ORIGINAL PAPER

A VARIATIONAL ANALYSIS OF THE GENERALIZED OFFSET CURVES FOR PSEUDO NULL CURVES

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Abstract. *In this study, the generalized offset curve of the pseudo-null curve is defined in terms of the vector elements of the original curve by using Serret Frenet and Bishop frames. The generalized curves are analyzed with a more precise approximation by using the $\epsilon-$neighborhood approach. This method allows for the examination of variations in arc length and curvatures between the generalized offset curves and the original pseudo-null curves, using the Serret Frenet and Bishop parameters. In conclusion, illustrative examples are presented.*

Keywords: *Minkowski space; generalized offset curves; pseudo null curves; arc length variation; torsion variation.*

1. INTRODUCTION

The representation of curves and surfaces in two- and three- dimensional spaces establishes a fundamental problem for computer-aided design (CAD)/computer-aided manufacturing (CAM) and computer graphics. Several approaches have been put to address this issue, focusing on both closed and non-closed parametric representations. One such approach is using offsets of curves and surfaces, defined as a geometric operation extending an object to a similar object along a specific dimension [1-5].

The calculation of a curve or surface at a fixed distance from a curve or surface is also applied in geometric modeling applications. Offset curves and surfaces are used to represent the formation of a solid body skeleton or to define its properties through the central axis [6-8]. It is established that offset curves and surfaces are employed in model-based manipulation systems to address the collision avoidance challenge posed by a rigid body's movement between polyhedral objects. In addition to the safe path planning of a rigid body in the plane, offset curves and surfaces are also considered in algorithms for collision-free space representations of unmanned aerial vehicles in space [9-11].

Physics and engineering applications require variational calculus to find a ground in the mathematical explanation. The calculus of variations is a mathematical field that extends the problem of finding extrema (maximum or minimum values) of functions of several variables to the determination of extrema of functionals. This calculus, forming the foundation of numerous contemporary mathematical physics theories, is employed to both generate intriguing differential equations and substantiate the existence of solutions in instances where the solution cannot be derived analytically [12-14].

Minkowski space geometry is a branch of geometry that provides theoretical models for Einstein's relativity theory. Especially, Minkowski 4-space has a very strong physical

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background for this theory. In particular, variational calculations from classical mechanics, where the principle of least action is analyzed, to quantum mechanics and general relativity are also based on Minkowski space [15-17]. For example, Feynman developed the "many paths" assumption for the motion of a quantum particle in the space-time continuum with variational principles and Minkowski space [18].

The study by Bulut and Çalışkan [19] provides motivation for defining the generalized offset curve of the pseudo null curve in terms of the $\epsilon-$neighborhood approach via S.Frenet and Bishop frames. Variations of arc length and curvatures between pseudo null curves and their generalized offset curves are analyzed in terms of both S. Frenet as well as Bishop frames. The findings of our study are anticipated to offer theoretical insights that will contribute to the ongoing discourse on parallel curves, pseudo-null curves, and also null Cartan curves [20-23]. Furthermore, the objective is to provide a novel perspective for studies within the scope of the Principle of Least Action, where variational studies are conducted based on Minkowski space.

2. PRELIMINARIES

The three-dimensional Lorentz-Minkowski space $E_1^3 = (R^3, <,>)$ is a real vector space provided with the following metric,

$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3, \quad u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \quad u, v \in E_1^3. \#$$

The metric symbol "$<,>$" is referred to as the Lorentzian metric [24]. In the case where $u \in E_1^3$, the norm of a vector u is given by $|u| = \sqrt{|\langle u, u\rangle|}$. A vector u is defined as a unit vector if its length is equal to 1, i.e. $|u| = 1$.

A vector u has three distinct causal characters, namely, spacelike if $\langle u, u\rangle > 0$ or $u = 0$; timelike if $\langle u, u\rangle < 0$ and lightlike if $\langle u, u\rangle = 0$ or $u \neq 0$ [25].

Let the curve $r(s)$ be a unit speed curve in Minkowski 3-space. If $r'(s)$ is a spacelike, timelike, and lightlike, then $r(s)$ is spacelike, timelike, and lightlike at s, respectively [26].

A spacelike curve $r: I \to E_1^3$ is defined as a pseudo-null curve if the principal normal vector field N and binormal vector field B are null vector fields that satisfy the condition $\langle N, B\rangle = 1$. The Frenet formula of a non-geodesic pseudo-null curve according to arc length parameter s, is given by:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & \tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \# \tag{1}$$

where T, N, B being the unit tangent vector, the unit normal vector, and the unit binormal vector, respectively, and also the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ of a pseudo-null curve $r(s)$ is an arbitrary function of s. The Frenet vector fields also satisfy the following equations [27];

$$\langle T, T\rangle = 1, \langle N, N\rangle = \langle B, B\rangle = 0,$$
$$\langle T, N\rangle = \langle T, B\rangle = 0, \langle N, B\rangle = 1,$$

and

$$T \times N = N, \qquad N \times B = T, \qquad B \times T = B. \# \tag{2}$$

The Bishop $\{\boldsymbol{T_1}, \boldsymbol{N_1}, \boldsymbol{N_2}\}$ and Frenet $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ frames of the pseudo-null curve $\boldsymbol{r}(\boldsymbol{s})$ are related by:

i.

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\kappa_2 & 0 \\ 0 & 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \# \tag{3}$$

and the derivative formulas corresponding to the Bishop frame,

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 1 & \kappa_2 & \kappa_1 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \# \tag{4}$$

where $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau(s) ds}, c_0 \in R_0^+$;

ii.

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -\kappa_1 \\ 0 & -1/\kappa_1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \# \tag{5}$$

and the derivative formulas via Bishop parameters,

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 1 & \kappa_2 & \kappa_1 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \# \tag{6}$$

where $\kappa_1(s) = c_0 e^{\int \tau(s) ds}, c_0 \in R_0^-$ and $\kappa_2(s) = 0$.

