*ORIGINAL PAPER*

# THE GEOMETRY INVARIANT PSEUDOPARALLEL SUBMANIFOLD OF SASAKIAN MANIFOLDS EQUIPPED WITH A GENERAL CONNECTION

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***Abstract.*** *The aim of the present paper is to study invariant pseudo-parallel submanifolds of a Sasakian manifold admitting a general connection. We investigate the necessary and sufficient conditions for an invariant pseudo-parallel submanifold to be totally geodesic with respect to the general connection, and the structures induced by the ambient manifold under certain conditions. The obtained results are evaluated.*

***Keywords:*** *Sasakian Manifold; pseudoparallel and Ricci pseudoparallel; Ricci-generalized pseudoparallel and 2-pseudoparallel submanifolds; general connection.*

## 1. INTRODUCTION

Sasakian manifolds are an important class in differential geometry and especially in Riemannian geometry. Sasakian manifolds can be thought of as Riemannian manifolds with a 1-dimensional contact structure and can be viewed as a strange but natural generalization of Kaehler manifolds. These manifolds have important applications in mathematics, physics, and engineering. Because it is related to contact geometry, it is used for analysis of mechanical systems such as robot arms and dynamic control systems. Especially, in areas such as contact mechanical systems and automatic motion planning, Sasakian geometry can be used. In geometric optics models, Sasakian manifolds can be used to understand how light travels in curved spaces. Particle motion under magnetic fields can be modeled using contact and Sasakian structures. Sasakian manifolds, which have applications in many similar areas, are a very important classes for differential geometry.

A connection on a manifold provides a way to differentiate vector fields along curves. More formally, a connection allows the definition of a derivative of a vector field along another vector field, facilitating the study of how vectors change in a manifold's curved geometry. Levi-Civita connection is the most common type of connection, uniquely determined for a Riemannian manifold. It is compatible with the metric and is torsion-free, meaning the connection does not introduce any twisting in the vectors.

General connection, often referred to as a connection on a differentiable manifold, is a fundamental concept in differential geometry and plays a crucial role in the study of curved spaces. General connections are a powerful tool in understanding the geometric structure of manifolds. They provide the framework for defining differentiation in curved spaces and have significant implications in both mathematics and physics. The study of connections continues

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to be an active area of research, leading to deeper insights into the geometry and topology of manifolds.

The aim of the present paper is to study invariant pseudo-parallel submanifolds of a Sasakian manifold admitting a general connection. We search the necessary and sufficient conditions for an invariant pesudo-parallel submanifold to be totally geodesic with respect to general connection and reduced structures by structures on the ambient manifold are investigated under the some conditions. The obtained results are evaluated.

## 2. PRELIMINARIES

An almost contact manifold is odd-dimensional manifold $\widetilde{M}^{2n+1}$ which carries a field $\phi$ of endomorphism of the tangent space, called the structure vector field $\xi$, and a 1-form $\eta$-satisfying;

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{1}$$

where $I$ denote the identity mapping of tangent space of each point at $M$. From $(1)$, it follows

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad rank(\phi) = 2n. \tag{2}$$

In this case, $\widetilde{M}^{2n+1}$ is said to be almost contact manifold [1]. An almost contact manifold $\widetilde{M}^{2n+1}$ is called an almost contact metric manifold if a Riemannian metric tensor $g$ satisfies

$$\begin{cases} g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \\ \eta(X) = g(X, \xi), \quad g(\phi X, Y) + g(X, \phi Y) = 0 \end{cases} \tag{3}$$

for any vector fields $X, Y$ on $\widetilde{M}^{2n+1}$ The structure $(\phi, \xi, \eta, g)$ on $\widetilde{M}^{2n+1}$ is said to be almost contact metric structure. In an almost contact metric structure $(\varphi, \xi, \eta, g)$, the Nijenhuis tensor and the fundamental form are, respectively, defined by

$$N_\phi(X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

and

$$\Phi(X, Y) = g(X, \phi Y)$$

for all $X, Y \in \Gamma(T\widetilde{M})$. An almost contact metric structure $(\varphi, \xi, \eta, g)$ is said to be normal if

$$N_\phi(X, Y) + 2d\eta(X, Y)\xi = 0.$$

A normal contact metric manifold is called a Sasakain manifold. It is well known that a contact metric structure is a Sasakian if and only if

$$(\widetilde{\nabla}_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{4}$$

for all $X, Y \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ denote the Levi-Civita connection on $\widetilde{M}^{2n+1}$. It follows that

$$(\widetilde{\nabla}_X\xi = -\phi X. \tag{5}$$

On the other hand, by $\tilde{R}$ and $\tilde{S}$ we denote the Riemannian curvature and Ricci tensors of $\widetilde{\nabla}$ in Sasakian manifold $\widetilde{M}^{2n+1}$, respectively, then we have

$$\tilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{6}$$

$$\tilde{R}(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{7}$$

$$\tilde{S}(X,\xi) = 2n\eta(X). \tag{8}$$

Recently, in [2] Biswas and Baishya introduced a new connection which is called general connection in the set of contact geometry as

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \lambda_1[(\widetilde{\nabla}_X \eta)(Y)\xi - \eta(Y)\widetilde{\nabla}_X \xi] + \lambda_2 \eta(X)\phi Y, \tag{9}$$

for all $X,Y \in \Gamma(T\widetilde{M})$, where $\lambda_1$ and $\lambda_2$ are real constants.

