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CANAL AND TUBE SURFACES WITH SPLIT QUATERNIONS IN MINKOWSKI 3-SPACE ACCORDING TO THE Q-FRAME

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Abstract. *This paper examines canal and tube surfaces generated by unit timelike split quaternions by means of q-frames in 3-dimensional Minkowski space. Moreover, these surfaces are derived under homothetic motion by utilizing orthogonal matrices corresponding to the unit split quaternions. Subsequently, the geometric characterizations of the obtained surfaces are analyzed.*

Keywords: *Canal surface; Split quaternions; q-frame; Tube surface.*
Mathematics Subject Classification: *49K05, 53A04, 53A35.*

1. INTRODUCTION

Canal surfaces were first studied by Monge in 1850. A canal surface is the envelope of a family of moving spheres with variable radii, lying along the central spin curve. The curve, which represents the trajectory of the centers of the moving spheres, is referred to as the central (spin) curve of the canal surface. Canal surfaces are commonly used in the geometric modeling of objects such as pipes, rods, and cables, and are widely studied in engineering and medical research.

If the radius function is constant, the canal surface is referred to as a tube or pipe surface [1-8]. Recently, canal and tube surfaces have been studied and characterised by some researchers by using different frames in 3-dimensional Euclidean space. For instance, the canal and tube surfaces have been analyzed with the help of the Serret-Frenet, Bishop, Darboux, and q-frames, and their geometric properties have been provided [3-11].

Additionally, geometric characterizations of curves and surfaces in Minkowski 3-space have been presented, and a tube surface surrounding a timelike or spacelike curve has been constructed [12-14]. On the other hand, an alternative frame to the Frenet frame has been defined by using a projection vector, and directed curves in 3-dimensional Minkowski space have been analyzed. The characterizations of directed timelike tube surfaces have been analyzed. Then, the different characterizations of quasi-canal and quasi-tube surfaces obtained through the quasi-orthonormal frame have been presented [15-17].

Moreover, some properties of one-parameter homothetic motion in Minkowski 3-space have been analyzed [18]. Quaternions and some applications of quaternions have also been provided [19-23]. The geometric modeling and applications of canal surfaces have also

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been obtained by using split quaternions in 3-dimensional Minkowski space [24,25]. The canal and tube surfaces parametrized according to the q-frame were previously derived with the help of unit quaternions [26].

In this study, canal and tube surfaces will be obtained in Minkowski 3-space according to the q-frame, using unit timelike split quaternions associated with the central curve.

2. MATERIALS AND METHODS

A canal surface is the envelope of a family of moving spheres with variable radii, defined by the radius $r(s)$ and orbit of whose centers lie on the spin curve $\Omega:(a,b)\to R^3$. Let the curvature $\Omega(s)$ be the unit-speed center curve of a canal surface with non-zero curvature. Then, the canal surface is parametrically defined as

$$K(s,\varphi)=\Omega(s)-r(s)r'(s)\boldsymbol{t}(s)\pm r(s)\sqrt{1-r'^2(s)}\left(\cos\varphi\boldsymbol{n}(s)+\sin\varphi\boldsymbol{b}(s)\right).$$

If the radius function $r(s)=s$ is constant, the canal surface is referred to as a tube surface or pipe surface, and it is expressed parametrically by the equation

$$T(s,\varphi)=\Omega(s)+r\left(\cos\varphi\boldsymbol{n}(s)+\sin\varphi\boldsymbol{b}(s)\right),\quad 0\le\varphi<2\pi.$$

Here, $\boldsymbol{t}(s),\boldsymbol{n}(s),\boldsymbol{b}(s)$ represent the tangent, principal normal, and binormal of the center curve Ω at the point $\Omega(s)$, respectively [1-5]. Now, let's introduce the q-frame as an alternative to the Serret-Frenet frame. The q-frame has two significant advantages over the Serret-Frenet frame. The first is that while the Serret-Frenet frame cannot be defined in cases where the second derivative is not present, the q-frame can still be defined. The second advantage is that the q-frame prevents unnecessary torsion that arises around the tangent vector of the curve. Let $\boldsymbol{k_z}=(0,0,1)$ and $D=(d,e,f)$. Since $D\wedge\boldsymbol{k_z}=(e,-d,0)$ the vector D is projected onto the xy-plane. Here, the symbol $\wedge$ denotes the cross product. Accordingly, the projection vector $\boldsymbol{k}$ is taken as a unit vector along the axes. The projection vectors are $\boldsymbol{k_z}=(0,0,1)$, $\boldsymbol{k_y}=(0,1,0)$, $\boldsymbol{k_x}=(1,0,0)$, respectively, and the q-frame is denoted in three types as $\{\boldsymbol{t},\boldsymbol{n}_q,\boldsymbol{b}_q,\boldsymbol{k_z}\}$, $\{\boldsymbol{t},\boldsymbol{n}_q,\boldsymbol{b}_q,\boldsymbol{k_y}\}$, $\{\boldsymbol{t},\boldsymbol{n}_q,\boldsymbol{b}_q,\boldsymbol{k_x}\}$. The q-frame is defined by the following equations along a space curve $\Omega(s)$:

$$\boldsymbol{t}(s)=\frac{\Omega'(s)}{\|\Omega'(s)\|},\ \boldsymbol{n}_q(s)=\frac{\boldsymbol{t}(s)\wedge\boldsymbol{k}}{\|\boldsymbol{t}(s)\wedge\boldsymbol{k}\|},\ \boldsymbol{b}_q(s)=\boldsymbol{t}(s)\wedge\boldsymbol{n}_q(s)$$

$\boldsymbol{k}$, $\boldsymbol{t}$, $\boldsymbol{n}_q$, $\boldsymbol{b}_q$ are the projection vector, tangent, quasi-normal, and quasi-binormal of the curve $\Omega(s)$ [9]. Thus, the derivative formula of the q-frame is given by:

$$\boldsymbol{t}'=k_1\boldsymbol{n}_q+k_2\boldsymbol{b}_q,\ \boldsymbol{n}_q'=-k_1\boldsymbol{t}+k_3\boldsymbol{b}_q,\ \boldsymbol{b}_q'=-k_2\boldsymbol{t}-k_3\boldsymbol{n}_q$$

[9-11]. Considering the q-frame given by Equation (3), the canal and tube surfaces are

$$\tilde{K}(s,\varphi)=\Omega(s)-r(s)r'(s)\boldsymbol{t}(s)\pm r(s)\sqrt{1-r'^2(s)}\left(\cos\varphi\boldsymbol{n}_q(s)+\sin\varphi\boldsymbol{b}_q(s)\right),$$

$$T(s,\varphi)=\Omega(s)+r\left(\cos\varphi\boldsymbol{n}_q(s)+\sin\varphi\boldsymbol{b}_q(s)\right),$$

respectively [9-11].

