ORIGINAL PAPER

REVISITING STIRLING'S FORMULA

CRISTINEL MORTICI[1,2,3]

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Abstract. *The aim of this paper is to establish a new, simple formula for estimating the factorial function. This formula is an improvement of the much-celebrated Stirling's formula. Some inequalities are given.*

Keywords: *Factorial function; Stirling's formula; inequalities.*
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1. INTRODUCTION

The factorial function, denoted by $n!$, is a key concept in many areas of mathematics, including combinatorics, probability, and analysis. As n increases, the factorial grows extremely rapidly, making exact computation difficult and often impractical. To address this, mathematicians have developed various approximations, among which *Stirling's formula:*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad n \geq 1,$$

is one of the most well-known and widely used. This approximation is attributed to the Scottish mathematician James Stirling, who published it in his 1730 work *Methodus Differentialis*. However, similar ideas had already been explored by Abraham de Moivre, particularly in the context of probability theory. Stirling refined the expression, improving its accuracy and establishing the version that has since become standard in mathematical literature.

Over the years, Stirling's formula has been the subject of numerous refinements that are not only of theoretical interest but also have practical value in numerical computations and algorithm design.

2. A FAMILY OF APPROXIMATIONS

We propose the following family of approximations:

$$n! \sim \rho_n(a,b) := \sqrt{2\pi n}\left(\frac{n+a}{e}\right)^{\frac{n}{2}+b}\left(\frac{n-a}{e}\right)^{\frac{n}{2}-b}, \quad \text{as } n \to \infty, \tag{1}$$

where a, b are real parameters. Note that the Stirling's formula is obtained for $a = 0$.

[1] Academy of Romanian Scientists, Ilfov 3, Bucharest, Romania.
[2] Valahia University of Targoviste, 130004 Targoviste, Romania.
[3] National University of Science and Technology Politehnica Bucuresti, 060042 Bucharest, Romania.
E-mail: cristinel.mortici@hotmail.com.

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We will find values for the parameters a, b such that new approximations more accurate than Stirling's formula are obtained.

In general, an approximation formula of the form

$$f(n) \sim g(n), \text{ as } n \to \infty,$$

(in the sense that $\lim_{n\to\infty}(f(n)/g(n)) = 1$) is more accurate as the sequence

$$w_n = \ln \frac{f(n)}{g(n)}$$

faster converges to zero. A useful tool for measuring the speed of convergence is the following result first stated in this form by Mortici [5].

Lemma 1. If the sequence $(\omega_n)_{n\geq 1}$ is convergent to zero and there exists the limit:

$$\lim_{n\to\infty} n^s(\omega_n - \omega_{n+1}) = l \in \mathbb{R}^*,$$

with $s > 1$, *then*

$$\lim_{n\to\infty} n^{s-1}\omega_n = \frac{l}{s-1}.$$

This lemma asserts that a sequence ω_n converges to zero as $1/\, n^{s-1}$, if $\omega_n - \omega_{n+1}$ converges to zero as $1/n^s$.

This Lemma is a main tool for establishing new approximation formulas or to accelerate some convergences [3-9].

The associated sequence w_n to the approximation (1): $n! \sim \rho_n(a, b)$ is

$$w_n = \frac{\ln n!}{\sqrt{2\pi n}\left(\frac{n+a}{e}\right)^{\frac{n}{2}+b}\left(\frac{n-a}{e}\right)^{\frac{n}{2}-b}},$$

namely

$$w_n = \ln n! - \frac{1}{2}\ln 2\pi n - \left(\frac{n}{2}+b\right)\ln\frac{n+a}{e} - \left(\frac{n}{2}-b\right)\ln\frac{n-a}{e}.$$

Thus

$$w_n - w_{n+1} = -\ln(n+1) - \frac{1}{2}\ln\frac{n}{n+1} - \left(\frac{n}{2}+b\right)\ln\frac{n+a}{e} - \left(\frac{n}{2}-b\right)\ln\frac{n-a}{e} + \left(\frac{n+1}{2}+b\right)\ln\frac{n+1+a}{e} - \left(\frac{n+1}{2}-b\right)\ln\frac{n+1-a}{e}.$$

To estimate the speed of convergence of the sequence $w_n - w_{n+1}$, we write it as a series in powers of $1/n$:

$$\begin{aligned} w_n(a,b) - w_{n+1}(a,b) &= \left(\frac{1}{2}a^2 - 2ab + \frac{1}{12}\right)\frac{1}{n^2} - \left(\frac{1}{2}a^2 - 2ab + \frac{1}{12}\right)\frac{1}{n^3} \\ &+ \left(\frac{3}{4}a^4 - 2a^3b + \frac{1}{2}a^2 - 2ab + \frac{3}{40}\right)\frac{1}{n^4} \\ &- \left(\frac{3}{2}a^4 - 4a^3b + \frac{1}{2}a^2 - 2ab + \frac{1}{15}\right)\frac{1}{n^5} + O\left(\frac{1}{n^6}\right). \end{aligned} \quad (2)$$