The general connection can be seen generalization other connections. Namely,

- Quater symmetric connection metric connection for $(\lambda_1,\lambda_2) = (0,-1)$ in [3,4];
- Schouten-van Kampen connection for $(\lambda_1,\lambda_2) = (1,0)$ in [5];
- Tanaka-Webster connection for $(\lambda_1,\lambda_2) = (1,-1)$ in [6];
- Zamkovoy connection for $(\lambda_1,\lambda_2) = (1,1)$ in [7].

Many authors have also conducted numerous studies on the geometry of Sasakian manifolds admitting a general connection [8-13]. In Sasakian geometry, making use of (5) and (9) one can easily to see

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + \lambda_1\{g(\phi X,Y)\xi + \eta(Y)\phi X\} + \lambda_2 \eta(X)\phi Y. \tag{10}$$

Now, we will calculate the covariant derivative of $\phi$ with respect to general connection $\overline{\nabla}$. By using (10), we have

$$\begin{aligned}(\overline{\nabla}_X \phi)Y &= \overline{\nabla}_X \phi Y - \phi \overline{\nabla}_X Y \\ &= \widetilde{\nabla}_X \phi Y + \lambda_1\{g(X,\phi^2 Y)\xi + \eta(\phi Y)\phi X\} + \lambda_2 \eta(X)\phi^2 Y \\ &\quad -\phi\{\widetilde{\nabla}_X Y + \lambda_1[g(X,\phi Y)\xi + \eta(Y)\phi X] + \lambda_2 \eta(X)\phi Y\} \\ &= (\widetilde{\nabla}_X \phi)Y - \lambda_1 g(\phi X,\phi Y)\xi - \lambda_1 \eta(Y)\phi^2 Y \\ &= (1-\lambda_1)\{g(X,Y)\xi - \eta(Y)X\}.\end{aligned} \tag{11}$$

which requires

$$\overline{\nabla}_X \xi = (\lambda_1 - 1)\phi X, \tag{12}$$

for all $X,Y \in \Gamma(T\widetilde{M})$. Furthermore, the Riemannian curvature and Ricci tensors $\overline{R}$ and $\overline{S}$ with respect to general connection $\overline{\nabla}$ are given by

$$\begin{aligned}\overline{R}(X,Y)Z &= \tilde{R}(X,Y)Z + (\lambda_1^2 - 2\lambda)\{g(\phi X,Z)\phi Y - g(Y,\phi Z)\phi X\} \\ &-2\lambda_2 g(Y,\phi X)\phi Z + (\lambda_1 - \lambda_1\lambda_2 + \lambda_2)\{g(X,Z)\eta(Y)\xi \\ &-g(Y,Z)\eta(X)\xi + \eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X\},\end{aligned} \tag{13}$$

and

$$\begin{aligned}\overline{S}(X,Y) &= \tilde{S}(X,Y) - (\lambda_1^2 - \lambda_1 - \lambda_2 - \lambda_1\lambda_2)g(X,Y) \\ &+[\lambda_1^2 + (2n-1)\lambda_1\lambda_2 - (2n+1)(\lambda_1+\lambda_2)]\eta(X)\eta(Y),\end{aligned} \tag{14}$$

for all $X,Y \in \Gamma(T\widetilde{M})$ [8].

Furthermore, (6) and (13), we observe

$$\overline{R}(X,Y)\xi = (\lambda_1 - 1)(1-\lambda_2)[\eta(Y)X - \eta(X)Y]. \tag{15}$$

(8) and (15) require

$$\overline{S}(X,\xi) = 2n(\lambda_1 - 1)(\lambda_2 - 1)\eta(X). \tag{16}$$

Now, let $M$ be an immersed submanifold of a semi-Riemannian manifold $(\widetilde{M},g)$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{17}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{18}$$

for all $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $\nabla$ and $\nabla^{\perp}$ are induced connections on $M$, $\Gamma(T^{\perp}M)$ and $\sigma$, $A$ denote the second fundamental form and shape operatory of $M$, respectively.

For a submanifold $M$ of a semi-Riemannian manifold $(\widetilde{M},g)$, the Gauss and Weingarten equations are given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \\ &+(\widetilde{\nabla}_X\sigma)(Y,Z) - (\widetilde{\nabla}_Y\sigma)(X,Z),\end{aligned} \tag{19}$$

and

$$g(\tilde{R}(X,Y)U,V) = g(R^{\perp}(X,Y)U,V) + g([A_V,A_U]X,Y), \tag{20}$$

for all $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$, where $R$ and $R^{\perp}$ are the Riemannian curvature tensors of $M$ and $\Gamma(T^{\perp}M)$, respectively.

