***ORIGINAL PAPER***

# EARTHQUAKE P-FUNCTIONS WITH SOME RELATED INEQUALITIES AND THEIR APPLICATIONS

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***Abstract.*** *As researchers, our primary commitment is to engage in the creation of scientific and academic knowledge. Recognizing the importance of societal impact, we also strive to contribute to social awareness through our work. Natural disasters, such as floods, earthquakes, avalanches, landslides, tsunamis, and others, frequently afflict regions globally, resulting in profound human losses and extensive material damage. Notably, the earthquake that struck our country on February 6, 2023, marked as the disaster of the century, deeply affected eleven provinces, leading to a tragic loss of tens of thousands of lives. In this paper, we investigate the concept of "earthquake P-functions". We establish the Hermite-Hadamard inequality for this class of functions and derive several refinements of the Hermite-Hadamard type inequality for functions whose first derivative, in absolute value, at a certain power, is an earthquake P-function. We present some applications of the trapezoidal formula. We also provide new bounds for special means of different non-negative real numbers.*

***Keywords:*** *Convex function; earthquake P-function; Hermite-Hadamard inequality; integral inequalities.*

## 1. INTRODUCTION

Unfortunately, Turkey experienced two significant seismic events on February 6, 2023, causing severe repercussions for eleven provinces. This unfortunate occurrence led to the tragic loss of tens of thousands of our fellow citizens and the destruction of thousands of structures. Throughout history, similar calamities have befallen various regions globally, and regrettably, they persist. To raise awareness among researchers reading our paper, we introduce a novel class of convex functions, which we term 'earthquake P-functions'.

Recent years have witnessed extensive exploration of the theory of convex functions and inequalities by numerous researchers. This exploration is driven by the recognition of its significance in diverse fields such as mathematics, engineering, physics, optimization theory, management sciences, mathematical finance, economics, algorithms, statistics, etc. [1–5].

A function $\Theta\colon \mathfrak{I} \subset \mathbb{R} \to \mathbb{R}$ is called convex if

$$\Theta(t\varkappa + (1-t)y) \le t\Theta(\varkappa) + (1-t)\Theta(y)$$

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is valid for all $\varkappa, y \in \Im$ and $t \in [0,1]$. If this inequality holds in the reverse direction, then $\Theta$ is called concave on $\Im \neq \emptyset$.

The theory of convexity offers robust principles and methodologies for investigating an extensive range of problems in both theoretical and practical branches of mathematics [6–14]. The theory of convexity and the theory of inequalities are closely related. By employing convexity, several extensions and generalizations of integral inequalities were discovered along with their useful applications.

Among the pivotal discoveries in convexity theory is the Hermite–Hadamard (H–H) inequality, unveiled by Hermite and Hadamard. This inequality stands as one of the cornerstones in the broader framework of convexity and inequalities, contributing significantly to various scientific disciplines. It is stated as follows ([15–16]):

Let $\Theta: \Im \to \mathbb{R}$ be a convex function. Then

$$\Theta\left(\frac{\nu+\omega}{2}\right) \leq \frac{1}{\omega-\nu}\int_{\nu}^{\omega} \Theta(\varkappa)d\varkappa \leq \frac{\Theta(\nu)+\Theta(\omega)}{2} \tag{1}$$

for all $\nu, \omega \in \Im$ with $\nu < \omega$. If the double inequality in (1) holds in the reversed direction, then the function $\Theta$ is concave. For some results that enhance, expand, and improve the H–H inequality [17–28].

In [29], Dragomir et al. defined the class of $P$-functions and established the related H–H inequality as follows:

**Definition 1.** A non-negative function $\Theta: \Im \subseteq \mathbb{R} \to \mathbb{R}$ is called P-function if

$$\Theta(t\nu + (1-t)\omega) \leq \Theta(\nu) + \Theta(\omega)$$

for all $\nu, \omega \in \Im$ and $t \in [0,1]$.

