w) \leq 0$$

According to Theorem 3.1, one has $w_n \to w$ in X and so Ψ_λ meets (PS)-condition. From (f_2), we have

$$F(x,t) \geq C|t|^\theta, \quad \text{for all } |t| \geq M, \text{ a.e. } x \in Q\# \tag{4.3}$$

So, for any fixed $w_0 \in X \setminus \{0\}, t > 1$ and assertion $(\mathbf{H}_2)$, we have

$$\begin{aligned}\Psi_\lambda(tw_0) &= \int_Q \left(K(x, t\Delta w_0) + \frac{b(x)}{p(x)}|tw_0|^{p(x)}\right)dx - \lambda\int_Q F(x, tw_0)dx \\ &\leq 2t^{p^+}c_3|d|_{p'(.)}|\Delta w_0|_{p(.)} + \frac{t^{p^+}c_3}{p^-}\int_Q \left(|\Delta w_0|^{p(x)} + b(x)|w_0|^{p(x)}\right)dx - \lambda C t^\theta \int_Q |w_0|^\theta dx \\ &\leq 2t^{p^+}c_3|d|_{p'(.)}|\Delta w_0|_{p(.)} + \frac{t^{p^+}c_3}{p^-}\|w_0\|_b^{p^+} - \lambda C t^\theta \int_Q |w_0|^\theta dx.\end{aligned}$$

Since $\theta > p^+$, the functional Ψ_λ is unbounded from below. Fix $\lambda \in (0, \lambda^*)$, using condition (H_2), one has

$$\frac{1}{p^+}\|w\|_b^{p^+} \leq \frac{1}{p^+}\rho_{p(x)}^b(w) \leq J(w)$$

for every $w \in X$ such that $w \in J^{-1}(]-\infty, 1[)$. It follows that

$$\|w\|_b < (p^+ r)^{\frac{1}{p^+}}$$

Moreover, the compact imbedding $X \hookrightarrow L^{q(x)}(Q)$, imply that for all $w \in J^{-1}(]-\infty, 1[)$, one has

$$\begin{aligned}\Phi(w) &\leq C|a|_{\frac{q(x)}{q(x)-1}}\|w\|_b + \frac{\alpha}{q^-}\max\left(C^{q^+}\|w\|_b^{q^+}, C^{q^-}\|w\|_b^{q^-}\right) \\ &< C|a|_{\frac{q(x)}{q(x)-1}}(p^+ r)^{\frac{1}{p^+}} + \frac{\alpha}{q^-}\max\left(C^{q^+}(p^+ r)^{\frac{q^+}{p^+}}, C^{q^-}(p^+ r)^{\frac{q^-}{p^+}}\right)\end{aligned}$$

choosing $r = 1$, one has

$$\sup_{J(w)<1}\Phi(w) \leq C|a|_{\frac{q(x)}{q(x)-1}}(p^+)^{\frac{1}{p^+}} + \frac{\alpha}{q^-}\max\left(C^{q^+}(p^+)^{\frac{q^+}{p^+}}, C^{q^-}(p^+)^{\frac{q^-}{p^+}}\right) = \frac{1}{\lambda^*} < \frac{1}{\lambda}\# \tag{4.4}$$

From (4.4), one has

$$\lambda \in]0, \lambda^*[\subseteq]0, \frac{1}{\sup_{J(w)<1}\Phi(w)}[$$

So all assertions of Theorem (1.2) hold, then for each $\lambda \in]0,\lambda^*[\Psi_\lambda$ admits two distinct critical points which represent the weak solutions of problem (1.1).