The covariant derivative of $\sigma$ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}\sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z), \tag{21}$$

for all $X,Y,Z \in \Gamma(TM)$.

## 3. INVARIANT PSEUDOPARALLEL SUBMANIFOLDS IN SASAKIAN MANIFOLDS ADMITTING A GENERAL CONNECTION

The geometry of submanifolds of a contact metric manifold is depend on the behaviour of contact metric structure $\phi$. A submanifold $M$ of a contact manifold is said to be invariant if the structure vector field $\xi$ is tangent to $M$ at every point of $M$ and $\phi X$ is tangent to $M$ for any vector feld $X$ tangent to $M$ at every point of $M$. In other words, $\phi(TM) \subset TM$ at each point of $M$.

**Theorem 1.** Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$. Then we have following equalities;
i. $\sigma(\phi X, Y) = \sigma(X, \phi Y) = \phi\sigma(X, Y)$,
ii. $\sigma(X, \xi) = 0, A_V\xi = 0$,
iii. The second fundamental forms $\sigma$ and $\bar{\sigma}$ of $M$ with respect to $\widetilde{\nabla}$ and $\overline{\nabla}$ are equal.
iv. $\bar{R}(X, Y)\xi = R(X, Y)\xi$,
where $R$ denote the Riemannian curvature tensor of submanifold $M$ with respect to $\overline{\nabla}$.

*Proof:* Since the proof is the direct results of direct calculations, we do not give the proof.

In the rest of this paper, we will assume that an invariant submanifold $M$ of a contact metric manifold $\widetilde{M}$. A submanifold $M$ of a semi-Riemannian manifold $(\widetilde{M}, g)$ is called Chaki-pseudo parallel if its second fundamental form $\sigma$ satisfies

$$(\widetilde{\nabla}_X\sigma)(Y, Z) = 2\gamma(X)\sigma(Y, Z) + \gamma(Y)\sigma(X, Z) + \gamma(Z)\sigma(X, Y), \tag{22}$$

for all $X, Y, Z \in \Gamma(TM)$ and $\gamma$ is a nowhere vanishing 1-form.

In particular, if $\gamma = 0$ then $M$ is said to be parallel submanifold of $\widetilde{M}$ [14]. For a submanifold $M$ of a semi-Riemannian manifold $(\widetilde{M}, g)$, the Gauss and Weingarten equations are given by

$$\begin{aligned}\tilde{R}(X, Y)Z &= R(X, Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \\ &+(\widetilde{\nabla}_X\sigma)(Y, Z) - (\widetilde{\nabla}_Y\sigma)(X, Z),\end{aligned} \tag{23}$$

and

$$g(\tilde{R}(X, Y)U, V) = g(R^{\perp}(X, Y)U, V) + g([A_V, A_U]X, Y), \tag{24}$$

for all $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, where $R$ and $R^{\perp}$ are the Riemannian curvature tensors of $M$ and $\Gamma(T^{\perp}M)$, respectively.

**Theorem 2.** Let M be an invariant submanifold of a sasakian manifold $\widetilde{M}^{2n+1}$. Then M is a Chaki pseudoparallel with respect to a general connection if and only if M is either totally geodesic submanifold or $\alpha(\xi) = 1 - \lambda$.

*Proof:* Let us suppose that $M$ is Chaki pseudoparallel. Then from $(22)$, there a exists 1-form $\alpha$ such that

$$(\overline{\nabla}_X\sigma)(Y, Z) = 2\alpha(X)\sigma(Y, Z) + \alpha(Y)\sigma(X, Z) + \alpha(Z)\sigma(X, Y),$$

for all $X, Y, Z \in \Gamma(TM)$. Here, setting $Z = \xi$ and making use of $(21)$ and $(5)$, we have

$$-\sigma(Y, \nabla_X\xi) = \alpha(\xi)\sigma(X, Y)$$

which follows

$$[\alpha(\xi)+\lambda-1]\phi\sigma(X,Y)=0.$$

This proves our assertion. We have following corollary since every totally geodesic submanifold is a Chaki pseudoparallel.

**Corollary 1.** Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection. Then M is a parallel submanifold if and only if M is a totally geodesic unless the general connection $\overline{\nabla}$ is a not Schouten-Van Kampen.

**Definition 1.** Let M be an immersed submanifold of a semi-Riemannian manifold $(\widetilde{M},g)$. If there exist forms ψ and θ such that

$$(\overline{\nabla}_X\overline{\nabla}_Y\sigma)(Z,U)=\psi(X,Y)\sigma(Z,U)+\theta(X)(\overline{\nabla}_Y\sigma)(Z,U), \tag{25}$$

then $M$ is said to be generalized 2-recurrent submanifold [15].