The Bishop's vectors also satisfy the equations below [27],

$$\begin{gathered} \langle T_1, T_1 \rangle = 1, \langle N_1, N_1 \rangle = \langle N_2, N_2 \rangle = 0, \\ \langle T_1, N_1 \rangle = \langle T_1, N_2 \rangle = 0, \langle N_1, N_2 \rangle = 1. \# \end{gathered} \tag{7}$$

3. GENERALIZED OFFSET CURVES OF PSEUDO NULL CURVES AND THEIR VARIATIONS

The generalized offset curve $r_*(s)$ of a pseudo-null parametric curve $r(s)$ with a variable offset distance and offset direction are defined as

$$r_*(s) = r(s) + d_1(s)T + d_2(s)N + d_3(s)B \# \tag{8}$$

such as $d_1(s)$, $d_2(s)$ and $d_3(s)$ are the functions of the arc length parameter s. The offset direction and the offset distance are determined by $d_1(s)T + d_2(s)N + d_3(s)B$.

The generalized offset curve can also be defined as using Blaschke's relation [28],

$$r_*(s) = r(s) + \epsilon \overline{d}_1(s)T + \epsilon \overline{d}_2(s)N + \epsilon \overline{d}_3(s)B, \#$$

where $d_1(s) = \epsilon\overline{d}_1(s)$, $d_2(s) = \epsilon\overline{d}_2(s)$ and $d_3(s) = \epsilon\overline{d}_3(s)$. The offset distance is represented as $\epsilon\sqrt{\left|\overline{d}_1(s)^2 + \overline{d}_2(s)^2 - \overline{d}_3(s)^2\right|}$.

The parametric derivatives of $\boldsymbol{r}_*(s)$ becomes by the Frenet formula (1) as follows;

$$\boldsymbol{r}_*' = \boldsymbol{T} + \epsilon\left[\left(\overline{d}_1' - \overline{d}_3\right)\boldsymbol{T} + \left(\overline{d}_1 + \overline{d}_2' + \overline{d}_2\tau\right)\boldsymbol{N} + \left(\overline{d}_3' - \overline{d}_3\tau\right)\boldsymbol{B}\right].\#$$

This equation can be written as

$$r_*' = T + \epsilon(\alpha T + \beta N + \gamma B)\# \tag{9}$$

where $\alpha = \left(\overline{d}_1' - \overline{d}_3\right)$, $\beta = \left(\overline{d}_1 + \overline{d}_2' + \overline{d}_2\tau\right)$, $\gamma = \left(\overline{d}_3' - \overline{d}_3\tau\right)$ and thus,

$$r_*'' = N + \epsilon[(\alpha' - \gamma)T + (\alpha + \beta' + \beta\tau)N + (\gamma' - \gamma\tau)B].\# \tag{10}$$

The conditions for the generalized offset curve $\boldsymbol{r}_*(s)$ being a pseudo-null curve;

i. $\beta\gamma > -\left(\frac{1+\epsilon\alpha}{\sqrt{2}\epsilon}\right)^2$
ii. $\epsilon^2(\alpha' - \gamma)^2 + 2\epsilon(\epsilon\alpha + \epsilon\beta' + \epsilon\beta\tau + 1)(\gamma' - \gamma\tau) = 0$
iii. $\alpha(\epsilon\alpha + 2)^2 + 2\epsilon\beta(\epsilon\alpha\gamma + 2\gamma) = 0$.

Let the Frenet vector field $\{T_*, N_*, B_*\}$ belong to generalized offset pseudo null curve $r_*(s)$, then the unit tangent vector T_* becomes by (9) and

$$\begin{gathered} |r_*'| = \sqrt{|\langle r_*', r_*'\rangle|} = \sqrt{|(1+\epsilon\alpha)^2 + 2\epsilon^2\beta\gamma|} = 1 \\ T_* = T + \epsilon(\alpha T + \beta N + \gamma B).\# \end{gathered} \tag{11}$$

By using the relation $N_*(s) = r_*''(s)$, the unit normal vector N_* takes the following form;

$$N_* = N + \epsilon[(\alpha' - \gamma)T + (\alpha + \beta' + \beta\tau)N + (\gamma' - \gamma\tau)B].\# \tag{12}$$

Finally, let the unit binormal vector B_* be represented as $\lambda_1 T + \lambda_2 N + \lambda_3 B$. The relations
i. $B_* \times T_* = B_*$
ii. $\langle N_*, B_*\rangle = 1$
should be solved for the coefficients $\lambda_1, \lambda_2, \lambda_3$:

i.
$$\begin{aligned} B_* \times T_* &= (\lambda_1 T + \lambda_2 N + \lambda_3 B) \times \left((\epsilon\alpha + 1)T + \epsilon\beta N + \epsilon\gamma B\right) \\ &= (\lambda_2\epsilon\gamma - \lambda_3\epsilon\beta)T + \left(\lambda_1\epsilon\beta - \lambda_2(\epsilon\alpha + 1)\right)N + (\lambda_3(\epsilon\alpha + 1) - \lambda_1\epsilon\gamma)B \\ &= \lambda_1 T + \lambda_2 N + \lambda_3 B, \end{aligned}$$

and thus, the following relations are obtained:

$$\lambda_2(\epsilon\alpha + 2) = \lambda_1\epsilon\beta \qquad \text{and} \qquad \lambda_1\gamma = \lambda_3\alpha.\# \tag{13}$$

ii.
$$\langle N_*, B_*\rangle = \langle\left((\epsilon\alpha' - \epsilon\gamma), (\epsilon\alpha + \epsilon\beta' + \epsilon\beta\tau + 1), (\epsilon\gamma' - \epsilon\gamma\tau)\right), (\lambda_1, \lambda_2, \lambda_3)\rangle$$

$$= (\epsilon\alpha' - \epsilon\gamma)\lambda_1 + (\epsilon\gamma' - \epsilon\gamma\tau)\lambda_2 + (\epsilon\alpha + \epsilon\beta' + \epsilon\beta\tau + 1)\lambda_3 \#$$

since $\langle N_*, B_* \rangle = 1$,

$$(\epsilon\alpha' - \epsilon\gamma)\lambda_1 + (\epsilon\gamma' - \epsilon\gamma\tau)\lambda_2 + (\epsilon\alpha + \epsilon\beta' + \epsilon\beta\tau + 1)\lambda_3 = 1. \# \tag{14}$$