**Theorem 1.** Let $\Theta$ be a P-function on $\Im$, $\nu, \omega \in \Im$ with $\nu < \omega$ and $\Theta \in L[\nu, \omega]$. Then

$$\Theta\left(\frac{\nu+\omega}{2}\right) \leq \frac{2}{\omega-\nu}\int_{\nu}^{\omega} \Theta(\varkappa)d\varkappa \leq 2[\Theta(\nu)+\Theta(\omega)].$$

In [30], Kadakal et al. introduced the notion of earthquake convex functions as follows:

**Definition 2.** A function $\Theta: \Im \subseteq \mathbb{R} \to \mathbb{R}$ is called an earthquake convex if

$$\Theta(t\nu + (1-t)\omega) \leq te^{1-t}\Theta(\nu) + (1-t)e^{t}\Theta(\omega)$$

for all $\nu, \omega \in \Im$ and $t \in [0,1]$.

Motivated by the above results and literature, this work is organized as follows:

In Section 2, a new concept called the earthquake $P$-function is introduced, and some algebraic properties of this new class of functions are presented. The connections of earthquake $P$-functions with other types of convex functions are also examined. In Section 3, the H–H inequality for earthquake $P$-functions is established. In Section 4, some refinements of the H–H type inequality for functions whose first derivative in absolute value at a certain power is earthquake $P$-functionare obtained. In Section 5, some applications of the obtained results of the trapezoidal formula are presented. Section 6 provides new bounds for special means of different non-negative real numbers. Section 7 is devoted to the conclusion.

## 2. THE DEFINITION OF EARTHQUAKE P-FUNCTIONS

In this section, a new class of functions called earthquake $P$-functions is introduced. The relations of our newly defined class of functions with some other classes of functions are also given.

**Definition 3.** A function $\Theta\colon \Im \subseteq \mathbb{R} \to \mathbb{R}$ is called earthquake P-function if

$$\Theta(t\nu + (1-t)\omega) \leq (te^{1-t} + (1-t)e^{t})[\Theta(\nu) + \Theta(\omega)] \tag{2}$$

for all $\nu, \omega \in \Im$ and $t \in [0,1]$.

We will denote by $EQP(\Im)$ the class of all earthquakes $P$-functions on $\Im$. Note that every earthquake $P$-function is an $h$-convex function with the function $h(t) = te^{1-t} + (1-t)e^{t}$. Therefore, if $\Theta, \Phi \in EQP(\Im)$, then
i) $\Theta + \Phi \in EQP(\Im)$ and $\alpha\Theta \in EQP(\Im)$ for $\alpha \in \mathbb{R}$ $(\alpha \geq 0)$ (see [31], Proposition 9).
ii) If $\Theta$ and $\Phi$ are similarly ordered on $\Im$, then $\Theta\Phi \in EQP(\Im)$ (see [31], Proposition 10).

Also, if $\Theta\colon \Im \to I$ is convex and $\Phi \in EQP(I)$ and is nondecreasing, then $\Phi \circ \Theta$ $EQP(\Im)$ (see [31], Theorem 15).

**Remark 1.** Every earthquake P-function is a non-negative function. Indeed, if $\Theta\colon \Im \subseteq \mathbb{R} \to \mathbb{R}$ is an earthquake P-function, then for $t = 0$ and $\nu = \omega$, we get

$$0 \leq 2\Theta(\nu)$$

for all $\nu \in \Im$. So, we have $\Theta(\nu) \geq 0$ for all $\nu \in \Im$.

**Proposition 1.** Every earthquake convex function is an earthquake P-function.

*Proof:* Let $\Theta\colon \Im \subseteq \mathbb{R} \to \mathbb{R}$ be an arbitrary earthquake convex function. Then $\Theta$ is non-negative, and the inequality

$$\Theta(t\nu + (1-t)\omega) \leq te^{1-t}\Theta(\nu) + (1-t)e^{t}\Theta(\omega)$$

for all $\nu, \omega \in \Im$ and $t \in [0,1]$. From the inequalities $\Theta(\nu) \leq \Theta(\nu) + \Theta(\omega)$ and $\Theta(\omega) \leq \Theta(\nu) + \Theta(\omega)$, we get the desired result.