The space $E_1^3 = \left(R^3,\langle,\rangle_L\right)$ defined by the metric $\langle v,w\rangle_L = v_1w_1+v_2w_2-v_3w_3$ for the vectors $v=(v_1,v_2,v_3)$, $w=(w_1,w_2,w_3)\in R^3$ in R^3 is called Minkowski 3-space. Here if $\langle v,v\rangle>0$ or $v=0$, the vector v is called spacelike, if $\langle v,v\rangle<0$ it is called time like, and if $\langle v,v\rangle=0$ and $v\neq 0$, it is referred to as null vector. The Minkowski cross product of the vectors $v,w\in E_1^3$ is $v\wedge_L w=(v_2w_3-v_3w_2, v_3w_1-v_1w_3, v_2w_1-v_1w_2)$ and the norm of the vector v is $\|v\|_L=\sqrt{\left|\langle v,v\rangle_L\right|}$. If $\|v\|_L=1$, then v is a unit vector. If the tangent vector of the curve is spacelike, timelike, or null, the curve itself is referred to as spacelike, timelike, or null, respectively.

The K Gauss and the H mean curvature of the surface $X(s,\varphi)$ in the space E_1^3 are referred to, respectively, as follows:

$$K=\varepsilon\left(\frac{LN-M^2}{EG-F^2}\right)$$

$$H=\frac{1}{2}\varepsilon\left(\frac{GL-2MF+EN}{EG-F^2}\right).$$

Here, the unit normal of the surface $X(u,v)$ is

$$n(s,\varphi)=\frac{X_s\wedge_L X_\varphi}{\left\|X_s\wedge_L X_\varphi\right\|}.$$

The coefficients of the first and the second fundamental forms are

$$E=\langle X_s,X_s\rangle_L,\ F=\langle X_s,X_\varphi\rangle_L,\ G=\langle X_\varphi,X_\varphi\rangle_L$$

$$L=\langle X_{ss},n\rangle_L,\ M=\langle X_{s\varphi},n\rangle_L,\ N=\langle X_{\varphi\varphi},n\rangle_L$$

where $\varepsilon=\langle n,n\rangle_L=\pm 1$ [12].

The principal curvatures of the regular surface $X(u,v)$

$$p_1=H+\sqrt{H^2-K},\ p_2=H-\sqrt{H^2-K}$$

respectively [1]. Second Gauss curvature K_{II} is

$$K_{II}=\frac{1}{\left(LN-M^2\right)^2}\left\{\begin{vmatrix}-\frac{1}{2}L_{\varphi\varphi}+M_{s\varphi}-\frac{1}{2}N_{ss} & \frac{1}{2}L_s & M_s-\frac{1}{2}L_\varphi\\ M_\varphi-\frac{1}{2}N_s & L & M\\ \frac{1}{2}N_\varphi & M & N\end{vmatrix}-\begin{vmatrix}0 & \frac{1}{2}L_\varphi & \frac{1}{2}N_s\\ \frac{1}{2}L_\varphi & L & M\\ \frac{1}{2}N_s & M & N\end{vmatrix}\right\}[6].$$

The center curve $\Omega(s)$ is a unit-speed spacelike curve, with a timelike principal normal and non-zero curvature in the space E_1^3. Thus, the parametrization of the canal and tube surfaces are

$$K_L(s,\varphi)=\Omega(s)-r(s)r'(s)\boldsymbol{t}(s)\pm r(s)\sqrt{1-r'^2(s)}\left(\sinh\varphi\boldsymbol{n}(s)+\cosh\varphi\boldsymbol{b}(s)\right),$$

$$T_L(s,\varphi)=\Omega(s)+r\left(\sinh\varphi\boldsymbol{n}(s)+\cosh\varphi\boldsymbol{b}(s)\right)$$

respectively. Similarly, if the center curve $\Omega(s)$ is a timelike curve with non-zero curvature, the parametric expressions of the canal and tube surfaces are

$$\begin{aligned}K_L(s,\varphi)&=\Omega(s)+r(s)r'(s)\boldsymbol{t}(s)\pm r(s)\sqrt{1+r'^2(s)}\ \left(\cos\varphi\boldsymbol{n}(s)+\sin\varphi\boldsymbol{b}(s)\right),\\ T_L(s,\varphi)&=\Omega(s)+r\left(\cos\varphi\boldsymbol{n}(s)+\sin\varphi\boldsymbol{b}(s)\right)\end{aligned}\tag{1}$$

respectively. Then, $\boldsymbol{t}(s)$, $\boldsymbol{n}(s)$, $\boldsymbol{b}(s)$ are the tangent, principal normal, and binormal vector fields at the point $\Omega(s)$ on the center curve Ω in Minkowski 3-space, respectively [13,14].

Let's represent the canal and tube surfaces formed with the help of the q-frame in Minkowski space E_1^3. The center curve $\Omega(s)$ is a regular spacelike curve, the projection vector $k=(0,1,0)$ is spacelike, the quasi-principal normal $\boldsymbol{n}_q(s)$ is a timelike curve, and the curvature is non-zero. Therefore, the parametrization of the canal and tube surfaces is given by

$$\tilde{K}_L(s,\varphi)=\Omega(s)-r(s)r'(s)\boldsymbol{t}(s)\pm r(s)\sqrt{1-r'^2(s)}\left(\sinh\varphi\boldsymbol{n}_q(s)+\cosh\varphi\boldsymbol{b}_q(s)\right),$$
$$\tilde{T}_L(s,\varphi)=\Omega(s)+r\left(\sinh\varphi\boldsymbol{n}_q(s)+\cosh\varphi\boldsymbol{b}_q(s)\right)$$

Here $\varphi\in[0,2\pi]$ and the derivative formulas of the q-frame are

$$\boldsymbol{t}'=-k_1\boldsymbol{n}_q+k_2\boldsymbol{b}_q,\ \boldsymbol{n}_q'=-k_1\boldsymbol{t}+k_3\boldsymbol{b}_q,\ \boldsymbol{b}_q'=-k_2\boldsymbol{t}+k_3\boldsymbol{n}_q$$

The cross-product of the vectors $\boldsymbol{t},\boldsymbol{n}_q,\boldsymbol{b}_q$ is

$$\boldsymbol{t}\wedge_L\boldsymbol{n}_q=-\boldsymbol{b}_q,\ \boldsymbol{n}_q\wedge_L\boldsymbol{b}_q=-\boldsymbol{t},\boldsymbol{b}_q\wedge_L\boldsymbol{t}=\boldsymbol{n}_q$$