From (13) and (14); the coefficients $\lambda_1, \lambda_2, \lambda_3$ are respectively;

$$\lambda_1 = \frac{\epsilon\alpha^2 + 2\alpha}{2\gamma + \epsilon(2\alpha\alpha' + 2\beta'\gamma + 2\beta\gamma\tau + \alpha\gamma) + \epsilon^2(\alpha^2\alpha' + \alpha\beta'\gamma + \alpha\beta\gamma')},$$

$$\lambda_2 = \frac{\epsilon\beta\gamma}{2\gamma + \epsilon(2\alpha\alpha' + 2\beta'\gamma + 2\beta\gamma\tau + \alpha\gamma) + \epsilon^2(\alpha^2\alpha' + \alpha\beta'\gamma + \alpha\beta\gamma')},$$

and

$$\lambda_3 = \frac{\epsilon\alpha\gamma + 2\gamma}{2\gamma + \epsilon(2\alpha\alpha' + 2\beta'\gamma + 2\beta\gamma\tau + \alpha\gamma) + \epsilon^2(\alpha^2\alpha' + \alpha\beta'\gamma + \alpha\beta\gamma')}.$$

Therefore, the unit binormal vector B_* are found as

$$\begin{aligned} B_* = &\left(\frac{\epsilon\alpha^2 + 2\alpha}{2\gamma + \epsilon(2\alpha\alpha' + 2\beta'\gamma + 2\beta\gamma\tau + \alpha\gamma) + \epsilon^2(\alpha^2\alpha' + \alpha\beta'\gamma + \alpha\beta\gamma')}\right) T + \\ &\left(\frac{\epsilon\beta\gamma}{2\gamma + \epsilon(2\alpha\alpha' + 2\beta'\gamma + 2\beta\gamma\tau + \alpha\gamma) + \epsilon^2(\alpha^2\alpha' + \alpha\beta'\gamma + \alpha\beta\gamma')}\right) N + \\ &\left(\frac{\epsilon\alpha\gamma + 2\gamma}{2\gamma + \epsilon(2\alpha\alpha' + 2\beta'\gamma + 2\beta\gamma\tau + \alpha\gamma) + \epsilon^2(\alpha^2\alpha' + \alpha\beta'\gamma + \alpha\beta\gamma')}\right) B. \# \end{aligned} \tag{15}$$

3.1. THE ARC LENGTH VARIATION OF GENERALIZED OFFSET CURVES

Let the curve $\boldsymbol{r}(s)$ be a pseudo-null curve. Hence, it's a generalized offset $\boldsymbol{r}_*(s)$ is given as follows

$$r_* = r + d_1T + d_2N + d_3B = r + y, \# \tag{16}$$

where $\boldsymbol{y}$ is the function of the arc length parameter s, and also $\boldsymbol{y} = \epsilon\overline{\boldsymbol{y}}(s)$.

Let $r_*(s)$ be expanded to the series according to ϵ

$$s_* = \int_{s_1}^{s_2} \sqrt{\left|\boldsymbol{r}'^{2}_*\right|} ds \, \# \tag{17}$$

and this expansion is demonstrated as a linear term δs, we obtain

$$s_* = s + \delta s + \cdots. \#$$

Thus, the increment δs is said as the first variation of the arc length s and is defined as

$$\delta s = \epsilon \lim_{\epsilon \to 0} \frac{s_* - s}{\epsilon}. \#$$

By the relations (1) and (16), the first parametric derivative of $\boldsymbol{y}(s)$ can be determined by

$$y' = (d_1' - d_3)T + (d_1 + d_2' + \tau d_2)N + (d_3' - \tau d_3)B. \# \tag{18}$$

From the differentiation of the generalized offset curve $\boldsymbol{r}_*(s)$, we have

$$\begin{gathered} r_*' = (r + y)' = r' + y', \\ {r_*'}^2 = {r'}^2 + 2r'y' + \cdots . \# \end{gathered} \tag{19}$$

When the second-degree terms to ϵ is eliminated, the equation (19) can be written as

$${r_*'}^2 = (1 + r'y')^2. \# \tag{20}$$

From Equations (17) and (20), we reach

$$s_* = \int_{s_1}^{s_2} 1 + \boldsymbol{r}'\boldsymbol{y}' ds + \cdots = s + \int_{s_1}^{s_2} \boldsymbol{r}'\boldsymbol{y}' ds + \cdots , \#$$

and also after some computations by using $r'(s) = T$ and the equation (18)

$$s_* = s + \int_{s_1}^{s_2} (d_1' - d_3) ds = s + [d_1]_{s_1}^{s_2} - \int_{s_1}^{s_2} d_3 ds,$$

we find

$$\delta s = [d_1]_{s_1}^{s_2} - \int_{s_1}^{s_2} d_3 ds.$$

which leads to the first variation of arc length s. The increment δs turns into

$$\delta s = - \int_{s_1}^{s_2} d_3 ds.$$

for the common boundary points $d_1(s_1) = 0$ and $d_1(s_2) = 0$. If δB is taken instead of d_3, we have

$$\delta s = - \int_{s_1}^{s_2} \delta B ds.$$

3.2. THE TORSION VARIATION OF THE GENERALIZED OFFSET CURVES

The torsion τ_* of the generalized offset curve r_* is expressed as $\tau_* = \langle N_*', B_* \rangle$. By equation (12), we obtain

$$N_*' = (\epsilon\alpha'' - 2\epsilon\gamma' + \epsilon\gamma\tau)T + (2\epsilon\alpha' - \epsilon\gamma + \epsilon\beta'' + 2\epsilon\beta'\tau + \epsilon\beta\tau' + \epsilon\alpha\tau + \epsilon\beta\tau^2 + \tau)N + (\epsilon\gamma'' - 2\epsilon\gamma'\tau - \epsilon\gamma\tau' + \epsilon\gamma\tau^2)B.$$

