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DEVELOPABILITY OF RULED SURFACES GENERATED BY CURVES AT A CONSTANT DISTANCE FROM THE NON-NULL CURVE

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Abstract. *This study aims to characterize curves at constant distance by using the tangent, normal, and binormal vectors of a unit speed non-null curve in Minkowski 3-space. The Frenet vectors and the curvature and torsion functions of these curves are calculated by using the curvature and torsion of the unit speed non-null curve. Then, the developability of ruled surfaces created with these curves and their tangents is investigated. Finally, the examples containing these curves and surfaces are given, and their graphs are drawn.*

Keywords: *Developable surface; non-null curve; ruled surface.*

1. INTRODUCTION

The most popular topics studied in differential geometry in recent years are curve theory and surface theory. Although two- and three-dimensional studies of curves are common, the most common space for surface studies is Euclidean 3-space. Surface theory was investigated first by Monge in the 18th century, and after this study, surfaces have been considered in different dimensions and spaces. Later, Guggenheimer and Hoschek examined ruled surfaces from different perspectives. Curves are expressed by characterizations of the Frenet vector that are defined on the curve. Ruled surfaces are surfaces that are formed by the continuous moving of a line along the base curve [1,2]. Although curves and surfaces in Minkowski space have similar properties to curves and surfaces in Euclidean space, there are some differences due to the Lorentzian inner product. One of the most important tools used to analyze a curve is the Frenet frame, which is a moving frame that provides a coordinate system at each point on the curve at a given point on the curve. By using the Frénet frame, curvature and torsion functions can be defined on the curve. Different space curves can be distinguished only with the help of curvature and torsion functions. These curvature and torsion functions are called differential invariants of the curve. The fundamental theorem of curves in differential geometry states that these invariants completely determine the curve. During the investigation of curves, numerous types of curves are introduced and examined in terms of their frame structures [3]. Creating curves from curves has been the subject of many studies, and one of these curves is the parallel curves. Parallel curves have also been

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investigated by using different spaces and frames [4-12]. Additionally, some studies on ruled surfaces have been investigated in detail in [13-27].

In this study, the curves are obtained at a constant distance from the curve by using the tangent, normal, and binormal vectors of a unit speed space curve in Minkowski space. The Frenet vectors and curvatures of these curves are investigated. The Frenet vectors, the curvature, and torsion functions of these curves are calculated by using the curvature and torsion of the given unit speed curves. Then, the developability of ruled surfaces created with these curves and their tangents is presented. Finally, examples containing these curves and surfaces are given, and their graphs are shown.

2. PRELIMINARIES

In Minkowski 3-space $\mathbb{R}_1^3$, the Lorentzian inner product and vector product are given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

and

$$x \times y = \left(x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1\right),$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbb{R}$. The norm of $x \in \mathbb{R}_1^3$ is $\|x\| = \sqrt{|\langle x, x\rangle|}$.

The characteristics of the vector are given as follows;

- the vector x is a spacelike, if $\langle x, x\rangle > 0$ or $x = 0$,
- the vector x is a timelike, if $\langle x, x\rangle < 0$,
- the vector x is a lightlike (or null), if $\langle x, x\rangle = 0$, $x \neq 0$.

Let $\alpha : I \to \mathbb{R}_1^3$ be a regular unit speed non-null curve with arc-length parameter s in $\mathbb{R}_1^3$. If the vectors t, n, and are tangent, principal normal, and binormal vectors of the non-null curve α, respectively. Then, the Frenet formulas are

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix}_s = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon_1\varepsilon_2\kappa & 0 & \tau \\ 0 & -\varepsilon_2\varepsilon_3\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{1}$$

where $\langle t, t\rangle = \varepsilon_1$, $\langle n, n\rangle = \varepsilon_2$ and $\langle b, b\rangle = \varepsilon_3$. Also, $n \times t = \varepsilon_3 b$, $b \times n = \varepsilon_1 t$, $t \times b = \varepsilon_2 n$ and $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$. The functions $\kappa(s)$, $\tau(s)$ are the curvature and the torsion of α, respectively. Let $\{t, n, b\}$ be a moving frame of α, it holds the following conditions [28,29]:

- $\varepsilon_1 = -1$, $\varepsilon_2 = 1$, $\varepsilon_3 = 1$ for the timelike curve,
- $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $\varepsilon_3 = 1$ for the spacelike curve with timelike normal,
- $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$ for the spacelike curve with timelike binormal.