As for $a = 0$, the Stirling's formula is obtained, the corresponding convergence speed sequence will be

$$w_n(0,b) - w_{n+1}(0,b) = \frac{1}{12n^2} - \frac{1}{12n^3} + \frac{3}{40n^4} - \frac{1}{15n^5} + O\left(\frac{1}{n^6}\right).$$

Then

$$\lim_{n\to\infty} n^2(w_n(0,b) - w_{n+1}(0,b)) = \frac{1}{12},$$

and, according to Lemma 1,

$$\lim_{n\to\infty} n \cdot\ w_n(0,b) = \frac{1}{12}.$$

We will say that the Stirling's approximation formula is of order n^{-1}. If $a \neq 0$ and

$$b \neq \frac{6a^2+1}{24a},$$

then the coefficient of n^{-2} in (2) is nonzero. Thus, the approximation formula (1): $n! \sim \rho_n(a,b)$ is of order n^{-2}.

We give the following:

Theorem 1. With a $\neq 0$ and

$$b = b_a := \frac{6a^2+1}{24a},$$

the approximation formula

$$n! \sim \rho_n(a,b_a) := \sqrt{2\pi n}\left(\frac{n+a}{e}\right)^{\frac{n}{2}+b_a}\left(\frac{n-a}{e}\right)^{\frac{n}{2}-b_a},\ \text{ as } n\to\infty, \tag{3}$$

is of order n^{-3} or n^{-4}.

Proof: For $b = b_a$, the first two coefficients in the series (2): $w_n(a,b_a) - w_{n+1}(a,b_a)$ vanish. Thus

$$\lim_{n\to\infty} n^4(w_n(a,b_a) - w_{n+1}(a,b_a)) = \frac{3}{4}a^4 - 2a^3b_a + \frac{1}{2}a^2 - 2ab_a + \frac{3}{40}.$$

According to Lemma 1, we deduce that

$$\lim_{n\to\infty} n^3 w_n(a,b_a) = \frac{1}{3}\left(\frac{3}{4}a^4 - 2a^3b_a + \frac{1}{2}a^2 - 2ab_a + \frac{3}{40}\right).$$

In case this limit is nonzero, the approximation (3) is of order n^{-3}. Otherwise, (3) is of order n^{-4}. This later case will be analized in the next Theorem 2. The proof is completed.

3. AN EXAMPLE

We give an example of approximation which is more accurate than the Stirling's approximation, namely

$$n! \sim \tau_n := \sqrt{2\pi n}\left(\frac{n+1}{e}\right)^{\frac{n}{2}+\frac{7}{24}}\left(\frac{n-1}{e}\right)^{\frac{n}{2}-\frac{7}{24}}, \text{ as } n\to\infty. \quad (1)$$

This is obtained by taking $a = 1$ in (1). By putting $a = 1$ and $b = 7/24$ in the expression (2) of $w_n(a,b) - w_{n+1}(a,b)$, we deduce that

$$\frac{\ln n!}{\tau_n} - \frac{\ln(n+1)!}{\tau_{n+1}} = \frac{19}{120n^4} - \frac{19}{60n^5} + \frac{197}{252n^6} - \frac{87}{56n^7} + O\left(\frac{1}{n^8}\right).$$

Some numerical comparisons of the approximation formula $n! \sim \tau_n$ with the Stirling's formula $n! \sim \sigma_n$ are given in the following Table 1.

Table 1. Numerical comparisons of the approximation formula $n! \sim \tau_n$ with Stirling's formula $n! \sim \sigma_n$.

n	$\ln(n!/\sigma_n)$	$\ln(n!/\tau_n)$
10	8.3306×10^{-3}	5.3290×10^{-5}
50	1.6666×10^{-3}	4.2238×10^{-7}
100	8.3333×10^{-4}	5.2783×10^{-8}
750	1.1111×10^{-4}	1.251×10^{-10}
2500	3.3333×10^{-5}	3.3778×10^{-12}

Related to this approximation formula (4), we give the following:

Theorem 2. The following inequality holds true, for all integers $n \geq 2$:

$$\lambda \cdot \sqrt{2\pi n}\left(\frac{n+1}{e}\right)^{\frac{n}{2}+\frac{7}{24}}\left(\frac{n-1}{e}\right)^{\frac{n}{2}-\frac{7}{24}} < n! \leq \mu \cdot \sqrt{2\pi n}\left(\frac{n+1}{e}\right)^{\frac{n}{2}+\frac{7}{24}}\left(\frac{n-1}{e}\right)^{\frac{n}{2}-\frac{7}{24}},$$

where the constants

$$\lambda = 1 \quad \textit{and} \quad \mu = \frac{2e^2}{(4\pi)^{\frac{1}{2}} \cdot 3^{\frac{31}{24}}} = 1.0086\ldots$$

are sharp. Moreover, equality in the right-hand side holds if and only if $n = 2$.