Then, using the vector $\boldsymbol{N}_*'$ and Eq. (15), we reach the torsion τ_*,

$$\tau_* = \frac{\begin{array}{c}2\gamma\tau + \epsilon(2\alpha\alpha'' - 4\alpha\gamma' + 5\alpha\gamma\tau + 2\beta''\gamma + 4\alpha'\gamma + 4\beta'\gamma\tau - 2\gamma^2 + 2\beta\gamma\tau' + 2\beta\gamma\tau^2) + \\ \epsilon^2(\alpha^2\alpha'' - 2\alpha^2\gamma' + \alpha\beta\gamma'' - 2\alpha\beta\gamma'\tau + 2\alpha^2\gamma\tau + 2\alpha\beta\gamma\tau^2 + \alpha\beta''\gamma + 2\alpha\alpha'\gamma + 2\alpha\beta'\gamma\tau - \alpha\gamma^2)\end{array}}{\left(2\gamma + \epsilon(2\alpha\alpha' + 2\beta'\gamma + 2\beta\gamma\tau + \alpha\gamma) + \epsilon^2(\alpha^2\alpha' + \alpha\beta'\gamma + \alpha\beta\gamma')\right)} \#$$

Furthermore first variation of the torsion is calculated according to ϵ approximation as follows;

$$\delta\tau = \frac{\epsilon(2\alpha\alpha'' - 4\alpha\gamma' + 5\alpha\gamma\tau + 2\beta''\gamma + 4\alpha'\gamma + 2\beta'\gamma\tau - 2\gamma^2 + 2\beta\gamma\tau' - 2\alpha\alpha'\tau - \alpha\gamma\tau)}{2\gamma + \epsilon(2\alpha\alpha' + 2\beta'\gamma + 2\beta\gamma\tau + \alpha\gamma)}.$$

3.3. APPLICATIONS

E.g. 3.1. Let $\boldsymbol{r}(s) = \left(\sqrt{2}s^2 + \frac{s}{\sqrt{2}},\ \sqrt{2}s^2 - \frac{s}{\sqrt{2}},\ 2s^2\right)$ be a pseudo-null curve with the following Serret-Frenet frame,

$$T(s) = \left(2\sqrt{2}s + \tfrac{1}{\sqrt{2}},\ 2\sqrt{2}s - \tfrac{1}{\sqrt{2}},\ 4s\right), \quad N(s) = \left(2\sqrt{2}, 2\sqrt{2}, 4\right),$$
$$B(s) = \left(-\sqrt{2}s^2 - \tfrac{s}{\sqrt{2}} + \tfrac{1}{8\sqrt{2}}, -\sqrt{2}s^2 + \tfrac{s}{\sqrt{2}} + \tfrac{1}{8\sqrt{2}}, -2s^2 - \tfrac{1}{8}\right).$$

Then the generalized offset curves with the different offset distances,

a) For the offset distances $d_1 = -\frac{1}{4}$, $d_2 = 1$ and $d_3 = 1$,

$$r_{a*}(s) = \left(-\frac{s}{\sqrt{2}} + \frac{31}{8\sqrt{2}}, -\frac{s}{\sqrt{2}} + \frac{35}{8\sqrt{2}}, -s + \frac{31}{8}\right),$$

b) For the offset distances $d_1 = -2$, $d_2 = \frac{1}{4}$ and $d_3 = 8$,

$$r_{b*}(s) = \left(-7\sqrt{2}s^2 - \frac{15}{\sqrt{2}}s, -7\sqrt{2}s^2 - \frac{s}{\sqrt{2}} + 2\sqrt{2}, -14s^2 - 8s\right),$$

c) For the offset distances $d_1 = \sqrt{2}s$, $d_2 = -32$ and $d_3 = 1$,

$$r_{c*}(s) = (4s^2 + s, 4s^2 - s\,, 4\sqrt{2}s^2 - \frac{1}{4})$$

are obtained, and the relationship between these curves is plotted in Fig. 3.1.

E.g. 3.2. Let $\boldsymbol{r}(s) = (e^{s+2}, s+3, e^{s+2})$ be a pseudo-null curve with the following Serret-Frenet frame,

$$T(s) = (e^{s+2}, 1, e^{s+2}), \quad N(s) = (e^{s+2}, 0, e^{s+2}), \quad B(s) = \left(\tfrac{e^{2s+4}-1}{2e^{s+2}}, -1, \tfrac{-e^{2s+4}-1}{2e^{s+2}}\right).$$

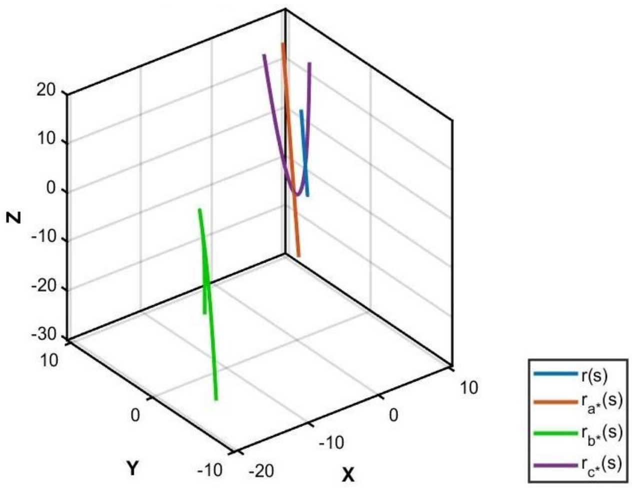

Figure 3.1. The original curve r (blue) and its generalized offset curves r_{a*}, r_{b*} and r_{c*} for different offset distances (red, green and purple, respectively).