Proof of Theorem 1.3. Since $p^+ < q^- < q(x) < p_2^*(x)$ and by Theorem 2.1, the embeddings $X \hookrightarrow L^{p^+}(Q)$ and $X \hookrightarrow L^{q(x)}(Q)$ are continuous, there exist $c_4, c_5 > 0$ with

$$|w|_{q(x)} \le c_4 \|w\|_b \quad |w|_{p^+} \le c_5 \|w\|_b \quad \text{for all } w \in X.\# \tag{4.5}$$

Let $\epsilon > 0$ small enough with $\lambda \frac{c_5^{p^+}\epsilon}{p^+}|a|_{\underline{p^+}} \le \frac{1}{2p^+-1}$. Using (f'_1) and (f'_2), we get

$$F(x,t) \le \epsilon \frac{a(x)}{p^+}|t|^{p^+} + \frac{\alpha}{q(x)}|t|^{q(x)} \quad \text{for all } (x,t) \in Q \times \mathbb{R}\# \tag{4.6}$$

Let $\rho \in (0,1)$ and $w \in X$ be such that $\|w\| = \rho$. By considering Proposition (2.1) and relations (4.5)-(4.6), we deduce that

$$\begin{aligned}\Psi_\lambda(w) &\ge \frac{c_2}{p^+}\|w\|_b^{p^+} - \lambda \frac{\epsilon}{p^+}\int_Q a(x)|w|^{p^+}dx - \lambda \frac{\alpha}{q^-}\int_Q \alpha |w|^{q(x)}dx \\ &\ge \frac{c_2}{p^+}\|w\|_b^{p^+} - \lambda \frac{c_5^{p^+}\epsilon}{p^+}|a|_{\frac{p^+}{p^+-1}}\|w\|_b^{p^+} - \lambda \frac{\alpha}{q^-}c_4^{q^-}\|w\|_b^{q^-} \\ &\ge \frac{c_2}{2p^+}\|w\|_b^{p^+} - \lambda c_6 \|w\|_b^{q^-} \\ &\ge \left(\frac{c_2}{2p^+} - \lambda c_6 \|w\|_b^{q^- - p^+}\right)\|w\|_b^{p^+}.\end{aligned}$$

Let $h_\lambda(s) = \frac{c_2}{2p^+} - \lambda c_6 s^{q^- - p^+}, t > 0$. It is not difficult to see that $h_\lambda(s) > 0$ for all $s \in (0, s_1)$, for $s_1 = \left(\frac{c_2}{2p^+\lambda c_6}\right)^{\frac{1}{q^- - p^+}}$. So, for all $\lambda > 0$ we can choose $\eta, r > 0$ such that

$$\Psi_\lambda(w) \ge r > 0, \text{ for all } w \in X \text{ with } \|w\| = \eta \in (0,1).$$

In view of (f_2) we have for all $|t| \ge M$

$$F(x,t) \ge C|t|^\theta \quad \text{a.e. } x \in Q\# \tag{4.7}$$

Let $u \in X \setminus \{0\}$ and $t > 1$. By (4.7) one has

$$\begin{aligned}\Psi_\lambda(tu) &= \int_Q \left(K(x,\Delta tu) + \frac{b(x)}{p(x)}|tu|^{p(x)}\right)dx - \lambda \int_Q F(x,tu)dx \\ &\le t^{p^+}\int_Q \left(K(x,\Delta u) + \frac{b(x)}{p(x)}|u|^{p(x)}\right)dx - \lambda C t^\theta \int_Q |u|^\theta dx\end{aligned}$$

Since $\theta > p^+$, we have $\Psi_\lambda(tu) \to -\infty (t \to +\infty)$. So, for $t > 1$ and large enough, we can choose $u_1 = tu$ such that $\|u_1\| > \rho$ and $\Psi_\lambda(u_1) < 0$.

5. CONCLUSIONS

Since $\Psi_\lambda(0) = 0$ and Ψ_λ satisfy the Palais-Smale condition, all conditions of Theorem (3.3) are fulfilled. Consequently, Ψ_λ has at least one nontrivial critical point which is a nontrivial weak solution.

In what follows, using the Fountain theorem, we will establish the existence of infinitely many pairs of weak solutions for (1.1).

Proof of Theorem 1.4. According to $(\mathbf{H}_5)$ and $(\mathbf{f}_3)$, Ψ_λ is even and satisfy (PS) condition. We shall prove that if k is large enough, then there exist $\rho_k > \gamma_k > 0$ such that (A) $b_k := \inf\{\Psi_\lambda(w): w \in Z_k, \|w\| = \gamma_k\} \to \infty$ as $k \to \infty$, (B) $a_k := \max\{\Psi_\lambda(w): w \in Y_k, \|w\| = \rho_k\} \leq 0$.