The center curve $\Omega(s)$ is a regular timelike curve, the projection vector $k=(0,1,0)$ is spacelike, and the curvature is non-zero, where the parametric expressions of the canal and tube surfaces are

$$\tilde{K}_L(s,\varphi)=\Omega(s)+r(s)r'(s)\boldsymbol{t}(s)\pm r(s)\sqrt{1+r'^2(s)}\left(\cos\varphi\boldsymbol{n}_{\mathrm{q}}(s)+\sin\varphi\boldsymbol{b}_{\mathrm{q}}(s)\right),$$
$$T_L(s,\varphi)=\Omega(s)+r\left(\cos\varphi\boldsymbol{n}_{\mathrm{q}}(s)+\sin\varphi\boldsymbol{b}_{\mathrm{q}}(s)\right) \tag{2}$$

respectively. Here $\varphi\in[0,2\pi]$, the derivative formulas of the q-frame along a timelike curve are given by

$$\boldsymbol{t}'=k_1\boldsymbol{n}_{\mathrm{q}}+k_2\boldsymbol{b}_{\mathrm{q}},\ \boldsymbol{n}_{\mathrm{q}}'=k_1\boldsymbol{t}+k_3\boldsymbol{b}_{\mathrm{q}},\ \boldsymbol{b}_{\mathrm{q}}'=k_2\boldsymbol{t}-k_3\boldsymbol{n}_{\mathrm{q}}$$

The cross-product of the vectors $\boldsymbol{t},\boldsymbol{n}_{\mathrm{q}},\boldsymbol{b}_{\mathrm{q}}$ is

$$\boldsymbol{t}\wedge_L\boldsymbol{n}_{\mathrm{q}}=-\boldsymbol{b}_{\mathrm{q}},\ \boldsymbol{n}_{\mathrm{q}}\wedge_L\boldsymbol{b}_{\mathrm{q}}=\boldsymbol{t},\ \boldsymbol{b}_{\mathrm{q}}\wedge_L\boldsymbol{t}=-\boldsymbol{n}_{\mathrm{q}}$$

[15-17]. The one-parameter homothetic motion in the 3-dimensional Minkowski space is given by the transformation

$$\begin{bmatrix}Y\\1\end{bmatrix}=\begin{bmatrix}hA & C\\0 & 1\end{bmatrix}\begin{bmatrix}X\\1\end{bmatrix} \tag{3}$$

Here, $A\in SO_1(3)$, $A^t=\varepsilon A^{-1}\varepsilon$, $\varepsilon=diag(1,1,-1)$, h and C represent the homothetic scaling and translation vectors, respectively, where A, h and C are continuously differentiable functions dependent on the parameter t. Additionally, here, Y and X are the position vectors of a point in the fixed R' and the moving space R, respectively [18].

Quaternions were introduced in 1843 by the Irish mathematician Sir William Rowan Hamilton as a generalization of complex numbers and have since been studied by many researchers. Unit quaternions, since they enable rotational motion in three-dimensional Euclidean space, have been widely used in mechanical engineering for spherical mechanisms, in robotics, and in game simulations. The term "split quaternion" was defined by James Cockle in 1849 [19-23].

$$e_1^2=-1,\ e_2^2=e_3^2=1$$

$$e_1e_2=e_3,\ e_3e_2=-e_1, e_3e_1=e_2$$

Thus, the quaternion $q=q_0.1+q_1e_1+q_2e_2+q_3e_3$ is referred to as a split quaternion. The set formed by split quaternions is denoted by

$$\hat{H}=\left\{q_0.1+q_1e_1+q_2e_2+q_3e_3 : e_1^2=-1,\ e_2^2=e_3^2=e_1e_2e_3=1,\ q_0,q_1,q_2,q_3\in IR\right\}$$

The set of split quaternions $\hat{H}$ is expressed as $q = S_q + V_q$ for each $q \in \hat{H}$. Let $q = q_0.1 + q_1e_1 + q_2e_2 + q_3e_3$ and $p = p_0.1 + p_1e_1 + p_2e_2 + p_3e_3$ be two split quaternions. In this case, the sum, conjugate, quaternion multiplication, norm, and inverse of split quaternions are respectively:

$$
\begin{aligned}
q + p &= \left(S_q + S_p\right) + \left(V_q + V_p\right), \\
\bar{q} &= S_q - V_q, \\
q \times_L p &= S_q S_p + \langle V_q, V_p \rangle_L + S_q V_p + S_p V_q + V_q \wedge_L V_p, \\
\|q\| &= \sqrt{\left|I_q\right|} = \sqrt{\left|q_0^2 - q_1^2 - q_2^2 + q_3^2\right|}, \\
q^{-1} &= \frac{\bar{q}}{\|q\|^2},\ q \neq 0.
\end{aligned}
\tag{4}
$$

Here, $\langle , \rangle_L$ and $\wedge_L$ represent the Lorentzian inner product and the Lorentzian cross product, respectively. $I_q = q \times_L \bar{q} = \bar{q} \times_L q = q_0^2 - q_1^2 - q_2^2 + q_3^2$, then, $I_q < 0$, $I_q > 0$ and $I_q = 0$ represent the spacelike, timelike, and null split quaternions, respectively. If $\|q\| = 1$, then q is referred to as a unit split quaternion. The vector part of any spacelike quaternion is a spacelike vector. However, the vector part of any timelike quaternion can either be a timelike vector or a spacelike vector.

(i) Let v be a unit spacelike vector, then every unit spacelike quaternion can be written in the form $p = \sinh\varphi + v\cosh\varphi$.

(ii) Let v be a unit spacelike vector, then each unit timelike split quaternion, whose vector part is a spacelike vector, can be written as $p = \cosh\varphi + v\sinh\varphi$.

(iii) Let v be a unit timelike vector, and each unit timelike split quaternion, whose vector part is a timelike vector, can be written as $p = \cos\varphi + v\sin\varphi$. Here, the unit vector v is the axis of rotation, and φ is the angle of rotation [20-24].

Now, let's consider the unit timelike split quaternions that induce rotation in Minkowski 3-space. Given $q = q_0.1 + q_1e_1 + q_2e_2 + q_3e_3$ as a unit timelike split quaternion, the matrix corresponding to the transformation

$$\left(q \times_L V_q \times_L q^{-1}\right)_i = \sum_{j=1}^{3} R_{ij} \left(V_q\right)_j$$

is given by

$$R_Q = \begin{bmatrix} q_0^2 + q_1^2 + q_2^2 + q_3^2 & 2q_0q_3 - 2q_1q_2 & -2q_0q_2 - 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & q_0^2 - q_2^2 - q_1^2 + q_3^2 & -2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_0q_1 - 2q_2q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 \end{bmatrix} \text{ [21-25].}$$

3. RESULTS AND DISCUSSION

In this section, the equations of the canal and tube surfaces $\tilde{K}_L(s,\varphi)$ and $\tilde{T}_L(s,\varphi)$ are derived using timelike split quaternions with the help of the q-frame in Minkowski 3-space, and some examples are provided.