**Proposition 2.** Every P-function is an earthquake P-function.

*Proof:* The proof is obvious from the inequalities

$$t \leq te^{1-t} \text{ \textbackslash and } 1-t \leq (1-t)e^{t}$$

for all $t \in [0,1]$. So, one has

$$1 \leq te^{1-t} + (1-t)e^{t},$$

which completes the proof.

**Corollary 1.** If $\Theta: \mathfrak{I} \subseteq \mathbb{R} \to \mathbb{R}$ is a non-negative convex function, then $\Theta$ is an earthquake P-function.

Hence, we can provide the following examples:

**Example 1.** The function $\Psi: [0, \infty) \to \mathbb{R}$, $\Psi(\varkappa) = \varkappa$ is an earthquake P-function.

**Example 2.** The function $\Omega: [0, \infty) \to \mathbb{R}$, $\Omega(\varkappa) = \varkappa^{\mathrm{n}}, \mathrm{n} \geq 1$ is an earthquake P-function.

**Example 3.** The function $\varphi: \mathbb{R} \to \mathbb{R}, \varphi(\varkappa) = \mathrm{e}^{\varkappa}$ is an earthquake P-function.

**Example 4.** The function $\zeta: (0,1] \to \mathbb{R}, \zeta(\varkappa) = -\ln \varkappa$ is an earthquake P-function.

**Theorem 2.** Let $\omega > \nu$ and $\Theta_{\alpha}: [\nu, \omega] \to \mathbb{R}$ be an arbitrary family of earthquakes P-functions and let $\Theta(\varkappa) = \sup_{\alpha} \Theta_{\alpha}(\varkappa)$. If $\mathrm{I} = \{s \in [\nu, \omega]: \Theta(\mathrm{s}) < \infty\}$ is nonempty, then I is an interval and $\Theta$ is an earthquake P-function on I.

*Proof:* Let $\mathrm{t} \in [0,1]$ and let $\nu, \omega \in \mathrm{I}$ be arbitrary. Then

$$\begin{aligned} \Theta(t\nu + (1-t)\omega) &= \sup_{\alpha} \Theta_{\alpha}(t\nu + (1-t)\omega) \\ &\leq \sup_{\alpha} \{(te^{1-t} + (1-t)e^{t})[\Theta_{\alpha}(\nu) + \Theta_{\alpha}(\omega)]\} \\ &\leq (te^{1-t} + (1-t)e^{t}) \left[\sup_{\alpha} \Theta_{\alpha}(\nu) + \sup_{\alpha} \Theta_{\alpha}(\omega)\right] \\ &= (te^{1-t} + (1-t)e^{t})[\Theta(\nu) + \Theta(\omega)] < \infty. \end{aligned}$$

This simultaneously demonstrates that I is an interval, as it encompasses every point between any two of its points, and that $\Theta$ is an earthquake P-function on I. So, the proof is completed.

**Theorem 3.** If $\Theta: [\nu, \omega] \to \mathbb{R}$ is an earthquake P-function, then $\Theta$ is bounded on $[\nu, \omega]$.

*Proof:* Let $\mathrm{K} = \max\{\Theta(\nu), \Theta(\omega)\}$ and $\varkappa \in [\nu, \omega]$ be an arbitrary point. Then there exists $\mathrm{t} \in [0,1]$ such that $\varkappa = t\nu + (1-t)\omega$. Since $e^{1-t} \leq e$ \ and $e^{t} \leq e$, we have

$$\begin{aligned} \Theta(\varkappa) &= \Theta(t\nu + (1-t)\omega) \\ &\leq (te^{1-t} + (1-t)e^{t})[\Theta(\nu) + \Theta(\omega)] \\ &\leq 2Ke = S. \end{aligned}$$

This demonstrates that $\Theta$ is bounded above by a real number $S$.