In Minkowski 3-space $\mathbb{R}_1^3$, a ruled surface M is a regular surface that is parameterized as:

$$X : I \times \mathbb{R} \to \mathbb{R}_1^3$$
$$(s,v) \to X(s,v) = \alpha(s) + vr(s),$$

where the curve $\alpha(s)$ and $r(s)$ are known as the base and director curves of a ruled surface, respectively [29]. Additionally, the parameter of the distribution of $X(s,v)$ is

$$P(s) = \frac{\det\left(\alpha'(s), r(s), r'(s)\right)}{\left\|r'(s)\right\|^2}.$$

Theorem 2.1. ([29]) Let $X(s,v)$ be a ruled surface, then the ruled surface is developable if and only if

$$\det\left(\alpha'(s), r(s), r'(s)\right) = 0. \tag{2}$$

3. CURVES AT A CONSTANT DISTANCE FROM THE NON-NULL CURVE

In this section, we describe curves that can be constructed using the tangent, principal normal, and binormal vectors of the Frenet frame along a space curve in $\mathbb{R}_1^3$.

Definition 3.1. Let α be a unit speed space curve with the Frenet frame $\{t,n,b\}$ in Minkowski 3-space and α_t, α_n, and α_b be curves at a constant distance from the point $\alpha(s)$ of α. Then the parametric equations of α_t, α_n, and α_b are defined by

$$\begin{aligned} \alpha_t(s^*) &= \alpha(s) + \lambda t(s), \\ \alpha_n(s^*) &= \alpha(s) + \lambda n(s), \\ \alpha_b(s^*) &= \alpha(s) + \lambda b(s), \end{aligned} \tag{3}$$

where s and s^* are parameters of the curves α and α_t, α_n, α_b, respectively. Also λ is presented a nonzero constant real number.

Theorem 3.2. Let $\alpha_t : I \to \mathbb{R}_1^3$ be a curve at constant distance from α with the Frenet frame $\{T_t, N_t, B_t\}$ and its curvature and torsion κ_t and τ_t in Minkowski 3-space. Then, the following relations satisfy:

$$T_t = \frac{t + \lambda\kappa n}{\sqrt{\left|-\varepsilon_1 + \lambda^2\kappa^2\varepsilon_2\right|}},$$

$$N_t = \frac{\left(\varepsilon_1\lambda^3\kappa^4 - \varepsilon_1\varepsilon_3\lambda\kappa^2 - \varepsilon_1\varepsilon_3\lambda^2\kappa\kappa'\right)t + \left(\kappa + \lambda\kappa' - \varepsilon_3\lambda^2\kappa^3\right)n + \left(\varepsilon_1\varepsilon_3\lambda^3\kappa^3\tau + \lambda\kappa\tau\right)b}{\sqrt{\left|\left(-\varepsilon_1 + \lambda^2\kappa^2\varepsilon_2\right)\left(-\varepsilon_1\lambda^4\kappa^4\tau^2 + \varepsilon_2\lambda^2\kappa^2\tau^2 + \varepsilon_3\left(\lambda^2\kappa^3 - \varepsilon_3\kappa - \varepsilon_3\lambda\kappa'\right)^2\right)\right|}},$$

$$B_t = \frac{-\varepsilon_1\lambda^2\kappa^2\tau t + \varepsilon_3\lambda\kappa\tau n + \left(-\varepsilon_3\kappa - \varepsilon_3\lambda\kappa' + \lambda^2\kappa^3\right)b}{\sqrt{\left|\varepsilon_2\lambda^2\kappa^2\tau^2 - \varepsilon_1\lambda^4\kappa^4\tau^2 + \varepsilon_3\left(\lambda^2\kappa^3 - \varepsilon_3\kappa - \varepsilon_3\lambda\kappa'\right)^2\right|}},$$

and

$$\kappa_t = \frac{\sqrt{\left|\varepsilon_2\lambda^2\kappa^2\tau^2 - \varepsilon_1\lambda^4\kappa^4\tau^2 + \varepsilon_3\left(\lambda^2\kappa^3 - \varepsilon_3\kappa - \varepsilon_3\kappa'\right)^2\right|}}{\left(\left|\varepsilon_2\lambda^2\kappa^2 - \varepsilon_1\right|\right)^{\frac{3}{2}}},$$

$$\tau_t = \frac{\varepsilon_2\varepsilon_3\lambda\kappa\tau\mu_2 - \lambda^2\kappa^2\tau\mu_1 + \left(\varepsilon_3\lambda^2\kappa^3 - \kappa - \lambda\kappa'\right)\mu_3}{\left|\varepsilon_2\lambda^2\kappa^2\tau^2 - \varepsilon_1\lambda^4\kappa^4\tau^2 + \varepsilon_3\left(\lambda^2\kappa^3 - \varepsilon_3\kappa - \varepsilon_3\kappa'\right)^2\right|},$$