Theorem 3.1. Let $\Omega(s)$ be a regular spacelike curve with a timelike quasi normal, and the projection vector $k=(0,1,0)$ be a spacelike. Let $\Omega(s)$ be the center curve of the canal surface $\tilde{K}_L(s,\varphi)$ and the unit timelike split quaternion with a spacelike vector part be $Q_1(s,\varphi)=\cosh\varphi-\sinh\varphi\boldsymbol{t}(s)$. Considering the split quaternion $Q_1(s,\varphi)\times_L \boldsymbol{b}_q(s)$, the parametric equation of the canal surface is given by

$$\tilde{K}_L(s,\varphi)=\Omega(s)-r(s)r'(s)\boldsymbol{t}(s)\pm r(s)\sqrt{1-r'^2(s)}\,Q_1(s,\varphi)\times_L \boldsymbol{b}_q(s) \tag{5}$$

where $\{\boldsymbol{t}(s),\boldsymbol{n}_q(s),\boldsymbol{b}_q(s)\}$ is the q-frame of the curve $\Omega(s)$ in the 3-dimensional Minkowski space. Moreover, the expression of the canal surface $\tilde{K}(s,\varphi)$ under homothetic motion, using the rotation matrix R_{Q_1} corresponding to the unit timelike split quaternion $Q_1(s,\varphi)$ with a spacelike vector part, is given by

$$\tilde{K}_L(s,\varphi)=\gamma(s)+\sigma(s)R_{Q_1}\boldsymbol{b}_q(s) \tag{6}$$

where $\gamma(s)=\Omega(s)-r(s)r'(s)\boldsymbol{t}(s)$, $\sigma(s)=\pm r(s)\sqrt{1-r'^2(s)}$ and $\boldsymbol{b}_q(s)$ are the quasi-binormal of the curve $\Omega(s)$.

Proof: Let unit timelike split quaternions with a spacelike vector part $Q_1(s,\varphi)=\cosh\varphi-\sinh\varphi\boldsymbol{t}(s)$ and $\boldsymbol{b}_q(s)$ be the pure quasi binormal vector of the regular spacelike curve $\Omega(s)$. Then, the split quaternion product of the quaternions $Q_1(s,\varphi)$ and $\boldsymbol{b}_q(s)$ yields (4), from which $Q_1(s,\varphi)\times_L \boldsymbol{b}_q(s)=\sinh\varphi\boldsymbol{n}_q(s)+\cosh\varphi\boldsymbol{b}_q(s)$ is obtained. Considering equation (1), the canal surface (5) and equation (3) lead to the derivation of (6).

Corollary 3.2. Let $\tilde{T}_L(s,\varphi)$ be a tube surface and $\Omega(s)$ be the regular spacelike center curve of this tube surface. Then, the unit timelike split quaternion with spacelike vector part $Q_1(s,\varphi)=\cosh\varphi-\sinh\varphi\boldsymbol{t}(s)$ and $\{\boldsymbol{t}(s),\boldsymbol{n}_q(s),\boldsymbol{b}_q(s)\}$ be the q-frame of the curvature $\Omega(s)$ in Minkowski 3-space. Thus, the parametric equation of the tube surface and its representation under homothetic motion using the split quaternion product $Q_1(s,\varphi)\times_L \boldsymbol{b}_q(s)$ are given by

$$\tilde{T}_L(s,\varphi)=\Omega(s)+r\left(Q_1(s,\varphi)\times_L \boldsymbol{b}_q(s)\right)$$

$$\tilde{T}_L(s,\varphi)=\Omega(s)+rR_{Q_1}\boldsymbol{b}_q(s)$$

Here, $\Omega(s)$ is the translation vector of the homothetic motion, the denotation r is the homothetic scale, and R_{Q_1} is the orthogonal matrix. To calculate the Gaussian and mean curvatures of the canal and tube surfaces given above, we state the following lemma.

Lemma 3.3. Let $\{\boldsymbol{t}(s), \boldsymbol{n}_{\mathrm{q}}(s), \boldsymbol{b}_{\mathrm{q}}(s)\}$ be the q-frame of the regular spacelike curve $\Omega(s)$ considered in the space E_1^3. Let $\Omega(s)$ be the central curve of the tube surface $\tilde{T}_L(s,\varphi)$ and $Q_{\mathbf{1}}(s,\varphi)=\cosh\varphi-\sinh\varphi\boldsymbol{t}(s)$ is a unit timelike split quaternion with a spacelike vector part. Then, the first- and the second-order partial derivatives of the split quaternion product $Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)$ for the parameters s and φ are given as

i) $\frac{\partial}{\partial s}\left(Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)=Q_{\mathbf{1s}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)+Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}'_{\mathrm{q}}(s)=\left(\cosh\varphi\boldsymbol{b}'_{\mathrm{q}}(s)+\sinh\varphi\boldsymbol{n}'_{\mathrm{q}}(s)\right)$

ii) $\frac{\partial}{\partial \varphi}\left(Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)=\left(Q_{\mathbf{1}\varphi}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)=\left(\cosh\varphi\boldsymbol{n}_{\mathrm{q}}(s)+\sinh\varphi\boldsymbol{b}_{\mathrm{q}}(s)\right)$

iii) $\frac{\partial^2}{\partial s^2}\left(Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)=\left(Q_{\mathbf{1}ss}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)+2Q_{\mathbf{1}s}(s,\varphi)\times_L \boldsymbol{b}'_{\mathrm{q}}(s)+Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}''_{\mathrm{q}}(s)\right)=\sinh\varphi\boldsymbol{n}''_{\mathrm{q}}(s)+\cosh\varphi\boldsymbol{b}''_{\mathrm{q}}(s)$

iv) $\frac{\partial}{\partial \varphi}\left(\frac{\partial}{\partial s}\left(Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)\right)=\left(Q_{\mathbf{1}s\varphi}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)+\left(Q_{\mathbf{1}\varphi}(s,\varphi)\times_L \boldsymbol{b}'_{\mathrm{q}}(s)\right)=\sinh\varphi\boldsymbol{b}'_{q}(s)+\cosh\varphi\boldsymbol{n}'_{\mathrm{q}}(s)$

v) $\frac{\partial^2}{\partial \varphi^2}\left(Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)=\left(Q_{\mathbf{1}\varphi\varphi}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)=\left(\cosh\varphi-\sinh\varphi\boldsymbol{t}(s)\right)\times_L \boldsymbol{b}_{\mathrm{q}}(s)=\cosh\varphi\boldsymbol{b}_{\mathrm{q}}(s)+\sinh\varphi\boldsymbol{n}_{\mathrm{q}}(s)$