For every $\varkappa \in [\nu, \omega]$, there exists a $\mu \in \left[0, \frac{\omega-\nu}{2}\right]$ such that either $\varkappa = \frac{\nu+\omega}{2} + \mu$ or $\varkappa = \frac{\nu+\omega}{2} - \mu$. Since it will lose nothing in generality, we suppose $\varkappa = \frac{\nu+\omega}{2} + \mu$. So, we have

$$\begin{aligned}\Theta\left(\frac{\nu+\omega}{2}\right) &= \Theta\left(\frac{1}{2}\left[\frac{\nu+\omega}{2}+\mu\right]+\frac{1}{2}\left[\frac{\nu+\omega}{2}-\mu\right]\right)\\ &\leq \sqrt{e}\left(\Theta(\varkappa)+\Theta\left(\frac{\nu+\omega}{2}-\mu\right)\right).\end{aligned}$$

By using $S$ as the upper bound, we get

$$\begin{aligned}\Theta(\varkappa) &\geq \frac{1}{\sqrt{e}}\Theta\left(\frac{\nu+\omega}{2}\right)-\Theta\left(\frac{\nu+\omega}{2}-\mu\right)\\ &\geq \frac{1}{\sqrt{e}}\Theta\left(\frac{\nu+\omega}{2}\right)-S=s.\end{aligned}$$

This completes the proof.

## 3. HERMITE-HADAMARD'S INEQUALITY FOR EARTHQUAKE P-FUNCTIONS

This section aims to derive H–H inequality for the earthquake $P$-functions. In the next sections, we denote by $L[\nu,\omega]$ the space of (Lebesgue) integrable functions on $[\nu,\omega]$.

**Theorem 4.** Let $\Theta: [\nu,\omega] \to \mathbb{R}$ be an earthquake P-function. If $\nu < \omega$ and $\Theta \in L[\nu,\omega]$, then

$$\Theta\left(\frac{\nu+\omega}{2}\right) \leq \frac{2\sqrt{e}}{\omega-\nu}\int_{\nu}^{\omega}\Theta(\varkappa)d\varkappa \leq 4\sqrt{e}(e-2)[\Theta(\nu)+\Theta(\omega)]. \tag{3}$$

*Proof:* Since $\Theta$ is an earthquake $P$-function, we have

$$\begin{aligned}\Theta\left(\frac{\nu+\omega}{2}\right) &= \Theta\left(\frac{1}{2}[t\nu+(1-t)\omega]+\frac{1}{2}[(1-t)\nu+t\omega]\right)\\ &\leq \sqrt{e}[\Theta(t\nu+(1-t)\omega)+\Theta((1-t)\nu+t\omega)].\end{aligned}$$

Now, integrating the last inequality with respect to $t \in [0,1]$, we obtain

$$\begin{aligned}\Theta\left(\frac{\nu+\omega}{2}\right) &\leq \sqrt{e}\left[\int_0^1\Theta(t\nu+(1-t)\omega)dt+\int_0^1\Theta((1-t)\nu+t\omega)dt\right]\\ &= \frac{2\sqrt{e}}{\omega-\nu}\int_{\nu}^{\omega}\Theta(\varkappa)d\varkappa,\end{aligned}$$

which completes the proof of the left-hand side of the inequality. For the proof of the right-hand side of the inequality, changing the variable of integration as $\varkappa = t\nu+(1-t)\omega$, and using the definition of earthquake $P$-function, we obtain

$$\begin{aligned}\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\Theta(\varkappa)d\varkappa &= \int_0^1 \Theta(t\nu+(1-t)\omega)dt\\ &\leq [\Theta(\nu)+\Theta(\omega)]\int_0^1 (te^{1-t}+(1-t)e^t)dt\\ &= (2e-4)[\Theta(\nu)+\Theta(\omega)],\end{aligned}$$

where

$$\int_0^1 te^{1-t}dt=\int_0^1 (1-t)e^t dt = e-2.$$

So, the proof is completed.