The assertion of Theorem 1.4 can be obtained from Fountain Theorem (see Theorem 3.5).

(A) For any $w \in Z_k$ with $\|w\| = \gamma_k > 1$, we have

$$\begin{aligned}\Psi_\lambda(w) &\geq \frac{c_2}{p^+}\|w\|_b^{p^-} - \lambda\alpha\int_Q \frac{|w|^{q(x)}}{q(x)}dx - \lambda\int_Q a(x)|w|dx \\ &\geq \frac{c_2}{p^+}\|w\|_b^{p^-} - \frac{\lambda}{q^-}\alpha\int_Q |w|^{q(x)}dx - \|A\|_{L^1} \\ &\geq \frac{c_2}{p^+}\|w\|_b^{p^-} - \frac{\lambda}{q^-}\alpha\int_Q |w|^{q(x)}dx - c_7\end{aligned}$$

If $|w|_{q(x)} \leq 1$ then $\int_Q |w|^{q(x)}dx \leq |w|_{q(x)}^{q^-} \leq 1$. So

$$\Psi_\lambda(w) \geq \frac{c_2}{p^+}\|w\|_b^{p^-} - (c_8 + c_7)\#(4.8)$$

However, if $|w|_{q(x)} > 1$ then $\int_Q |w|^{q(x)}dx \leq |w|_{q(x)}^{q^+} \leq (\beta_k\|w\|_b)^{q+}$ and

$$\Psi_\lambda(w) \geq \frac{c_2}{p^+}\|w\|_b^{p^-} - c_8(\beta_k\|w\|_b)^{q+} - c_7\# \tag{4.9}$$

From (4.8) and (4.9), we deduce that

$$\Psi_\lambda(w) \geq \frac{c_2}{p^+}\|w\|_b^{p^-} - c_9(\beta_k\|w\|_b)^{q+} - c_{10}$$

Choose $\gamma_k = \left(\frac{c_9 q^+}{c_2}\beta_k^{q^+}\right)^{\frac{1}{p^- - q^+}}$. Then $\gamma_k \to \infty$ since $p^- < q^+$ and $\beta_k \to 0$ as $k \to \infty$. Hence, if $w \in Z_k$ and $\|w\|_b = \gamma_k$, we deduce that

$$\Psi_\lambda(w) \geq c_2\left(\frac{1}{p^+} - \frac{1}{q^+}\right)\gamma_k^{p^-} - c_{10} \to 0 \text{ as } k \to \infty$$

which implies (A).

To show (B), let $w \in Y_k$ with $\|w\|_b > 1$ and from (f_2), we write

$$\Psi_\lambda(w) \leq \frac{1}{p^-}\|w\|_b^{p^+} - \lambda\int_Q F(x,w)dx$$
$$\leq \frac{c_2}{p^-}\|w\|_b^{p^+} - \lambda C\int_Q |w|^\theta dx$$

Since $\dim Y_k < \infty$ (which implies all norms are equivalent) and $\theta > p^+$, we get that

$$\Psi_\lambda(w) \leq \frac{c_2}{p^-}\|w\|_b^{p^+} - \lambda C\|w\|_b^\theta \to -\infty \text{ as } \|w\|_b \to \infty$$

Thus, we can choose $\rho_k > \gamma_k > 0$ such that

$$\max_{w\in Y_k, \|w\|_b=\rho_k} \Psi_\lambda(w) \leq 0$$

Apply Theorem 3.5 to $\Psi_\lambda \in C^1(X,\mathbb{R})$, so there is a sequence of critical values of Ψ_λ converging to $+\infty$. As a consequence, there is a sequence $(\pm w_n)_{n\in\mathbb{N}}$ of critical points for Ψ_λ such that $\Psi_\lambda(\pm w_n) \to +\infty$ as $n \to \infty$. This ends the proof.