Proof: i) Let the unit timelike split quaternion $Q_{\mathbf{1}}(s,\varphi)=\cosh\varphi-\sinh\varphi\boldsymbol{t}(s)$ and $\boldsymbol{b}_{\mathrm{q}}(s)$ with a spacelike vector part be pure quasi binormal vector. Then, when the split quaternion product of the quaternions $Q_{\mathbf{1s}}(s,\varphi)$, $\boldsymbol{b}_{\mathrm{q}}(s)$ and $Q_{\mathbf{1}}(s,\varphi)$, $\boldsymbol{b}'_{\mathrm{q}}(s)$ is used by considering equation (4), it is observed that

$$\left(Q_{\mathbf{1s}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)=-k_2\sinh\varphi-k_1\sinh\varphi\boldsymbol{t}(s) \tag{7}$$

$$\left(Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}'_{\mathrm{q}}(s)\right)=k_2\sinh\varphi+\cosh\varphi\boldsymbol{b}'_{\mathrm{q}}(s)+k_3\sinh\varphi\boldsymbol{b}_{\mathrm{q}}(s) \tag{8}$$

By adding the corresponding sides of equations (7) and (8),

$$\left(Q_{\mathbf{1s}}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)+Q_{\mathbf{1}}(s,\varphi)\times_L \boldsymbol{b}'_{\mathrm{q}}(s)\right)=\cosh\varphi\boldsymbol{b}'_{\mathrm{q}}(s)+\sinh\varphi\boldsymbol{n}'_{\mathrm{q}}(s)$$

is obtained. The others are similarly obtained.

Lemma 3.4. Let the regular spacelike curve $\Omega(s)$ considered in the space E_1^3 be the central curve of the tube surface $\tilde{T}_L(s,\varphi)$. Then let the q-frame of the unit timelike split quaternion with spacelike vector part $Q_{\mathbf{1}}(s,\varphi)=\cosh\varphi-\sinh\varphi\boldsymbol{t}(s)$ and the curve $\Omega(s)$ be $\{\boldsymbol{t}(s), \boldsymbol{n}_{\mathrm{q}}(s), \boldsymbol{b}_{\mathrm{q}}(s)\}$. Thus, the normal of the tube surface $\tilde{T}_L(s,\varphi)$ is

$$\boldsymbol{n}_{\mathrm{q}}(s,\varphi)=-\left(Q_{\mathbf{1}\varphi\varphi}(s,\varphi)\times_L \boldsymbol{b}_{\mathrm{q}}(s)\right)=\left(-\cosh\varphi\boldsymbol{b}_{\mathrm{q}}(s)-\sinh\varphi\boldsymbol{n}_{\mathrm{q}}(s)\right)$$

The coefficients of the first fundamental form are

$$E_q = (1 - rk_2 \cosh\varphi - rk_1 \sinh\varphi)^2 - r^2 k_3^2,\ F_q = -r^2 k_3,\ G_q = -r^2$$

and the coefficients of the second form are as

$$L_q = -(k_1 \sinh\varphi + k_2 \cosh\varphi)(1 - r(k_1 \sinh\varphi + k_2 \cosh\varphi)) - rk_3^2,\ M_q = -rk_3,\ N_q = -r \tag{9}$$

Similarly, the normal of the canal surface, the coefficients of the first and second basic forms can be calculated by taking into account Lemma 3.3.

Theorem 3.5. Let the regular spacelike curve $\Omega(s)$ considered in the space E_1^3 be the central curve of the tube surface $\tilde{T}_L(s,\varphi)$. Then, let the q-frame of the unit timelike split quaternion with spacelike vector part $Q_1(s,\varphi) = \cosh\varphi - \sinh\varphi \boldsymbol{t}(s)$ and the curve $\Omega(s)$ be $\{\boldsymbol{t}(s), \boldsymbol{n}_q(s), \boldsymbol{b}_q(s)\}$. The Gaussian and mean curvatures of the tube surface $\tilde{T}_L(s,\varphi)$ are

$$K_q = -\frac{Q}{rP},\ H_q = \frac{2P-1}{2rP}$$

respectively, where $P = (1 - r(k_2 \cosh\varphi + k_1 \sinh\varphi))$ and $Q = (k_2 \cosh\varphi + k_1 \sinh\varphi)$.

Theorem 3.6. The surface $\tilde{T}_L(s,\varphi)$ has the relation

$$H = \frac{1}{2}\left(\frac{1}{r} + Kr\right)$$

between its Gaussian curvature K_q and the mean curvature H_q. Additionally, the principal curvatures p_1 and p_2 are

$$p_1 = H - \sqrt{H^2 - K} = Kr$$
$$p_2 = H + \sqrt{H^2 - K} = \frac{1}{r}$$

The second Gauss curvature K_{II} of the tube surface $\tilde{T}_L(s,\varphi)$ can be represented as

$$K_{II} = \frac{1}{r^2 P^2 Q^2}\left(\sum_{i=0}^{4} v_i(s)\cosh^i\varphi + \sum_{j=0}^{3} w_j(s)\cosh^j\varphi \sinh\varphi\right)$$

Here, the coefficients $v_i\ (i = 0,1,2,3,4)$ and $w_j\ (j = 0,1,2,3)$ are given by

$$
\begin{aligned}
v_0 &= r^3k_1^4 + 2r^3k_1^2k_2^2 - \frac{1}{2}rk_1^2,\\
v_1 &= \left(\frac{9}{2}r^2k_1^2k_2\right),\\
v_2 &= \left(-12r^3k_1^2k_2^2 - 2r^3k_1^4 + \frac{1}{2}rk_2^2 + \frac{1}{4}rk_1^2\right),\\
v_3 &= \left(-\frac{3}{2}r^2k_2^3 - \frac{9}{2}r^2k_1^2k_2\right)\\
v_4 &= \left(r^3k_2^4 + r^3k_1^4 + 10r^3k_1^2k_2^2\right),\\
w_0 &= \left(\frac{3}{2}r^2k_1^3 - r^2k_1k_2^2 + \frac{1}{2}rk_2^2\right),\\
w_1 &= \left(-4r^3k_1^3k_2 + \frac{1}{2}rk_1k_2\right),\\
w_2 &= \left(-\frac{7}{2}r^2k_1k_2^2 - \frac{3}{2}r^2k_1^3 - \frac{1}{4}rk_2^2\right),\\
w_3 &= \left(4r^3k_1^3k_2 + 4r^3k_1k_2^3\right).
\end{aligned}
$$

The equation

$$r^2P^2Q^2 = \sum_{i=0}^{4} m_i(s)\cosh^i\varphi + \sum_{j=0}^{3} n_j(s)\cosh^j\varphi\sinh\varphi$$

is obtained from equation (9). Here,

$$
\begin{aligned}
&m_0 = -r^2k_1^2 + r^4k_1^4,\ m_1 = 6r^3k_1^2k_2,\ m_2 = r^2k_2^2 + r^2k_1^2 - 6r^4k_1^2k_2^2 - 2r^4k_1^4,\ m_3 = -2r^3k_2^3 - 6r^3k_1^2k_2\\
&m_4 = r^4k_2^4 + 6r^4k_1^2k_2^2 + r^4k_1^4,\ n_0 = 2r^3k_1^3,\ n_1 = 2r^2k_1k_2 - 4r^4k_1^3k_2,\ n_2 = -6r^3k_1k_2^2 - 2r^3k_1^3,\ n_3 = 4r^4k_1k_2^3 + 4r^4k_1^3k_2.
\end{aligned}
$$