Then the generalized offset curves for the different offset distances are displayed as follows;

a) For the offset distances $d_1 = -\frac{1}{2}$, $d_2 = -\frac{1}{2}$ and $d_3 = 2e^{s+2}$,

$$r_{a*}(s) = \left(e^{2s+4} - 1, -2e^{s+2} + s + \frac{5}{2}, -e^{2s+4} - 1\right),$$

b) For the offset distances $d_1 = e^{s+2}$, $d_2 = e^{s+2}$ and $d_3 = -4e^{s+2}$,

$$r_{b*}(s) = (e^{s+2} + 2, 5e^{s+2} + s + 3, 4e^{2s+4} + e^{s+2} + 2),$$

c) For the offset distances $d_1 = 1$, $d_2 = -2$ and $d_3 = 2e^{s+2}$,

$$r_{c*}(s) = (e^{2s+4} - 1, -2e^{s+2} + s + 4, -e^{2s+4} - 1).$$

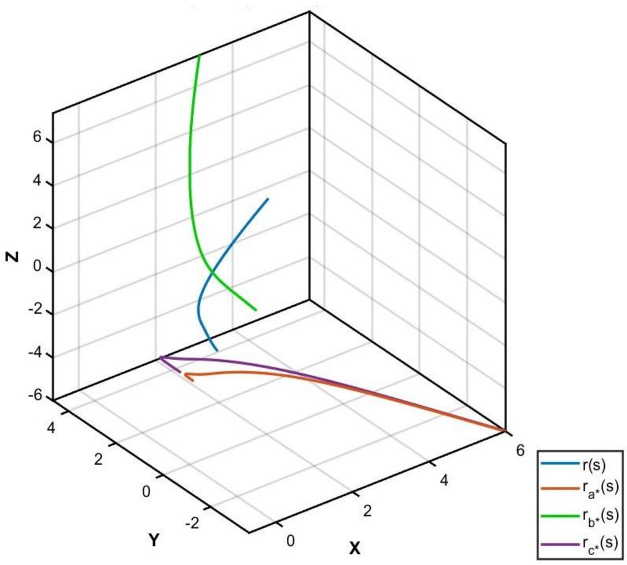

Figure 3.2. The original curve r (blue) and its generalized offset curves r_{a*}, r_{b*} and r_{c*} for different offset distances (red, green and purple, respectively).

4. GENERALIZED OFFSET CURVES OF PSEUDO NULL CURVES VIA BISHOP FRAME AND THEIR VARIATIONS

The generalized offset curve $\boldsymbol{r}_*(s)$ of a pseudo-null curve $\boldsymbol{r}(s)$ via the Bishop frame is defined as

$$r_*(s) = r(s) + d_1(s)T_1 + d_2(s)N_1 + d_3(s)N_2 \# \tag{21}$$

The offset direction is determined via $d_1(s)T_1 + d_2(s)N_1 + d_3(s)N_2$. The generalized offset curve can also be defined by using Blaschke's relation,

$$r_*(s) = r(s) + \epsilon\overline{d}_1(s)T_1 + \epsilon\overline{d}_2(s)N_1 + \epsilon\overline{d}_3(s)N_2 \#$$

where $d_1(s) = \epsilon\overline{d}_1(s),\ d_2(s) = \epsilon\overline{d}_2(s)$ and $d_3(s) = \epsilon\overline{d}_3(s)$. The offset distance is represented as $\epsilon\sqrt{\left|\overline{d}_1(s)^2 + \overline{d}_2(s)^2 - \overline{d}_3(s)^2\right|}$.

There are two cases arising from the Bishop frame of pseudo null curves:

Case 1. From equations (3), (4), and (21), we get

$$r_*' \ = \ T_1 + \epsilon\left[\overline{d}_1'T_1 + \left(\overline{d}_1\kappa_2 + \overline{d}_2'\right)N_1 + \left(\overline{d}_3' - \overline{d}_3\kappa_2\right)N_2\right]. \#$$

The rearrangement of this equation gives

$$r_*' = T_1 + \epsilon(\alpha T_1 + \beta N_1 + \gamma N_2) \# \tag{22}$$

where $\alpha = \overline{d}_1',\ \ \beta = \left(\overline{d}_1\kappa_2 + \overline{d}_2'\right),\ \gamma = \left(\overline{d}_3' - \overline{d}_3\kappa_2\right)$ and thus,

$$r_*'' \ = \kappa_2 N_1 + \epsilon[(\alpha' - \gamma\kappa_2)T_1 + (\alpha\kappa_2 + \beta')N_1 + \gamma' N_2]. \# \tag{23}$$

The conditions for the generalized offset curve $\boldsymbol{r}_*(s)$ to be a pseudo-null curve by using the Bishop frame;

i. $\beta\gamma > -\left(\frac{1+\epsilon\alpha}{\sqrt{2}\epsilon}\right)^2$
ii. $\epsilon^2(\alpha' - \gamma\kappa_2)^2 + 2\epsilon\gamma'(\epsilon\alpha\kappa_2 + \epsilon\beta' + \kappa_2) = 0$
iii. $\lambda_1^2 + 2\lambda_2\lambda_3 = 0.$