## 4. SOME NEW INEQUALITIES FOR THE EARTHQUAKE P-FUNCTIONS

Let us recall the following crucial Lemma given by Dragomir and Agarwal [17], which will be used in the sequel.

**Lemma 1.** Let $\Theta:\mathfrak{J}^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $\mathfrak{J}^{\circ}$, and let $\nu,\omega\in\mathfrak{J}^{\circ}$ with $\nu<\omega$. If $\Theta'\in L[\nu,\omega]$, then

$$\frac{\Theta(\nu)+\Theta(\omega)}{2}-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\Theta(\varkappa)d\varkappa=\frac{\omega-\nu}{2}\int_0^1 (1-2t)\Theta'(t\nu+(1-t)\omega)dt \tag{4}$$

**Theorem 5.** Let $\Theta:\mathfrak{J}\to\mathbb{R}$ be a differentiable function on $\mathfrak{J}^{\circ}$, let $\nu,\omega\in\mathfrak{J}^{\circ}$ with $\nu<\omega$ and assume that $\Theta'\in L[\nu,\omega]$. If $|\Theta'|$ is an earthquake P-function on the interval $[\nu,\omega]$, then

$$\left|\frac{\Theta(\nu)+\Theta(\omega)}{2}-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\Theta(\varkappa)d\varkappa\right|\leq(\omega-\nu)\left(10\sqrt{e}-3e-8\right)(|\Theta'(\nu)|+|\Theta'(\omega)|). \tag{5}$$

*Proof:* From Lemma 1 and the inequality

$$|\Theta'(t\nu+(1-t)\omega)|\leq(te^{1-t}+(1-t)e^t)[|\Theta'(\nu)|+|\Theta'(\omega)|],$$

we get

$$\begin{aligned}&\left|\frac{\Theta(\nu)+\Theta(\omega)}{2}-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\Theta(\varkappa)d\varkappa\right|\\ \leq\ &\frac{\omega-\nu}{2}[|\Theta'(\nu)|+|\Theta'(\omega)|]\int_0^1 |1-2t|(te^{1-t}+(1-t)e^t)dt\\ =\ &(\omega-\nu)\left(10\sqrt{e}-3e-8\right)(|\Theta'(\nu)|+|\Theta'(\omega)|),\end{aligned}$$

where

$$\int_0^1 |1-2t|te^{1-t}dt=\int_0^1 |1-2t|(1-t)e^t dt=10\sqrt{e}-3e-8.$$

Thus, the proof is completed.

**Theorem 6.** Let $\Theta: \mathfrak{J} \to \mathbb{R}$ be a differentiable function on $\mathfrak{J}^\circ$, $\nu, \omega \in \mathfrak{J}^\circ$ with $\nu < \omega$ and $\Theta' \in L[\nu, \omega]$. If $|\Theta'|^q$ is an earthquake P-function on $[\nu, \omega]$, where $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} &\left|\frac{\Theta(\nu)+\Theta(\omega)}{2} - \frac{1}{\omega-\nu}\int_\nu^\omega \Theta(\varkappa)d\varkappa\right| \\ &\leq (\omega-\nu)\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}(e-2)^{\frac{1}{q}}(|\Theta'(\nu)|^q + |\Theta'(\omega)|^q)^{\frac{1}{q}}. \end{aligned} \tag{6}$$

*Proof:* From Lemma 1, the Hölder's inequality, the property of earthquake *P*-function of $|\Theta'|^q$ and the properties of modulus, we have