Theorem 3.7. Let $\Omega(s)$ be a regular timelike curve and the projection vector $k=(0,1,0)$ be spacelike. Then central curve of the canal surface $\tilde{K}_L(s,\varphi)$ is $\Omega(s)$ and the unit timelike split quaternion with a timelike vector part is $Q_2(s,\varphi)=\cos\varphi-\sin\varphi\boldsymbol{t}(s)$. Considering the split quaternion $Q_2(s,\varphi)\times_L \boldsymbol{n}_q(s)$, the parametric equation of the canal surface and its expression under homothetic motion is

$$\tilde{K}_L(s,\varphi)=\Omega(s)+r(s)r'(s)\boldsymbol{t}(s)\pm r(s)\sqrt{1+r'^2(s)}\ Q_1(s,\varphi)\times_L \boldsymbol{n}_q(s), \tag{10}$$

$$\tilde{K}_L(s,\theta)=\gamma_1(s)+\sigma_1(s)R_{Q_2}\boldsymbol{n}_q(s) \tag{11}$$

Here, $\gamma_1(s)$, $\sigma_1(s)=\pm r(s)\sqrt{1+r'^2(s)}$ and $\boldsymbol{n}_q(s)$ are the quasi principal normal of the curve $\Omega(s)$.

Proof: Let $Q_2(s,\varphi)=\cos\varphi-\sin\varphi\boldsymbol{t}(s)$ be the unit timelike split quaternion with a timelike vector part and $\boldsymbol{n}_q(s)$ be the pure quasi-normal vector of the regular spacelike curve $\Omega(s)$. If the split quaternion product of quaternions $Q_2(s,\varphi)$ and $\boldsymbol{n}_q(s)$ is substituted into (4), it follows that $Q_2(s,\varphi)\times\boldsymbol{n}_q(s)=\cos\varphi\boldsymbol{n}_q(s)+\sin\varphi\boldsymbol{b}_q(s)$. Considering Equation (2), the canal surface given in (10) is obtained. On the other hand, considering the translation vector, homothetic scale, and orthogonal matrix of the homothetic motion in Equation (3) as $\gamma_1(s)$, $\sigma_1(s)=r(s)\sqrt{1+r'^2(s)}$ and R_{Q_2}, respectively, Equation (11) is obtained.

Corollary 3.8. Let $\tilde{T}_L(s,\theta)$ be a tube surface and $\Omega(s)$ be the regular timelike central curve of this tube surface. Then, let the unit timelike split quaternion with a timelike vector part be $Q_2(s,\varphi)=\cos\varphi-\sin\varphi\boldsymbol{t}(s)$. Using the split quaternion product $Q_2(s,\varphi)\times_L\boldsymbol{n}_q(s)$, the parametric equation of the tube surface and its expression under homothetic motion are

$$\tilde{T}_L(s,\varphi)=\Omega(s)+r\left(Q_2(s,\varphi)\times_L\boldsymbol{n}_q(s)\right),$$

$$\tilde{T}_L(s,\varphi)=\Omega(s)+rR_{Q_2}\boldsymbol{n}_q(s)$$

Here, the denotation $\Omega(s)$ is the translation vector of the homothetic motion, r is the homothetic scalar, and R_{Q_2} is the orthogonal matrix.

Lemma 3.9. Let the q-frame of the regular timelike curve $\Omega(s)$ considered in the space E_1^3 be $\{\boldsymbol{t}(s),\boldsymbol{n}_q(s),\boldsymbol{b}_q(s)\}$. Then, the central curve of the tube surface $\tilde{T}_L(s,\varphi)$ is $\Omega(s)$ and the unit timelike split quaternion with a timelike vector part is $Q_2(s,\varphi)=\cos\varphi-\sin\varphi\boldsymbol{t}(s)$. Thus, the first-order partial derivatives of the product for the split quaternion $Q_2(s,\varphi)\times_L\boldsymbol{n}_q(s)$ are

i) $\frac{\partial}{\partial s}\left(Q_2(s,\varphi)\times_L\boldsymbol{n}_q(s)\right)=Q_{2s}(s,\varphi)\times_L\boldsymbol{n}_q(s)+Q_2(s,\varphi)\times_L\boldsymbol{n}'_q(s)=\left(\cos\varphi\boldsymbol{n}'_q(s)+\sin\varphi\boldsymbol{b}'_q(s)\right)$

ii) $\frac{\partial}{\partial\varphi}\left(Q_2(s,\varphi)\times_L\boldsymbol{n}_q(s)\right)=Q_{2\varphi}(s,\varphi)\times_L\boldsymbol{n}_q(s)=\left(-\sin\varphi\boldsymbol{n}_q(s)+\cos\varphi\boldsymbol{b}_q(s)\right)$

iii) $\frac{\partial^2}{\partial s^2}\left(Q_2(s,\varphi)\times_L\boldsymbol{n}_q(s)\right)=Q_{2ss}(s,\varphi)\times_L\boldsymbol{n}_q(s)+2Q_{2s}(s,\varphi)\times_L\boldsymbol{n}'_q(s)+Q_2(s,\varphi)\times_L\boldsymbol{n}''_q(s)=\cos\varphi\boldsymbol{n}''_q(s)+\sin\varphi\boldsymbol{b}''_q(s)$

iv) $\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial s}\left(Q_2(s,\varphi)\times_L\boldsymbol{n}_q(s)\right)\right)=\left(Q_{2s\varphi}(s,\varphi)\times_L\boldsymbol{n}_q(s)\right)+\left(Q_{2\varphi}(s,\varphi)\times_L\boldsymbol{n}'_q(s)\right)=-\sin\varphi\boldsymbol{n}'_q(s)+\cos\varphi\boldsymbol{b}'_q(s)$

v) $\frac{\partial^2}{\partial\varphi^2}\left(Q_2(s,\varphi)\times_L\boldsymbol{n}_q(s)\right)=Q_{2\varphi\varphi}(s,\varphi)\times_L\boldsymbol{n}_q(s)=-\cos\varphi\boldsymbol{n}_q(s)-\sin\varphi\boldsymbol{b}_q(s)$

Proof: i) Let $Q_2(s,\varphi)=\cos\varphi-\sin\varphi\boldsymbol{t}(s)$ and $\boldsymbol{n}_q(s)$ be the unit timelike split quaternion with a timelike vector part, and the curve $\Omega(s)$ be the pure quasi-normal vector. In this case, by considering equation (4),

$$Q_{2s}(s,\varphi)\times_L\boldsymbol{n}_q(s)=-k_1\sin\varphi+k_2\sin\varphi\boldsymbol{t}(s) \tag{12}$$

$$Q_2(s,\varphi)\times_L \boldsymbol{n}'_q(s) = k_1\sin\varphi + \cos\varphi \boldsymbol{n}'_q(s) - k_3\sin\varphi \boldsymbol{n}_q(s) \tag{13}$$

are obtained. It follows from equations (12) and (13) that

$$Q_{2s}(s,\varphi)\times_L \boldsymbol{n}_q(s) + Q_2(s,\varphi)\times_L \boldsymbol{n}'_q(s) = \cos\varphi \boldsymbol{n}'_q(s) + \sin\varphi \boldsymbol{b}'_q(s).$$

The others can be observed similarly.