Given that $\{T_{1_*}, N_{1_*}, N_{2_*}\}$ is the Bishop frame of $r_*(s)$. Then the relationship between the Bishop frame $\{T_{1_*}, N_{1_*}, N_{2_*}\}$ and the Frenet frame $\{T_*, N_*, B_*\}$ of the curve $r_*(s)$ becomes

$$\begin{bmatrix} T_{1_*} \\ N_{1_*} \\ N_{2_*} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\kappa_{2_*} & 0 \\ 0 & 0 & \kappa_{2_*} \end{bmatrix} \begin{bmatrix} T_* \\ N_* \\ B_* \end{bmatrix}. \# \tag{24}$$

The unit tangent vector T_{1_*} of the curve $r_*(s)$ comes from (22), (24), and

$$|\boldsymbol{r}_*'| = \sqrt{|\langle \boldsymbol{r}_*', \boldsymbol{r}_*' \rangle|} = \sqrt{|(1+\epsilon\alpha)^2 + 2\epsilon^2\beta\gamma|} = 1,$$

$$T_{1_*} = T_1 + \epsilon(\alpha T_1 + \beta N_1 + \gamma N_2). \#$$

The vector $N_{1_*}(s)$ becomes by (23) and (24)

$$N_{1_*} = \frac{1}{\kappa_{2_*}}\left[\kappa_2 N_1 + \epsilon\big((\alpha' - \gamma\kappa_2)T_1 + (\alpha\kappa_2 + \beta')N_1 + \gamma' N_2\big)\right].$$

The vector $N_{2_*}(s)$ is obtained by equation (24)

$$N_{2_*} = \kappa_{2_*}\left(\lambda_1 T_1 + \lambda_2\kappa_2 N_1 + \frac{\lambda_3}{\kappa_2}N_2\right), \#$$

where $\kappa_{2_*} = c_0 e^{\int \tau_*(s)ds}$ and $\kappa_2(s) = c_0 e^{\int \tau(s)ds}, c_0 \in R_0^+$.

Case 2. The differentiation of $\boldsymbol{r}_*(s)$ occurs

$$r_*' = -T_1 + \epsilon\left[\left(\overline{d}_1' - \overline{d}_2\kappa_1\right)T_1 + \overline{d}_2' N_1 + \left(\overline{d}_1\kappa_1 + \overline{d}_3'\right)N_2\right] \#$$

by using (5) and (6). This equation can be rearrange,

$$r_*' = -T_1 + \epsilon(\alpha T_1 + \beta N_1 + \gamma N_2) \# \tag{25}$$

becomes, where $\alpha = \left(\overline{d}_1' - \overline{d}_2\kappa_1\right)$, $\beta = \overline{d}_2'$, $\gamma = \left(\overline{d}_1\kappa_1 + \overline{d}_3'\right)$ and thus,

$$r_*'' = -\kappa_1 N_2 + \epsilon[(\alpha' - \beta\kappa_1)T_1 + \beta' N_1 + (\alpha\kappa_1 + \gamma')N_2]. \# \tag{26}$$

The conditions for the generalized offset curve $\boldsymbol{r}_*(s)$ to be a pseudo-null curve by using the Bishop frame are listed:

i. $\beta\gamma > -\left(\frac{1-\epsilon\alpha}{\sqrt{2}\epsilon}\right)^2$
ii. $\epsilon^2(\alpha' - \beta\kappa_1)^2 + 2\epsilon\beta'(\epsilon\alpha\kappa_1 + \epsilon\gamma' - \kappa_1) = 0$
iii. $\lambda_1^2 + 2\lambda_2\lambda_3 = 0.$

Given that $\{T_{1_*}, N_{1_*}, N_{2_*}\}$ be the Bishop frame of the generalized offset $r_*(s)$. Then the relation between the Bishop frame $\{T_{1_*}, N_{1_*}, N_{2_*}\}$ and the Frenet frame $\{T_*, N_*, B_*\}$ appears

$$\begin{bmatrix} T_{1_*} \\ N_{1_*} \\ N_{2_*} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -\kappa_{1_*} \\ 0 & -1/\kappa_{1_*} & 0 \end{bmatrix} \begin{bmatrix} T_* \\ N_* \\ B_* \end{bmatrix}. \# \tag{27}$$

Here the unit tangent vector $\boldsymbol{T}_{\boldsymbol{1}_*}$ comes from (25), (27), and

$$|\boldsymbol{r}_*'| = \sqrt{|\langle \boldsymbol{r}_*', \boldsymbol{r}_*' \rangle|} = \sqrt{|(1+\epsilon\alpha)^2 + 2\epsilon^2\beta\gamma|} = 1,$$

$$T_{1_*} = T_1 - \epsilon(\alpha T_1 + \beta N_1 + \gamma N_2). \#$$

The vector $\boldsymbol{N_{1}}_{*}(s)$ becomes by equation (27)

$$N_{1_*} = \kappa_{1_*}\left(\lambda_1 T_1 + \frac{\lambda_3}{\kappa_1}N_1 + \lambda_2\kappa_1 N_2\right), \#$$

The vector $\boldsymbol{N_{2}}_{*}(s)$ is, by (26) and (27), found

$$N_{2_*} = -\frac{1}{\kappa_{1_*}}\left[-\kappa_1 N_2 + \epsilon\big((\alpha' - \beta\kappa_1)T_1 + \beta'^{N_1} + (\alpha\kappa_1 + \gamma')N_2\big)\right],$$

where $\kappa_{1_*} = c_0 e^{\int \tau_*(s)ds}$ and $\kappa_1(s) = c_0 e^{\int \tau(s)ds}, c_0 \in R_0^-$.