Proof of Theorem 1.5. By using (f_5), we can write that for each $\epsilon > 0$, there exists $t_2 = t_2(\epsilon) > 0$ verifying

$$|f(x,t)| \leq \epsilon|t|^{p^- -1}, \text{ for a.e. } x \in Q \text{ and any } |t| > t_2$$

If we consider this relation and (f_4), we obtain

$$|f(x,t)| \leq C(\epsilon) + \epsilon|t|^{p^- -1}, \text{ for a.e. } x \in Q \text{ and any } t \in \mathbb{R}\# \tag{4.10}$$

where $C(\epsilon) = \operatorname*{esssup}_{x\in Q, |t|\leq t_2} |f(x,t)|$. Moreover, one has

$$|F(x,t)| \leq C(\epsilon)|t| + \frac{\epsilon}{p^-}|t|^{p^-}, \text{ for a.e. } x \in Q \text{ and all } t \in \mathbb{R}.\# \tag{4.11}$$

The continuous embedding $X \hookrightarrow L^{p^-}(Q)$ and $X \hookrightarrow L^1(Q)$ assures, the existence of $C_7, C_8 > 0$ verifying

$$|w|_{p^-} \leq C_7\|w\|_b, \quad |w|_{L^1(Q)} \leq C_8\|w\|_b, \quad \text{for all } w \in X$$

Then, by (H_4) and relation (4.11), for $w \in X$, we have

$$\Psi_\lambda(w) \geq \frac{c_2}{p^+}\|w\|_b^{p^-} - \lambda C(\epsilon)\int_Q |w|dx - \lambda\frac{\epsilon}{p^-}\int_Q |w|^{p^-}dx$$
$$\geq \left(\frac{c_2}{p^+} - \frac{\lambda\epsilon C_7^{p^-}}{p^-}\right)\|w\|_b^{p^-} - \lambda C(\epsilon)C_8\|w\|.$$

Thus, choosing $\epsilon > 0$ such that $\frac{c_2}{p^+} - \frac{\lambda \epsilon C_7^{p^-}}{p^-} > 0$, since $p^- > 1$ we obtain that Ψ_λ is coercive, and since Ψ_λ is sequentially weakly lower semicontinuous. Therefore, Ψ_λ admits a global minimizer w_1 on X.

In what follows, and in order to end our proof, let us show the existence $w_\epsilon \in X$ such that $\Psi_\lambda(w_\epsilon) < 0$. For any $\epsilon > 0$, put $B_\epsilon = \{x \in Q, \mathrm{dist}(x, B) \leq \epsilon\}$ where B is defined in assertion (f_6). Get $\epsilon > 0$ small enough such that $\overline{B_\epsilon} \in Q$. So there is $w_\epsilon \in C_c^1(Q)$ such that

$$w_\epsilon(x) := \begin{cases} t_0, & x \in B, \\ 0, & x \in Q \setminus B_\epsilon, \end{cases}$$

and $0 \leq w_\epsilon(x) \leq t_0, \forall x \in Q$, where t_0 is given in (f_6). Thus, $w_\epsilon \in X$ and for a.e. $x \in Q$,

$$|F(x, w_\epsilon)| \leq \int_0^{w_\epsilon(x)} |f(x,s)| ds \leq |f|_{L^\infty(Q \times [-t_0, t_0])} w_\epsilon(x) \leq t_0 |f|_{L^\infty(Q \times [-t_0, t_0])}$$

Thus, we estimate

$$\begin{aligned} \Psi_\lambda(w_\epsilon) &= \int_Q \left(K(x, \Delta w_\epsilon) + \frac{b(x)}{p(x)} |w_\epsilon|^{p(x)} \right) dx - \lambda \int_Q F(x, w_\epsilon) dx \\ &= \int_Q \left(K(x, \Delta w_\epsilon) + \frac{b(x)}{p(x)} |w_\epsilon|^{p(x)} \right) dx - \lambda \int_B F(x, t_0) dx - \lambda \int_{B_\epsilon \setminus B} F(x, w_\epsilon(x)) dx \\ &\leq \int_Q \left(K(x, \Delta w_\epsilon) + \frac{b(x)}{p(x)} |w_\epsilon|^{p(x)} \right) dx - \lambda \left(\int_B F(x, t_0) dx - t_0 |f|_{L^\infty(Q \times [-t_0, t_0])} |B_\epsilon \setminus B| \right), \end{aligned}$$