Lemma 3.10. Let the regular timelike curve $\Omega(s)$ considered in the space E_1^3 be the central curve of the tube surface $\tilde{T}_L(s,\varphi)$. Then let the unit timelike split quaternion with a timelike vector part $Q_2(s,\varphi) = \cos\varphi - \sin\varphi \boldsymbol{t}(s)$ and $\{\boldsymbol{t}(s), \boldsymbol{n}_q(s), \boldsymbol{b}_q(s)\}$ be the q-frame of the curve $\Omega(s)$, and $\boldsymbol{n}_q(s)$ be the quasi principal normal vector of the curve $\Omega(s)$. Thus, the unit normal vector of the tube surface $\tilde{T}_L(s,\varphi)$ is

$$\boldsymbol{n}_q(s,\varphi) = -\left(Q_{2\varphi\varphi}\times_L \boldsymbol{n}_q(s)\right) = \left(\sin\varphi \boldsymbol{b}_q(s) + \cos\varphi \boldsymbol{n}_q(s)\right)$$

the coefficients of the first fundamental form are

$$E_q = -\left(1 + rk_1\cos\varphi + rk_2\sin\varphi\right)^2 + r^2k_3^2,\ F_q = r^2k_3,\ G_q = r^2$$

and the coefficients of the second fundamental form are

$$L_q = \left(k_1\cos\varphi + k_2\sin\varphi\right)\left(1 + r\left(k_1\cos\varphi + k_2\sin\varphi\right)\right) - rk_3^2,\ M_q = -rk_3,\ N_q = -r \tag{14}$$

Theorem 3.11. The regular timelike curve $\Omega(s)$ considered in the space E_1^3 is the central curve of the tube surface $\tilde{T}_L(s,\varphi)$. Then, let the unit timelike split quaternion with a timelike vector part $Q_2(s,\varphi) = \cos\varphi - \sin\varphi \boldsymbol{t}(s)$ and $\{\boldsymbol{t}(s), \boldsymbol{n}_q(s), \boldsymbol{b}_q(s)\}$ be the q-frame of the curve $\Omega(s)$ and $\boldsymbol{n}_q(s)$ be the quasi-principal normal of the curve $\Omega(s)$. Thus, the Gauss and the mean curvature of the tube surface $\tilde{T}_L(s,\varphi)$ are

$$K_q = \frac{\Omega}{r\Psi},\ H_q = -\frac{2\Psi - 1}{2r\Psi}$$

respectively, where $\Psi = \left(1 + rk_1\cos\varphi + rk_2\sin\varphi\right)$ and $\Omega = \left(k_1\cos\varphi + k_2\sin\varphi\right)$.

Theorem 3.12. The surface $\tilde{T}_L(s,\varphi)$ has the relation

$$H = -\frac{1}{2}\left(\frac{1}{r} + Kr\right)$$

between the Gauss curvature K_q and the mean curvature H_q. Moreover, the principal curvatures p_1 and p_2 are given by

$$p_1 = H - \sqrt{H^2 - K} = -Kr$$

$$p_2 = H + \sqrt{H^2 - K} = -\frac{1}{r}.$$

The second Gauss curvature K_{II} of the tube surface $\tilde{T}_L(s,\varphi)$ can be represented as

$$K_{II} = \frac{1}{r^2P^2Q^2}\left(\sum_{i=0}^{4} p_i(s)\cos^i\varphi + \sum_{j=0}^{3} r_j(s)\cos^j\varphi\sin\varphi\right).$$

Here, the coefficients $p_i\,(i = 0,1,2,3,4)$ and $r_j\,(j = 0,1,2,3)$ are given by

$$\begin{aligned}
p_0 &= -r^3k_2^4 - \frac{1}{2}rk_2^2 - \frac{1}{4}rk_1^2,\\
p_1 &= -\frac{9}{2}r^2k_1k_2^2,\\
p_2 &= 2r^3k_2^4 - \frac{1}{4}rk_1^2 - 6r^3k_1^2k_2^2 + \frac{1}{4}rk_2^2,\\
p_3 &= -\frac{3}{2}r^2k_1^3 + \frac{9}{2}r^2k_1k_2^2\\
p_4 &= 6r^3k_1^2k_2^2 - r^3k_1^4 - r^3k_2^4,\\
r_0 &= -\frac{3}{2}r^2k_2^3,\\
r_1 &= -4r^3k_1k_2^3 - \frac{1}{2}rk_1k_2,\\
r_2 &= -\frac{9}{2}r^2k_1^2k_2 + \frac{3}{2}r^2k_2^3,\\
r_3 &= 4r^3k_1k_2^3 - 4r^3k_1^3k_2.
\end{aligned}$$

$$r^2P^2Q^2 = \sum_{i=0}^{4} t_i(s)\cos^i\varphi + \sum_{j=0}^{3} u_j(s)\cos^j\varphi\sin\varphi$$

is obtained from equation (14). Here,

$$\begin{aligned}
t_0 &= r^2k_2^2 + r^4k_2^4,\\
t_1 &= 6r^3k_1k_2^2,\\
t_2 &= r^2k_1^2 - r^2k_2^2 + 6r^4k_1^2k_2^2 - 2r^4k_2^4,\\
t_3 &= 2r^3k_1^3 - 6r^3k_1k_2^2\\
t_4 &= r^4k_1^4 + r^4k_2^4 - 6r^4k_1^2k_2^2,
\end{aligned}$$

$$\begin{aligned} u_0 &= 2r^3k_2^3, \\ u_1 &= 4r^4k_1k_2^3 + 2r^2k_1k_2, \\ u_2 &= 6r^3k_1^2k_2 - 2r^3k_2^3, \\ u_3 &= 4r^4k_1^3k_2 - 4r^4k_1k_2^3. \end{aligned}$$