Proof: Let us define the function $f: [2,\infty) \to \mathbb{R}$ by the formula:

$$f(x) = \ln\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x+1}{e}\right)^{\frac{x}{2}+\frac{7}{24}}\left(\frac{x-1}{e}\right)^{\frac{x}{2}-\frac{7}{24}}},$$

where Γ is the Euler's gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x > 0.$$

The gamma function is an extension of the factorial function to all positive real numbers, since

$$\Gamma(n) = (n-1)!, \quad n = 1,2,3,\ldots .$$

We have

$$f(x) = \ln\Gamma(x) + \ln x - \frac{1}{2}\ln 2\pi x - \left(\frac{x}{2}+\frac{7}{24}\right)\ln\frac{x+1}{e} - \left(\frac{x}{2}-\frac{7}{24}\right)\ln\frac{x-1}{e}.$$

Note that

$$f(2) = \ln 2 - \frac{31}{24} \ln \frac{3}{e} - \frac{1}{2} \ln 4\pi + \frac{17}{24}$$

and

$$\exp f(2) = \mu.$$

We have

$$f''(x) = \psi'(x) - \frac{6x^5 + 3x^4 - 11x^3 - 6x^2 + 3}{6(x^2 - 1)^2}.$$

Here, ψ is the digamma function, namely the logarithmic derivative of the gamma function:

$$\psi(x) = (\ln \Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Its derivative, also called the trigamma function, has the following asymptotic expansion:

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \cdots, \text{ as } x \to \infty.$$

The entire series can be given in terms of Bernoulli's numbers. Moreover, by truncation this series at a given term, alternatively, under- and upper approximations for the trigamma function are obtained. See, e.g., [3]. We use in the sequel the inequality

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5},$$

which is the underapproximation for the trigamma function $\psi'(x)$ obtained by truncation of its series at the fourth term. We deduce that

$$f''(x) > \left(\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5}\right) - \frac{6x^5 + 3x^4 - 11x^3 - 6x^2 + 3}{6(x^2 - 1)^2} = \frac{19x^4 + 7x^2 - 1}{30x^5(x^2 - 1)^2} > 0.$$

Now, $f'' > 0$, so f' is strictly increasing. As $\lim_{x\to\infty} f'(x) = 0$, it results that $f' < 0$. Thus f is strictly decreasing, and consequently, for all $x \geq 2$, we have:

$$0 = \lim_{t\to\infty} f(t) < f(x) \leq f(2).$$

By exponentiating, we deduce that

$$1 < \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x+1}{e}\right)^{\frac{x}{2}+\frac{7}{24}}\left(\frac{x-1}{e}\right)^{\frac{x}{2}-\frac{7}{24}}} \leq \exp f(2),$$

which is the conclusion.

4. THE BEST CONSTANTS IN OUR NEW APPROXIMATION FORMULA

In this section we use Lemma 1 to derive the best constants which gives the most accurate approximation formula of type (1).

Theorem 3. a) Let α be a real solution of the equation

$$\frac{1}{4}\alpha^4 - \frac{1}{12}\alpha^2 - \frac{1}{120} = 0$$

and let

$$\beta = \frac{6\alpha^2 + 1}{24\alpha}.$$

Then the approximation formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n+\alpha}{e}\right)^{\frac{n}{2}+\frac{6\alpha^2+1}{24\alpha}}\left(\frac{n-\alpha}{e}\right)^{\frac{n}{2}-\frac{6\alpha^2+1}{24\alpha}}, \quad \text{as } n \to \infty,$$

is of order n^{-4}.

Proof: First remark that

$$\alpha = \pm\frac{1}{2}\sqrt{\frac{2}{3} + \frac{2\sqrt{55}}{15}} = \pm 0.64333\ldots, \quad \beta = \pm\frac{1+\sqrt{55}}{48}\sqrt{\frac{2}{3} + \frac{2\sqrt{55}}{15}} = \pm 0.225600\ldots$$

with corresponding signs.

For the values $(a,b) = (\alpha,\beta)$, the first three terms of the $w_n(\alpha,\beta) - w_{n+1}(\alpha,\beta)$ vanish, since (α,β) is solution of the following system formed by the coefficients of n^{-2}, n^{-3} and n^{-4}:

$$\begin{cases} \frac{1}{2}a^2 - 2ab + \frac{1}{12} = 0 \\ \frac{3}{4}a^4 - 2a^3 b + \frac{1}{2}a^2 - 2ab + \frac{3}{40} = 0 \end{cases}$$

Thus

$$\lim_{n\to\infty} n^5\left(w_n(a,b_a) - w_{n+1}(a,b_a)\right) = \frac{3}{2}\alpha^4 - 4\alpha^3\beta + \frac{1}{2}\alpha^2 - 2\alpha\beta + \frac{1}{15}.$$

According to Lemma 1, we deduce that

$$\lim_{n\to\infty} n^4 w_n(a,b_a) = \frac{1}{4}\left(\frac{3}{2}\alpha^4 - 4\alpha^3\beta + \frac{1}{2}\alpha^2 - 2\alpha\beta + \frac{1}{15}\right) \neq 0.$$

This means that the approximation (2) is of order n^{-4}. The proof is completed.

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