4.1. THE ARC LENGTH VARIATION OF GENERALIZED OFFSET CURVES

The generalized offset curve $\boldsymbol{r}_*(s)$ is given

$$r_* = r + d_1T_1 + d_2N_1 + d_3N_2 = r + y_1 \# \tag{28}$$

where $\boldsymbol{y_1} = \epsilon\overline{\boldsymbol{y}}_{\boldsymbol{1}}(s)$. There are two cases for the arc length variations:

Case 1. From (4) and (28), we have

$$y_1' = (d_1' - d_3\kappa_2)T_1 + (d_1\kappa_2 + d_2')N_1 + d_3'N_2. \# \tag{29}$$

We have the following equation by Eq. (17) and $\boldsymbol{r}_*'^{\boldsymbol{2}} = (1 + \boldsymbol{r}'\boldsymbol{y}_{\boldsymbol{1}}')^{\boldsymbol{2}}$

$$s_* = \int_{s_1}^{s_2} 1 + r'y_1'ds + \cdots = s + \int_{s_1}^{s_2} r'y_1'ds + \cdots. \#$$

After some computations

$$s_* = s + \int_{s_1}^{s_2} (d_1' - d_3\kappa_2)ds = s + [d_1]_{s_1}^{s_2} - \int_{s_1}^{s_2} d_3\kappa_2 ds,$$

is obtained, which leads to the first variation of arc length

$$\delta s = [d_1]_{s_1}^{s_2} - \int_{s_1}^{s_2} d_3\kappa_2 ds.$$

For the common boundary points, the first variation turns into

$$\delta s = -\int_{s_1}^{s_2} d_3 \kappa_2 ds.$$

If N_2 is taken instead of the offset distance d_3, we obtain

$$\delta s = -\int_{s_1}^{s_2} \delta N_2 \kappa_2 ds.$$

Case 2. Differentiating the vector $\boldsymbol{y_1}(s)$ according to formula (6) appears as

$$y_1' = (d_1' - d_2\kappa_1)T_1 + d_2' N_1 + (d_1\kappa_1 + d_3')N_2. \#$$

In a similar methodology to that in Case 1, we get the following expressions

$$s_* = s - \int_{s_1}^{s_2} (d_1' - d_2\kappa_1) ds = s - [d_1]_{s_1}^{s_2} + \int_{s_1}^{s_2} d_2\kappa_1 ds,$$

and

$$\delta s = -[d_1]_{s_1}^{s_2} + \int_{s_1}^{s_2} d_2\kappa_1 ds.$$

The increment δs turns into,

$$\delta s = \int_{s_1}^{s_2} d_2\kappa_1 ds.$$

in the conditions for the boundary points to be zero. However, δN_1 is taken instead of the offset distance d_2, we get

$$\delta s = -\int_{s_1}^{s_2} \delta N_1 \kappa_1 ds.$$

4.2. THE BISHOP CURVATURES VARIATIONS OF GENERALIZED OFFSET CURVES

We take the Bishop curvatures and variations of the generalized offset curve $\boldsymbol{r}_*$ into consideration as in the following cases:

Case 1.

$$\delta\kappa_2 = \kappa_{2_*} - \kappa_2 = c_0 e^{\int \tau_*(s)ds} - c_0 e^{\int \tau(s)ds}, \qquad c_0 \in R_0^+$$
$$\delta\kappa_2 = c_0\left(e^{\int \tau_*(s)ds} - e^{\int \tau(s)ds}\right).$$

Case 2.

$$\delta\kappa_1 = \kappa_{1_*} - \kappa_1 = c_0 e^{\int \tau_*(s)ds} - c_0 e^{\int \tau(s)ds}, \qquad c_0 \in R_0^-$$
$$\delta\kappa_1 = c_0\left(e^{\int \tau_*(s)ds} - e^{\int \tau(s)ds}\right).$$

4.3. APPLICATIONS

E.g. 4.1. Let $\boldsymbol{r}(s)=\left(\sqrt{2}s^2+\frac{s}{\sqrt{2}},\ \sqrt{2}s^2-\frac{s}{\sqrt{2}},\ 2s^2\right)$ be a pseudo-null curve. For the first case κ_1 vanishes, and then the Bishop frame fields and the curvature κ_2 are found as follows;

$$T_1(s)=\left(2\sqrt{2}s+\frac{1}{\sqrt{2}},2\sqrt{2}s-\frac{1}{\sqrt{2}},4s\right),\quad N_1(s)=\frac{1}{c_0}\left(2\sqrt{2},2\sqrt{2},4\right),$$
$$N_2(s)=c_0\left(-\sqrt{2}s^2-\frac{s}{\sqrt{2}}+\frac{1}{8\sqrt{2}},-\sqrt{2}s^2+\frac{s}{\sqrt{2}}+\frac{1}{8\sqrt{2}},-2s^2-\frac{1}{8}\right)\text{ and }\kappa_2=c_0,\quad c_0\in R_0^+.$$

For $c_0=8\sqrt{2}$, then the Bishop frame becomes specifically as,

$$T_1(s)=\left(2\sqrt{2}s+\frac{1}{\sqrt{2}},2\sqrt{2}s-\frac{1}{\sqrt{2}},4s\right),\qquad N_1(s)=\left(\frac{1}{4},\frac{1}{4},\frac{1}{2\sqrt{2}}\right),$$

$$N_2(s)=\left(-16s^2-8s+1,-16s^2+8s+1,-16\sqrt{2}s^2-\sqrt{2}\right).$$

We find the generalized offset curves with different offset distances as follows;

a) For the offset distances $d_1=-\frac{1}{2}s,\ d_2=-4$ and $d_3=1$,

$$r_{a*}(s)=\left(-16s^2+\frac{\left(1-16\sqrt{2}\right)}{2\sqrt{2}}s,-16s^2+\frac{\left(-1+16\sqrt{2}\right)}{2\sqrt{2}}s,-16\sqrt{2}s^2-2\sqrt{2}\right),$$

b) For the offset distances $d_1=2\sqrt{2}s,\ d_2=16s$ and $d_3=\frac{1}{2}$,

$$r_{b*}(s)=\left(\sqrt{2}s^2+\frac{\left(2\sqrt{2}+1\right)}{\sqrt{2}}s+\frac{1}{2},\sqrt{2}s^2+\frac{\left(6\sqrt{2}-1\right)}{\sqrt{2}}s+\frac{1}{2},(4\sqrt{2}+2)s^2-\frac{1}{\sqrt{2}}\right).$$