$$\begin{aligned} &\left|\frac{\Theta(\nu)+\Theta(\omega)}{2} - \frac{1}{\omega-\nu}\int_\nu^\omega \Theta(\varkappa)d\varkappa\right| \\ \leq\ & \frac{\omega-\nu}{2}\int_0^1 |1-2t||\Theta'(t\nu+(1-t)\omega)|dt \\ \leq\ & \frac{\omega-\nu}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\Theta'(t\nu+(1-t)\omega)|^q dt\right)^{\frac{1}{q}} \\ \leq\ & \frac{\omega-\nu}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left([|\Theta'(\nu)|^q + |\Theta'(\omega)|^q]\int_0^1 (te^{1-t} + (1-t)e^t)dt\right)^{\frac{1}{q}} \\ \leq\ & (\omega-\nu)\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}(e-2)^{\frac{1}{q}}(|\Theta'(\nu)|^q + |\Theta'(\omega)|^q)^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} \int_0^1 |1-2t|^p dt &= \frac{1}{p+1}, \\ \int_0^1 te^{1-t}dt &= \int_0^1 (1-t)e^t dt = e-2. \end{aligned}$$

So, we get the desired result.

**Theorem 7.** Assume $\Theta: \mathfrak{J} \to \mathbb{R}$ is a differentiable function on $\mathfrak{J}^\circ$, $\nu, \omega \in \mathfrak{J}^\circ$ with $\nu < \omega$ and $\Theta' \in L[\nu, \omega]$. If $|\Theta'|^q$ is an earthquake P-function on $[\nu, \omega]$, where $q \geq 1$, then

$$\begin{aligned} &\left|\frac{\Theta(\nu)+\Theta(\omega)}{2} - \frac{1}{\omega-\nu}\int_\nu^\omega \Theta(\varkappa)d\varkappa\right| \\ &\leq \frac{\omega-\nu}{2^{2-\frac{2}{q}}}\left(10\sqrt{e} - 3e - 8\right)^{\frac{1}{q}}(|\Theta'(\nu)|^q + |\Theta'(\omega)|^q)^{\frac{1}{q}}. \end{aligned} \tag{7}$$

*Proof:* Firstly, assume that $q > 1$. From Lemma 1, the Hölder's inequality, the property of the earthquake *P*-function of $|\Theta'|^q$ and the properties of modulus, we get

$$\begin{aligned}
&\left|\frac{\Theta(\nu)+\Theta(\omega)}{2}-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\Theta(\varkappa)d\varkappa\right| \\
\leq\ & \frac{\omega-\nu}{2}\int_0^1 |1-2t||\Theta'(t\nu+(1-t)\omega)|dt \\
\leq\ & \frac{\omega-\nu}{2}\left(\int_0^1 |1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t||\Theta'(t\nu+(1-t)\omega)|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{\omega-\nu}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left([|\Theta'(\nu)|^q+|\Theta'(\omega)|^q]\int_0^1 |1-2t|(te^{1-t}+(1-t)e^t)dt\right)^{\frac{1}{q}} \\
=\ & \frac{\omega-\nu}{2^{2-\frac{2}{q}}}\left(10\sqrt{e}-3e-8\right)^{\frac{1}{q}}(|\Theta'(\nu)|^q+|\Theta'(\omega)|^q)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof of the theorem.

**Corollary 2.** Under the assumptions of Theorem 7, if we take $q=1$, then we get the following inequality

$$\begin{aligned}
&\left|\frac{\Theta(\nu)+\Theta(\omega)}{2}-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\Theta(\varkappa)d\varkappa\right| \\
\leq\ & (\omega-\nu)\left(10\sqrt{e}-3e-8\right)(|\Theta'(\nu)|+|\Theta'(\omega)|).
\end{aligned}$$

This inequality coincides with (5) in Theorem 5.

## 5. APPLICATIONS TO THE TRAPEZOIDAL FORMULA

Assume that $\sigma$ is a division of the interval $[\rho,\eta]$ such that

$$\sigma\colon \rho=\varkappa_0<\varkappa_1<\ldots<\varkappa_{n-1}<\varkappa_n<\eta.$$

For a function $\Theta\colon[\rho,\eta]\to\mathbb{R}$ let us consider the quadrature formula

$$\int_{\rho}^{\eta}\Theta(\varkappa)d\varkappa=T(\Theta,\sigma)+E(\Theta,\sigma),$$

where

$$T(\Theta,\sigma)=\sum\frac{\Theta(\varkappa_i)+\Theta(\varkappa_{i+1})}{2}(\varkappa_{i+1}-\varkappa_i)$$

is the trapezoidal formula and $E(\Theta,\sigma)$ denotes the approximation error of the integral $\int_{\rho}^{\eta}\Theta(\varkappa)d\varkappa$.