Let $\epsilon > 0$ small enough to assure

$$t_0 |f|_{L^\infty(Q \times [-t_0, t_0])} |B_\epsilon \setminus B| \leq \frac{1}{2} \int_B F(x, t_0) dx.$$

so

$$\Psi_\lambda(w_\epsilon) \leq \int_Q \left(K(x, \Delta w_\epsilon) + \frac{b(x)}{p(x)} |w_\epsilon|^{p(x)} \right) dx - \frac{\lambda}{2} \int_B F(x, t_0) dx.$$

Then $\Psi_\lambda(w_\epsilon) < 0$ for any $\lambda > \lambda_*$, with

$$\lambda_* := \frac{2 \int_Q \left(K(x, \Delta w_\epsilon) + \frac{b(x)}{p(x)} |w_\epsilon|^{p(x)} \right) dx}{\int_B F(x, t_0) dx}.$$

Therefore, for each $\lambda > \lambda_*$, one has $\Psi_\lambda(w_1) < 0 = \Psi_\lambda(0)$. So, for each $\lambda > \lambda_*, w_1$ is a weak solution of problem (1.1). Now, suppose that the condition (f_7) is fulfilled. We recall also that the fact that Ψ_λ is coercive and Ψ'_λ of type (S_+) assures that Ψ_λ fulfill the (PS) properties. So Ψ_λ has a mountain pass geometry. Now, the fact that $p^+ < p_2^*(x)$ for all $x \in \overline{Q}$, assures the existence of q such that $p^+ < q < p_2^*(x)$ for all $x \in \overline{Q}$. Consequently $X \hookrightarrow L^q \hookrightarrow L^{p^+}$, then there exists $C_{p^+}, C_q > 0$ such that

$$|w|_{p^+} \leq C_{p^+} \|w\|_b, |w|_q \leq C_q \|w\|_b \quad \text{for all } w \in X$$

From (f_5) and (f_7) we have that for $u = \frac{1}{2\lambda C_{p^+}^{p^+} p^+}$ there exist $\mu_1 > 0$ and $\mu_2 > 0$ verifying

$$|f(x,t)| \leq u|t|^{p^- -1} \leq u|t|^{q-1}, \text{ for a.e. } x \in Q \text{ and all } |t| > \mu_1$$

and

$$|f(x,t)| \leq p^+ u|t|^{p^+ -1}, \text{ for a.e. } x \in Q \text{ and all } |t| < \mu_2$$

these inequalities together with (f_4) assert that

$$|F(x,t)| \leq C_9|t|^q + u|t|^{p^+}, \text{ for a.e. } x \in Q \text{ and all } t \in \mathbb{R}.$$

So, for each $w \in X$ with $\|w\|_b < 1$, one has

$$\begin{aligned}\Psi_\lambda(w) &\geq \frac{c_2}{p^+}\|w\|_b^{p^+} - \lambda u \int_Q |w|^{p^+} dx - \lambda C_9 \int_Q |w|^q dx \\ &\geq \frac{c_2}{p^+}\|w\|_b^{p^+} - \lambda u C_{p^+}^{p^+}\|w\|_b^{p^+} - \lambda C_9 C_q^q \|w\|_b^q \\ &= \frac{c_2}{2p^+}\|w\|_b^{p^+} - \lambda C_9 C_q^q \|w\|_b^q.\end{aligned}$$

We choose $0 < r < min\left\{1, \|w_1\|, \left(\frac{c_2}{2p^+\lambda C_9 C_q^q}\right)^{\frac{1}{q-p^+}}\right\}$ and letting $\rho = \frac{c_2}{2p^+} r^{p^+} - \lambda C_9 C_q^q r^q$, we deduce that

$$\Psi_\lambda(w) \geq \rho, \forall w \in X \text{ with } \|w\|_b = r$$

Consequently, w_2, is a critical point of Ψ_λ has the second critical point w_2, so it will also be a weak solution to (1.1). moreover, one has $\Psi_\lambda(w_2) \geq \rho > 0 = \Psi_\lambda(0)$, so $w_2 \neq w_1$ and $w_2 \neq 0$.

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