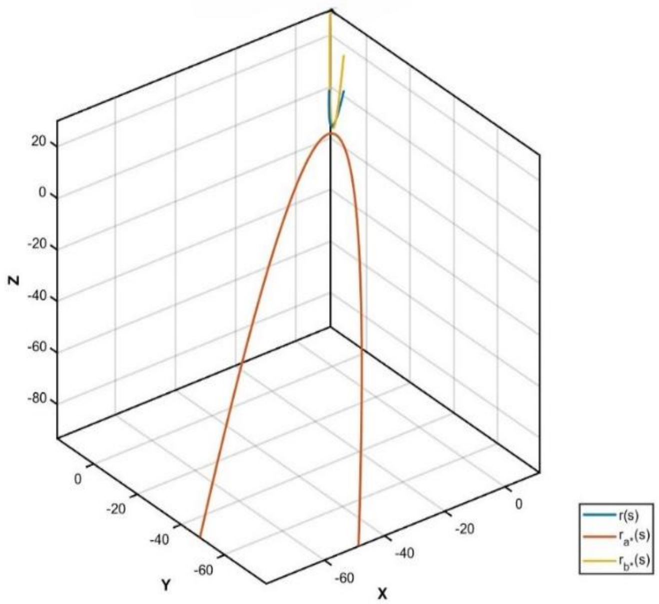

Figure 4.1. The original curve r (blue), and its generalized offset curves r_{a*} and r_{b*} for different offset distances (red and yellow, respectively).

E.g. 4.2. Let $\boldsymbol{r}(s) = (e^{s+2},\ s+3\ , e^{s+2})$ be a pseudo-null curve. Due to $\kappa_1 = 0$, the Bishop frame vectors and the curvature κ_2 are

$$T_1(s) = (e^{s+2}, 1, e^{s+2})\ ,\ \ N_1(s) = \frac{1}{c_0}(\mathrm{e}^2, 0, \mathrm{e}^2),\ N_2(s) = c_0\left(\frac{e^{2s+4}-1}{2e^2}, -e^s, \frac{-e^{2s+4}-1}{2e^2}\right)$$

and $\kappa_2 = c_0 e^s,\ \ c_0 \in R_0^+$.

For $c_0 = 2e^2$, the vectors become as follows;

$$T_1(s) = (e^{s+2}, 1, e^{s+2}),\ \ \ N_1(s) = \left(\frac{1}{2}, 0, \frac{1}{2}\right),\ \ \ N_2(s) = (e^{2s+4}-1, -2e^{s+2}, -e^{2s+4}-1).$$

The generalized offset curves with different offset distances are obtained.

a) For the offset distances $d_1 = -1,\ \ d_2 = 2$ and $d_3 = 1$

$$\boldsymbol{r}_{\boldsymbol{a}*}(s) = (e^{2s+4}, -2e^{s+2}+s+2, -e^{2s+4}).$$

b) For the offset distances $d_1 = e^{s+2},\ \ d_2 = 2s$ and $d_3 = -1$

$$\boldsymbol{r}_{\boldsymbol{b}*}(s) = (e^{s+2}+s+1, 3e^{s+2}+s+3, 2e^{2s+4}+e^{s+2}+s+1).$$

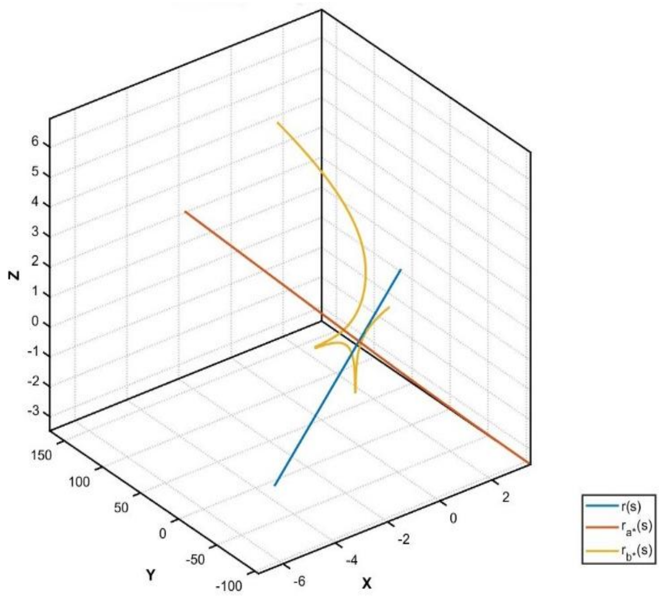

Figure 4.2. The original curve r (blue), and its generalized offset curves r_{a*} and r_{b*} for different offset distances (red and yellow, respectively).

5. CONCLUSIONS

The work of Bulut and Çalışkan (2015) aims to enrich the idea of generalized offset curves and their variations in Minkowski-3 space. Therefore, generalized offset curves are studied for pseudo-null curves by the ϵ-neighborhood approach. In the study, the generalized version of pseudo-null curves is studied by two different frames: Frenet and Bishop. The analysis of the pseudo curves by Frenet and Bishop parameters reveals interesting results in terms of the quantities characterizing the curves. The results about the variations of the arc length and the torsion of a pseudo-null curve in terms of the Frenet and Bishop parameters will make a theoretical contribution to differential geometry. Furthermore, it is anticipated that the variational results will provide a basis for developing new perspectives in the field of applied sciences.

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