Example 1. The q-frame for a regular spacelike curve with a spacelike binormal $\Omega(s) = \left(\frac{2}{\sqrt{13}}\cos s, \frac{2}{\sqrt{13}}\sin s, 0\right)$ and a spacelike projection vector $k = (0,1,0)$ is obtained as

$$\boldsymbol{t}(s) = (-\sin s, \cos s, 0),\ \boldsymbol{n}_q(s) = (0,0,1),\ \boldsymbol{b}_q(s) = (-\cos s, -\sin s, 0).$$

Therefore, the matrix R_{Q_1} corresponding to the unit quaternion $Q_1(s,\varphi) = \cosh\varphi - \sinh\varphi\boldsymbol{t}(s)$ is

$$R_{Q_1} = \begin{bmatrix} -\cosh 2\varphi\cos s - \sinh^2\varphi\sin 2s\sin s \\ \sinh^2\varphi\sin 2s\cos s - \sin s \\ -\sinh 2\varphi \end{bmatrix}.$$

Thus, based on Corollary (3.2), the tube surface is

$$\tilde{T}_L(s,\varphi) = \left(\frac{2}{\sqrt{13}}\cos s, \frac{2}{\sqrt{13}}\sin s, 0\right) + r\left(-\cosh 2\varphi\cos s - \sinh^2\varphi\sin 2s\sin s, \sinh^2\varphi\sin 2s\cos s - \sin s, -\sinh 2\varphi\right)$$

Considered $r = 1$, and the tube surface illustrated in Figure 1 is obtained.

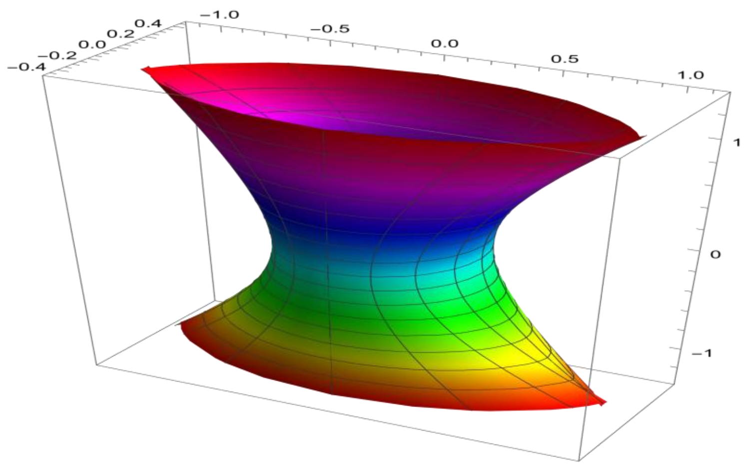

Figure 1. The tube surface around the spacelike curve in Example 1.

Example 2. The q-frame for the regular timelike curve $\Omega(s) = (3\sin s, 3\cos s, 5s)$ and the spacelike projection vector $k(t) = (0,1,0)$ is

$$\boldsymbol{t}(s)=\left(\frac{3}{4}\cos s,-\frac{3}{4}\sin s,\frac{5}{4}\right),$$

$$\boldsymbol{n}_q(s)=\left(-\frac{5}{\sqrt{25-9\cos^2 s}},0,-\frac{3\cos s}{\sqrt{25-9\cos^2 s}}\right),$$

$$\boldsymbol{b}_q(s)=\left(-\frac{9\sin s\cos s}{4\sqrt{25-9\cos^2 s}},\frac{25-9\cos^2 s}{4\sqrt{25-9\cos^2 s}},-\frac{15\sin s}{4\sqrt{25-9\cos^2 s}}\right).$$

Therefore, the matrix R_{Q_2} for the unit quaternion $Q_2(s,\varphi)=\cos\varphi-\sin\varphi\boldsymbol{t}(s)$ is

$$R_{Q_2}=\begin{bmatrix} \cos^2\varphi+\frac{17}{8}\sin^2\varphi & -\frac{5}{4}\sin 2\varphi+\frac{9}{16}\sin^2\varphi\sin 2s & -\frac{3}{4}\sin s\sin 2\varphi-\frac{15}{8}\cos s\sin^2\varphi \\ -\frac{5}{4}\sin 2\varphi-\frac{9}{16}\sin^2\varphi\sin 2s & 1 & \frac{15}{8}\sin s\sin^2\varphi+\frac{3}{4}\cos s\sin 2\varphi \\ \frac{15}{8}\sin^2\varphi\cos s-\frac{3}{4}\sin s\sin 2\varphi & -\frac{3}{4}\cos s\sin 2\varphi+\frac{15}{8}\sin s\sin^2\varphi & \cos^2\varphi-\frac{9}{16}\sin^2\varphi\cos 2s-\frac{25}{16}\sin^2\varphi \end{bmatrix}$$

Thus, based on Corollary (3.8), the tube surface is

$$\tilde{T}(s,\varphi)=\Omega(s)+rR_{Q_2}\boldsymbol{n}_q(s)$$

$$=(3\sin s,3\cos s,5s)+\frac{r}{16\sqrt{25-9\cos^2 s}}\begin{pmatrix}-80\cos^2\varphi-170\sin^2\varphi+36\sin s\cos s\sin 2\varphi+90\cos^2 s\sin^2\varphi, \\ 100\sin 2\varphi+45\sin^2\varphi\sin 2s-90\sin s\cos s\sin^2\theta-36\cos^2 s\sin 2\varphi, \\ 150\sin^2\varphi\cos s+60\sin s\sin 2\varphi-48\cos s\cos^2\varphi+27\sin^2\varphi\cos 2s\cos s+75\sin^2\varphi\cos s\end{pmatrix}$$

Consider $r=16$, and the tube surface illustrated in Figure 2 is obtained.

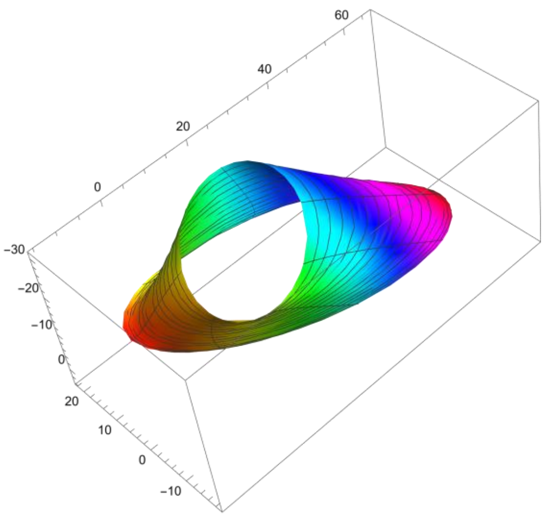

Figure 2. The tube surface over the timelike curve in Example 2.

4. CONCLUSION

In this study, a new perspective was introduced by utilizing the split quaternion representation to construct canal and tube surfaces in Minkowski 3-space using the q-frame. Canal and tube surfaces are widely used in fields such as medicine, engineering, and design. Therefore, it is expected that the results obtained in this study will make a significant contribution to the literature.

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