**Proposition 3.** Assume that $\rho,\eta\in\mathbb{R}$ with $\rho<\eta$ and $\Theta\colon[\rho,\eta]\to\mathbb{R}$ is a differentiable mapping on $(\rho,\eta)$. If $|\Theta'|$ is an earthquake P-function on $[\rho,\eta]$, then for every division $\sigma$ of $[\rho,\eta]$, we have

$$|E(\Theta,\sigma)| \leq 2e\left(10\sqrt{e}-3e-8\right)(|\Theta'(\rho)|+|\Theta'(\eta)|)\sum (\varkappa_{i+1}-\varkappa_i)^2. \tag{8}$$

*Proof:* We apply Theorem 5 on the sub-intervals $[\varkappa_i,\varkappa_{i+1}]$, $i=0,1,\dots,n-1$ given by the division $\sigma$. Adding from $i=0$ to $i=n-1$ we obtain

$$\left|T(\Theta,\sigma)-\int_\rho^\eta \Theta(\varkappa)d\varkappa\right| \leq \left(10\sqrt{e}-3e-8\right)(|\Theta'(\varkappa_i)|+|\Theta'(\varkappa_{i+1})|)\sum (\varkappa_{i+1}-\varkappa_i)^2. \tag{9}$$

On the other hand, for each $\varkappa_i \in [\rho,\eta]$ there exists $t_i \in [0,1]$ such that $\varkappa_i = t_i\rho + (1-t_i)\eta$. Since $|\Theta'|$ is an earthquake $P$-function and $e^{1-t} \leq e$ and $e^t \leq e$ for all $t \in [0,1]$, we deduce

$$|\Theta'(\varkappa_i)| \leq (t_ie^{1-t_i}+(1-t_i)e^{t_i})[\Theta'(\rho)+\Theta'(\eta)] \leq e.(|\Theta'(\rho)|+|\Theta'(\eta)|) \tag{10}$$

for each $i=0,1,\dots,n-1$. Relations (9) and (10) implies that the relation (8) holds. So, the proof is completed.

A method analogous to the one employed in the proof of Proposition 3, but relying on Theorem 6 and 7, demonstrates the validity of the subsequent results.

**Proposition 4.** Assume that $\rho,\eta \in \mathbb{R}$ with $\rho < \eta$ and $\Theta: [\rho,\eta] \to \mathbb{R}$ is a differentiable mapping on $(\rho,\eta)$. If $|\Theta'|^q$ is an earthquake P-function on $[\rho,\eta]$, then for each division $\sigma$ of the interval $[\rho,\eta]$, we have

$$|E(\Theta,\sigma)| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}} e.(2e-4)^{\frac{1}{q}}(|\Theta'(\rho)|^q+|\Theta'(\eta)|^q)^{\frac{1}{q}}\sum (\varkappa_{i+1}-\varkappa_i)^2,$$

where $\frac{1}{p}+\frac{1}{q}=1$.

**Proposition 5.** Assume that $\rho,\eta \in \mathbb{R}$ with $\rho < \eta$ and $\Theta: [\rho,\eta] \to \mathbb{R}$ is a differentiable mapping on $(\rho,\eta)$. If $|\Theta'|^q$ is an earthquake P-function on $[\rho,\eta]$, then for each division $\sigma$ of the interval $[\rho,\eta]$, we have

$$|E(\Theta,\sigma)| \leq \frac{e}{2^{1-\frac{2}{q}}}\left(10\sqrt{e}-3e-8\right)^{\frac{1}{q}}(|\Theta'(\nu)|^q+|\Theta'(\omega)|^q)^{\frac{1}{q}}\sum (\varkappa_{i+1}-\varkappa_i)^2,$$

where $q \geq 1$.