**Theorem 3.** Let M be an invariant submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection $\overline{\nabla}$. M is a generalized 2-recurrent submanifold with respect to a general connection if and only if M is a totally geodesic submanifold unless the general connection $\overline{\nabla}$ is not a Schouten-Van Kampen.

*Proof:* Let us suppose that $M$ be an invariant generalized 2-recurrent submanifold with respect to general connection $\overline{\nabla}$. Then there exist forms $\psi$ and $\theta$ such that

$$(\overline{\nabla}_X\overline{\nabla}_Y\sigma)(Z,U)=\psi(X,Y)\sigma(Z,U)+\theta(X)(\overline{\nabla}_Y\sigma)(Z,U), \tag{26}$$

for all $X,Y,Z,U\in\Gamma(TM)$. Here, putting $Z=U=\xi$ in $(26)$, this yields to

$$\begin{aligned}&\nabla^{\perp}(\overline{\nabla}_X\sigma)(\xi,\xi)-(\overline{\nabla}_Y\sigma)(\nabla_X\xi,\xi)-(\overline{\nabla}_X\sigma)(\xi,\nabla_Y\xi)\\&-(\overline{\nabla}_{\nabla_XY}\sigma)(\xi,\xi)=\psi(X,Y)\sigma(\xi,\xi)+\theta(X)(\overline{\nabla}_Y\sigma)(\xi,\xi).\end{aligned} \tag{27}$$

Taking into account Theorem 2.1 and making use of $(21)$, after the necessary adjustments, $(27)$ takes the form

$$\sigma(\nabla_X\xi,\nabla_Y\xi)=(\lambda-1)^2\sigma(\phi X,\phi Y)=0.$$

This completes of the proof. A submanifold $M$ of a Riemannian manifold $(\widetilde{M},g)$ is called Deszcz pseudoparallel if its second fundamental form $\sigma$ satisfies

$$\overline{R}\cdot\sigma=\ell_\sigma Q(g,\sigma), \tag{28}$$

where $\ell_\sigma$ is a function on $\widetilde{M}$. In particular, if $\ell_\sigma=0$, then $M$ is said to be semiparallel[14].

**Theorem 4.** Let M be an invariant pseudoparallel submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection $\overline{\nabla}$. Then M is either totally geodesic or the function $\ell=(\mu-1)(\lambda-1)$.

*Proof:* Since $M$ is a Deszcz pseudoparallel, form $(28)$, there exists a function such that

$$(\overline{R}(X,Y)\cdot\sigma)(U,V)=\ell Q(g,\sigma)(U,V;X,Y),$$

for all $X,Y,U,V\in\Gamma(TM)$. This implies that

$$R^{\perp}(X,Y)\sigma(U,V)-\sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)$$
$$=-\ell\{\sigma((X\wedge_g Y)U,V)+\sigma(U,(X\wedge_g Y)V)\}. \quad (29)$$

Taking $V=\xi$ in $(29)$ and by using $(15)$ and Theorem 2.1, we obtain

$$\sigma(U,R(X,Y)\xi)=\ell\sigma(U,(X\wedge_g Y)\xi),$$

that is,

$$(\lambda-\mu\lambda+\mu-1)\sigma(U,\eta(X)Y-\eta(Y)X)=\ell\sigma(U,\eta(Y)X-\eta(X)Y).$$

This proves our assertion. We have the following corollary from Theorem.

**Corollary 2.** Let M be an invariant semiparallel submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection $\overline{\nabla}$. Then M is either totally geodesic or the $\overline{\nabla}$ is reduced Zamkovoy connection.

**Definition 2.** Let M be a submanifold of a semi-Riemannian manifold $(\widetilde{M},g)$. If the tensors $\widetilde{R}\cdot\sigma$ and $Q(\widetilde{S},\sigma)$ are linearly dependent, then M is said to be generalized Ricci-pseudoparallel submanifold. In particular $\widetilde{R}\cdot\sigma=0$, it is called generalized Ricci symmetric [16-18].

**Theorem 5.** Let M be an invariant generalized Ricci-pseudoparallel submanifold $\widetilde{M}^{2n+1}$ admitting a general connection $\overline{\nabla}$. Then at least one of the following is true:

i. $M$ is a totally geodesic submanifold.
ii. $\ell_{\overline{S}}=\frac{1}{2n}$.
iii. $\overline{\nabla}$ is reduced Schouten-Van Kampen connection.
iv. $\overline{\nabla}$ is reduced Tanaka-Webster connection.
v. $\overline{\nabla}$ is reduced Zamkovoy connection.

*Proof:* If $M$ is an invariant generalized Ricci-pseudoparallel submanifold with respect to a general conncetion, then there exists a function on $\ell_{\overline{S}}$ such that

$$(\overline{R}(X,Y)\cdot\sigma)(U,V)=\ell_{\overline{S}}Q(\overline{S},\sigma)(U,V,X;Y),$$

for all $X,Y,U,V\in\Gamma(TM)$. This means that

$$R^{\perp}(X,Y)\sigma(U,V)-\sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)$$
$$=-\ell_{\overline{S}}\{\sigma((X\wedge_{\overline{S}} Y)U,V)+\sigma(U,(X\wedge_{\overline{S}} Y)V)\}. \quad (30)$$

Taking $V=\xi$ in $(30)$, consider Theorem 2.1, we have

$$\sigma(U,R(X,Y)\xi)=\ell_{\overline{S}}\sigma(U,(X\wedge_{\overline{S}} Y)\xi),$$

or

$$(\lambda_1-1)(1-\lambda_2)\eta(U,\eta(Y)X-\eta(X)Y)=\ell_{\overline{S}}\{U,\overline{S}(Y,\xi)X-\overline{S}(X,\xi)Y\}.$$

By means of $(15)$ and $(16)$, we infer

$$(\lambda_1 - 1)(1 - \lambda_2)\sigma(U, \eta(X)Y - \eta(Y)X)$$

$$= 2n(\lambda_1 - 1)(\lambda_2 - 1)\ell_{\overline{S}}\sigma(U, \eta(Y)X - \eta(X)Y),$$

that is,

$$(\lambda_1 - 1)(\lambda_2 - 1)[2n\ell_{\overline{S}} - 1]\sigma(U, \eta(X)Y - \eta(Y)X) = 0. \quad (31)$$

This completes the proof. From the Theorem 2.9, we can give the following corollary.

**Corollary 3.** Let M be an invariant generalized Ricci-symmetric submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection $\overline{\nabla}$. Then at least one of the following is true:
i. $M$ is a totally geodesic submanifold.
ii. The general connection $\overline{\nabla}$ is reduced to at least one of the Schouten-Van Kampen, Zamkovoy, Tanaka-Webster connections.

**Definition 3.** Let M be a submanifold of a semi-Riemannian manifold $(\widetilde{M}, g)$. If the tensors $\widetilde{R} \cdot \widetilde{\nabla}\sigma$ and $Q(g, \widetilde{\nabla}\sigma)$ are linearly dependent, then M is said to be generalized 2-pseudoparallel submanifold.

In particular, $\tilde{R} \cdot \widetilde{\nabla}\sigma = 0$, it is called generalized 2-pseudosymmetric submanifold.

**Theorem 6.** Let M be an invariant generalized 2-pseudoparallel submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection $\overline{\nabla}$. Then at least one of the following is true:
i. $M$ is totally geodesic submanifold.
ii. $\ell_{\overline{\nabla}} = (\lambda_1 - 1)(\lambda_2 - 1)$.
iii. $\xi$ is Killing vector field with respect to $\overline{\nabla}$.

*Proof:* Since $M$ be an invariant 2-pseudoparallel submanifold, we can write

$$(\overline{R}(X, Y) \cdot \overline{\nabla}\sigma)(U, V, Z) = \ell_{\overline{\nabla}\sigma} Q(g, \overline{\nabla} \cdot \sigma)(U, V, Z; X, Y),$$

for all $X, Y, U, V, Z \in \Gamma(TM)$, where $\ell_{\overline{\nabla}\sigma}$ is a function $\widetilde{M}$. This leads to

$$\begin{aligned} &R^{\perp}(X, Y)(\overline{\nabla}_U\sigma)(V, Z) - (\overline{\nabla}_{R(X,Y)U}\sigma)(V, Z) \\ &-(\overline{\nabla}_U\sigma)(V, R(X, Y)Z) - (\overline{\nabla}_U\sigma)(R(X, Y)V, Z) \\ &= -\ell_{\overline{\nabla}\sigma}\left\{(\overline{\nabla}_{(X\wedge_g Y)U}\sigma)(V, Z) + (\overline{\nabla}_U\sigma)((X \wedge_g Y)V, Z)\right. \\ &\left.+(\overline{\nabla}_U\sigma)(V, (X \wedge_g Y)Z)\right\}. \end{aligned} \quad (32)$$

Here if $X = Z = \xi$ is taken in $(32)$, we have

$$
\begin{aligned}
&R^{\perp}(\xi,Y)(\overline{\nabla}_U\sigma)(V,\xi)-(\overline{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi)\\
&-(\overline{\nabla}_U\sigma)(R(\xi,Y)V,\xi)-(\overline{\nabla}_U\sigma)(V,R(\xi,Y)\xi)\\
&=-\ell_{\overline{\nabla}\sigma}\Big\{(\overline{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi)+(\overline{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi)\\
&+(\overline{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi)\Big\}
\end{aligned}
\tag{33}
$$

Nex, we will calculate these statement. Non-zero component of the first term takes the form

$$
\begin{aligned}
R^{\perp}(\xi,Y)(\overline{\nabla}_U\sigma)(V,\xi) &= R(\xi,Y)^{\perp}\{\nabla_U^{\perp}\sigma(V,\xi)-\sigma(\nabla_U V,\xi)-\sigma(\nabla_U\xi,V)\}\\
&=-R^{\perp}(\xi,Y)\sigma(\nabla_U V)\\
&=-(\lambda_1-1)R^{\perp}(\xi,Y)\phi\sigma(U,V).
\end{aligned}
\tag{34}
$$