## 5. APPLICATIONS FOR SPECIAL MEANS

Let us remember the following special means for two non-negative numbers $\nu$ and $(\nu \neq \omega)$:

1. The arithmetic mean

$$A := A(\nu,\omega) = \frac{\nu+\omega}{2}, \nu,\omega \geq 0.$$

2. The geometric mean

$$G := G(\nu, \omega) = \sqrt{\nu\omega}, \nu, \omega \geq 0.$$

3. The harmonic mean

$$H := H(\nu, \omega) = \frac{2\nu\omega}{\nu + \omega}, \nu, \omega > 0.$$

4. The logarithmic mean

$$L := L(\nu, \omega) = \begin{cases} \dfrac{\omega - \nu}{\ln \omega - \ln \nu}, & \nu \neq \omega \\ a, & \nu = \omega \end{cases}; \ \nu, \omega > 0.$$

5. The $p$-logarithmic mean

$$L_p := L_p(\nu, \omega) = \begin{cases} \left(\dfrac{\omega^{p+1} - \nu^{p+1}}{(p+1)(\omega - \nu)}\right)^{\frac{1}{p}}, & \nu \neq \omega, p \in \mathbb{R} \setminus \{-1,0\} \\ \nu, & \nu = \omega \end{cases}; \ \nu, \omega > 0.$$

6. The identric mean

$$I := I(\nu, \omega) = \frac{1}{e}\left(\frac{\omega^{\omega}}{\nu^{\nu}}\right)^{\frac{1}{\omega - \nu}}, \nu, \omega > 0.$$

The following relation exists among special means:

$$H \leq G \leq L \leq I \leq A.$$

**Proposition 6.** Let $\nu, \omega \in [0, \infty)$ with $\nu < \omega$ and $n \geq 1$. Then,

$$A^n(\nu, \omega) \leq 2\sqrt{e} L_n^n(\nu, \omega) \leq 4\sqrt{e}(e - 2) A(\nu^n, \omega^n). \tag{11}$$

*Proof:* Choosing $\Theta(\varkappa) = \varkappa^n$ for $\varkappa \in [0, \infty)$ in (3), then inequality (11) is easily derived.

**Proposition 7.** Let $\nu, \omega \in (0, \infty)$ with $\nu < \omega$. Then, the following inequalities are obtained:

$$A^{-1}(\nu, \omega) \leq 2\sqrt{e} L^{-1}(\nu, \omega) \leq 4\sqrt{e}(e - 2) H^{-1}(\nu, \omega). \tag{12}$$

*Proof:* Taking $\Theta(\varkappa) = \varkappa^{-1}$ for $\varkappa \in [1, \infty)$ in (3), then inequality (12) is easily obtained.

**Proposition 8.** Let $\nu, \omega \in (0,1]$ with $\nu < \omega$. Then, the following inequalities are obtained:

$$4\sqrt{e}(e - 2) \ln[G(\nu, \omega)] \leq 2\sqrt{e} \ln I(\nu, \omega) \leq A(\nu, \omega). \tag{13}$$

*Proof:* Taking $\Theta(\varkappa) = -\ln \varkappa$ for $\varkappa \in (0,1]$ in (3), then inequality (13) is easily captured.

## 6. CONCLUSION

In this paper, we introduced the class of earthquake $P$-functions and gave some of their algebraic properties. We examined the connections of earthquake $P$-functions with other types of convex functions. We obtained a new version of H-H type inequality for this class of functions. We also obtained several refinements of the H-H type for functions whose first derivative in absolute value at a certain power is an earthquake $P$-function. Furthermore, we presented some applications of our findings to the e-trapezoidal formula. Finally, we provided new bounds for special means of different non-negative real numbers.

We anticipate extensive exploration within this class of functions, foreseeing profound investigations in this captivating and engaging realm of inequalities, as well as in various domains of both theoretical and practical sciences. Furthermore, we are confident that our innovative methodologies and brilliant concepts will inspire additional research among scholars dedicated to this discipline.

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