As the second term, one can see

$$
\begin{aligned}
(\overline{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) &= -\sigma(\nabla_{R(\xi,Y)U}\xi,V)\\
&=-(\lambda_1-1)\phi\sigma(R(\xi,Y)U,V)\\
&=(\lambda_1-1)\phi\sigma(\eta(U)Y-g(Y,U)\xi,V)(\lambda_1-1)(\lambda_2-1)\\
&=(\lambda_1-1)^2(\lambda_2-1)\eta(U)\phi\sigma(Y,V).
\end{aligned}
\tag{35}
$$

The non-zero component of the third term is

$$
\begin{aligned}
(\overline{\nabla}_U\sigma)(R(\xi,Y)V,\xi) &= -\sigma(\nabla_U\sigma,R(\xi,Y)V)\\
&=-(\lambda_1-1)\phi\sigma(U,R(\xi,Y)V)\\
&=(\lambda_1-1)^2(1-\lambda_2)\eta(V)\phi\sigma(Y,U).
\end{aligned}
\tag{36}
$$

In the same way, the non-zero components of the last term of the left side of the equality are

$$
\begin{aligned}
(\overline{\nabla}_U\sigma)(V,R(\xi,Y)\xi) &= (\overline{\nabla}_U\sigma)(V,Y-\eta(Y)\xi)(\lambda_1-1)(1-\lambda_2)\\
&=(\lambda_1-1)(1-\lambda_2)\{(\overline{\nabla}_U\sigma)(V,Y)-(\overline{\nabla}_U\sigma)(V,\eta(Y)\xi)\}\\
&=(\lambda_1-1)(1-\lambda_2)\{(\overline{\nabla}_U\sigma)(V,Y)+\sigma(U[\eta(Y)]\xi+\eta(Y)\nabla_U\xi,V)\}\\
&=(\lambda_1-1)(1-\lambda_2)\{(\overline{\nabla}_U\sigma)(V,Y)+(\lambda_1-1)\eta(Y)\phi\sigma(U,V)\}.
\end{aligned}
\tag{37}
$$

On the other hand, non-zero the first components of the right side of the equality are also

$$
\begin{aligned}
(\overline{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) &= -\sigma(\nabla_{(\xi\wedge_g Y)U}\xi, V) \\
&= -(\lambda_1 - 1)\phi\sigma(g(Y,U)\xi - \eta(U)Y, V) \\
&= (\lambda_1 - 1)\eta(U)\phi\sigma(Y,V).
\end{aligned} \tag{38}
$$

As second term is also us

$$
\begin{aligned}
(\overline{\nabla}_U\sigma)((\xi \wedge_g Y)V,\xi) &= -\sigma(\nabla_U\xi, (\xi \wedge_g Y)V) \\
&= -(\lambda_1 - 1)\phi\sigma(U, g(Y,V)\xi - \eta(V)Y) \\
&= (\lambda_1 - 1)\eta(V)\eta(V)\phi\sigma(U,Y).
\end{aligned} \tag{39}
$$

Finally, for the last term of the left of equality, we have

$$
\begin{aligned}
(\overline{\nabla}_U\sigma)(V,(\xi \wedge_g Y)\xi) &= (\overline{\nabla}_U\sigma)(V,\eta(Y)\xi - Y) \\
&= (\overline{\nabla}_U\sigma)(V,\eta(Y)\xi) - (\overline{\nabla}_U\sigma)(V,Y) \\
&= -\sigma(\nabla_U\eta(Y)\xi, V) - (\overline{\nabla}_U\sigma)(V,Y) \\
&= (\lambda_1 - 1)\eta(Y)\phi\sigma(U,V) - (\overline{\nabla}_U\sigma)(V,Y).
\end{aligned} \tag{40}
$$

Thus the statements $(34-40)$ are put in $(33)$ we reach at

$$
\begin{aligned}
&-(\lambda_1 - 1)R^{\perp}(\xi,Y)\phi\sigma(U,V) - (\lambda_1 - 1)^2(1-\lambda_2)\eta(U)\phi\sigma(Y,V) \\
&+(\lambda_1 - 1)^2(1-\lambda_2)\eta(V)\phi\sigma(Y,U) \\
&-(\lambda_1 - 1)(1-\lambda_2)\{(\overline{\nabla}_U\sigma)(V,Y) + (\lambda_1 - 1)\eta(Y)\phi\sigma(U,V)\} \\
&= -\ell_{\overline{\nabla}\sigma}\{(\lambda_1 - 1)\eta(U)\phi\sigma(V,Y) + (\lambda_1 - 1)\eta(V)\phi\sigma(U,Y) \\
&+(\lambda_1 - 1)\eta(Y)\phi\sigma(U,V) - (\overline{\nabla}_U\sigma)(V,Y)\}.
\end{aligned}
$$

In the last equality, taking $V = \xi$ and after the necessary adjustments are made, we conclude that

$$
(\lambda_1 - 1)[(\lambda_1 - 1)(\lambda_2 - 1) - \ell_{\overline{\nabla}\sigma}]\phi\sigma(U,Y) = 0,
$$

which proves our assertions.

We can give the following corrollary from the Theorem 2.12.

**Corollary 4.** Let M be an invariant generalized 2-pseudosymmetric submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection $\overline{\nabla}$. Then at least one of the following is true:
i. $M$ is totally geodesic.
ii. The general connection $\overline{\nabla}$ is reduced to at least one of the Schouten-Van Kampen, Zamkovoy, Tanaka-Webster connections.

**Definition 4.** Let M be a submanifold of a semi-Riemannian manifold $(\widetilde{M}, g)$. If the tensors $\overline{R}\cdot\overline{\nabla}\sigma$ and $Q(\overline{S},\overline{\nabla}\sigma)$ are linearly dependent, then M is said to be generalized 2-Ricci pseudoparallel submanifold.

**Theorem 7.** Let M be an invariant generalized 2-Ricci pseudoparallel submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection $\overline{\nabla}$. Then at least one of the following is true:
i. $M$ is totally geodesic submanifold.
ii. $\ell_{\overline{S}} = -\frac{1}{2n}$.
iii. $\overline{\nabla}$ is reduced Zamkovoy connection.

*Proof:* If $M$ is an invariant generalized 2-Ricci pseudoparallel submanifold of a Sasakian manifold $\widetilde{M}^{2n+1}$ admitting a general connection, then there is a function such on $\widetilde{M}^{2n+1}$ that

$$(\overline{R}(X,Y)\overline{\nabla}\sigma)(U,V,Z) = \ell_{\overline{S}}Q(\overline{S},\overline{\nabla}\sigma)(U,V,Z;X,Y),$$

for all $X,Y,Z,U,V \in \Gamma(TM)$. Explanation of this expression is

$$R^{\perp}(X,Y)(\overline{\nabla}_U\sigma)(V,Z) - (\overline{\nabla}_{R(X,Y)U}\sigma)(V,Z)$$
$$-(\overline{\nabla}_U\sigma)(R(X,Y)V,Z) - (\overline{\nabla}_U\sigma)(V,R(X,Y)Z)$$
$$= -\ell_{\overline{S}}\Big\{(\overline{\nabla}_{(X\wedge_{\overline{S}}Y)U}\sigma)(V,Z) + (\overline{\nabla}_U\sigma)((X\wedge_{\overline{S}}Y)V,Z)$$
$$+(\overline{\nabla}_U\sigma)(V,(X\wedge_{\overline{S}}Y)Z)\Big\}.$$

If this expression is re-written for $X = V = \xi$, we observe

$$R^{\perp}(\xi,Y)(\overline{\nabla}_U\sigma)(\xi,Z) - (\overline{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z)$$
$$-(\overline{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) - (\overline{\nabla}_U\sigma)(\xi,R(\xi,Y)Z)$$
$$= -\ell_{\overline{S}}\Big\{(\overline{\nabla}_{(\xi\wedge_{\overline{S}}Y)U}\sigma)(\xi,Z) + (\overline{\nabla}_U\sigma)((\xi\wedge_{\overline{S}}Y)\xi,Z) \tag{41}$$
$$+(\overline{\nabla}_U\sigma)(\xi,(\xi\wedge_{\overline{S}}Y)Z)\Big\}.$$

Now we will calculate each of these expressions each of these statements separately. For this reason, as non-zero components of the first term are

$$\begin{aligned} R^{\perp}(\xi,Y)(\overline{\nabla}_U\sigma)(\xi,Z) &= -R^{\perp}(\xi,Y)\sigma(\nabla_U\xi,Z) \\ &= -(\lambda_1-1)R^{\perp}(\xi,Y)\phi\sigma(U,Z). \end{aligned} \tag{42}$$

Non-zero components of the second term give us

$$\begin{aligned} (\overline{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z) &= -\sigma(\nabla_{R(\xi,Y)U}\xi,Z) \\ &= -(\lambda_1-1)\phi\sigma(R(\xi,Y)U,Z) \\ &= -(\lambda_1-1)^2(1-\lambda_2)\phi\sigma(\eta(U)Y-g(Y,U)\xi,Z) \\ &= -(\lambda_1-1)^2(1-\lambda_2)\eta(U)\phi\sigma(Y,Z). \end{aligned} \tag{43}$$

If we calculate non-zero components in the third term from the right of the equality, it takes the form

$$\begin{aligned} (\overline{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) &= (\lambda_1-1)(1-\lambda_2)(\overline{\nabla}_U\sigma)(Y-\eta(Y)\xi,Z) \\ &= (\lambda_1-1)(1-\lambda_2)\{(\overline{\nabla}_U\sigma)(Y,Z) \\ &+\sigma(U\eta(Y)\xi+\eta(Y)\nabla_U\xi,Z)\} \\ &= (\lambda_1-1)(1-\lambda_2)(\overline{\nabla}_U\sigma)(Y,Z) \\ &+(\lambda_1-1)^2(1-\lambda_2)\eta(Y)\phi\sigma(U,Z). \end{aligned} \tag{44}$$

For non-zero components of the last term of the right, one can easily to see

$$\begin{aligned} (\overline{\nabla}_U\sigma)(\xi,R(\xi,Y)Z) &= -\sigma(\nabla_U\xi,R(\xi,Y)Z) \\ &= -(\lambda_1-1)\phi\sigma(U,R(\xi,Y)Z) \\ &= -(\lambda_1-1)^2(1-\lambda_2)\eta(Z)\phi\sigma(U,Y). \end{aligned} \tag{45}$$

For the non-zero components of the first term of the left of equality, we have

$$\begin{aligned} (\overline{\nabla}_{(\xi\wedge_{\overline{S}}Y)U}\sigma)(\xi,Z) &= -\sigma(\nabla_{(\xi\wedge_S Y)U}\xi,Z) \\ &= -(\lambda_1-1)\phi\sigma((\xi\wedge_S Y)U,Z) \\ &= -(\lambda_1-1)\phi\sigma(\overline{S}(Y,U)\xi-\overline{S}(\xi,U)Y,Z) \\ &= 2n(\lambda_1-1)^2(\lambda_2-1)\eta(U)\phi\sigma(Y,Z). \end{aligned} \tag{46}$$

Non-zero components of the second term of the left of equality are

$$\begin{aligned}(\overline{\nabla}_U\sigma)((\xi\wedge_{\overline{S}}Y)\xi,Z) &= -\sigma(\nabla_U\overline{S}(Y,\xi)\xi,Z)-(n-1)(\lambda-1)(\mu-1)(\overline{\nabla}_U\sigma)(Y,Z)\\ &= -2n(\lambda_1-1)(\lambda_2-1)\{(\overline{\nabla}_U\sigma)(Y,Z)+\sigma(U\eta(Y)\xi+\eta(Y)\nabla_U\xi,Z)\}\\ &= -2n(\lambda_1-1)(\lambda_2-1)\{(\overline{\nabla}_U\sigma)(Y,Z)+(\lambda_1-1)\eta(Y)\phi\sigma(U,Z)\},\end{aligned} \tag{47}$$

and the last term is give us

$$\begin{aligned}(\overline{\nabla}_U\sigma)(\xi,(\xi\wedge_{\overline{S}}Y)Z) &= -\sigma(\nabla_U\xi,\overline{S}(Y,Z)\xi-\overline{S}(\xi,Z)Y)\\ &= -(\lambda_1-1)\phi\sigma(U,-(\lambda_1-1)(\lambda_2-1)\eta(Z)Y)\\ &= (\lambda_1-1)^2(\lambda_2-1)2n\eta(Z)\phi\sigma(U,Y).\end{aligned} \tag{48}$$

Consequently, (42)-(48) are put (41) we have

$$\begin{aligned}&-(\lambda_1-1)R^{\perp}(\xi,Y)\phi\sigma(U,Z)+(\lambda_1-1)^2(1-\lambda_2)\eta(U)\phi\sigma(Y,Z)\\ &-(\lambda_1-1)(1-\lambda_2)(\overline{\nabla}_U\sigma)(Y,Z)-(\lambda_1-1)^2(1-\lambda_2)\eta(Y)\phi\sigma(U,Z)\\ &+(\lambda_1-1)^2(1-\lambda_2)\eta(Z)\phi\sigma(U,Y)=-\ell_{\overline{S}}\{2n(\lambda_1-1)^2(\lambda_2-1)\eta(U)\sigma(Y,Z)\\ &-2n(\lambda_1-1)(\lambda_2-1)(\overline{\nabla}_U\sigma)(Y,Z)-2n(\lambda_1-1)^2(\lambda_2-1)\eta(Y)\phi\sigma(U,Z)\\ &+(\lambda_1-1)^2(\lambda_2-1)2n\eta(Z)\phi\sigma(U,Y)\}.\end{aligned} \tag{49}$$

Putting $Z=\xi$ in (49) and after the necessary reductions are made, we observe

$$(\lambda_1-1)^2(1-\lambda_2)[2n\ell_{\overline{S}}+1]\phi\sigma(U,Y)=0,$$

which proves our assertions.

## 4. CONCLUSIONS

This paper aims to examine invariant pseudo-parallel submanifolds within a Sasakian manifold equipped with a general connection. The study focuses on identifying the necessary and sufficient conditions under which such an invariant pseudo-parallel submanifold becomes totally geodesic with respect to the general connection. Additionally, the paper investigates the structures induced by the ambient manifold under certain conditions. The findings obtained through this analysis are thoroughly discussed.

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