where

$$\mu_1 = -3\varepsilon_1\varepsilon_2\lambda\kappa\kappa' - \varepsilon_1\varepsilon_2\kappa^2,\ \mu_2 = \kappa' + \lambda\kappa'' - \varepsilon_1\varepsilon_2\lambda\kappa\tau^2 - \varepsilon_1\varepsilon_2\lambda\kappa^3,\ \mu_3 = \kappa\tau + 2\lambda\kappa'\tau + \lambda\tau'\kappa.$$

Proof: Let α_t be a curve with parameter s^* at constant distance from α in $\mathbb{R}^3_1$, then from (1) and (3), it is known that

$$\frac{d\alpha_t}{ds^*}\frac{ds^*}{ds} = \frac{d\alpha}{ds} + \lambda\frac{dt}{ds} = t + \lambda\kappa n \tag{4}$$

or

$$T_t\frac{ds^*}{ds} = t + \lambda\kappa n. \tag{5}$$

If we take the Lorentzian norm of this expression, then it holds that

$$\frac{ds^*}{ds} = \sqrt{\left|\lambda^2\kappa^2\varepsilon_2 - \varepsilon_1\right|}.$$

Substituting the last equation into (5), the unit tangent vector of the curve α_t is obtained as

$$T_t = \frac{t + \lambda\kappa n}{\sqrt{\left|\lambda^2\kappa^2\varepsilon_2 - \varepsilon_1\right|}}$$

for $s \in I$.

By differentiating (4), we get

$$\frac{d^2\alpha_t}{ds^2} = -\lambda\varepsilon_1\varepsilon_2\kappa^2 t + \left(\kappa + \lambda\kappa'\right)n + \lambda\kappa\tau b$$

and

$$\frac{d^3\alpha_t}{ds^3} = \left(-3\lambda\varepsilon_1\varepsilon_2\kappa\kappa' - \varepsilon_1\varepsilon_2\kappa^2\right)t + \left(-\lambda\varepsilon_1\varepsilon_2\kappa^3 + \kappa' + \lambda\kappa'' - \lambda\varepsilon_1\varepsilon_2\kappa\tau^2\right)n + \left(\kappa\tau + 2\lambda\kappa'\tau + \lambda\tau'\kappa\right)b.$$

Considering the derivative equations of the curve α_t, we can calculate the normal and binormal vectors N_t and B_t of α_t as follows:

$$B_t=\frac{\frac{d\alpha_t}{ds}\times\frac{d^2\alpha_t}{ds^2}}{\left\|\frac{d\alpha_t}{ds}\times\frac{d^2\alpha_t}{ds^2}\right\|}=\frac{-\varepsilon_1\lambda^2\kappa^2\tau t+\varepsilon_3\lambda\kappa\tau n+\left(\lambda^2\kappa^3-\varepsilon_3\left(\kappa+\lambda\kappa'\right)\right)b}{\sqrt{\left|\lambda^2\kappa^2\tau^2\left(\varepsilon_2-\varepsilon_1\lambda^2\kappa^2\right)+\varepsilon_3\left(\lambda^2\kappa^3-\varepsilon_3\left(\kappa+\lambda\kappa'\right)\right)^2\right|}}$$

and

$$N_t=B_t\times T_t=\frac{\varepsilon_1\lambda\kappa\left(\lambda^2\kappa^3-\varepsilon_3\left(\kappa+\lambda\kappa'\right)\right)t+\left(\kappa+\lambda\kappa'-\varepsilon_3\lambda^2\kappa^3\right)n+\lambda\kappa\tau\left(\varepsilon_1\varepsilon_3\lambda^2\kappa^2+1\right)b}{\sqrt{\left|\lambda^2\kappa^2\varepsilon_2-\varepsilon_1\right|\left|\lambda^2\kappa^2\tau^2\left(\varepsilon_2-\varepsilon_1\lambda^2\kappa^2\right)+\varepsilon_3\left(\lambda^2\kappa^3-\varepsilon_3\left(\kappa+\lambda\kappa'\right)\right)^2\right|}}.$$

On the other hand, the curvature and torsion functions of α_t are obtained as:

$$\kappa_t=\frac{\left\|\frac{d\alpha_t}{ds}\times\frac{d^2\alpha_t}{ds^2}\right\|}{\left\|\frac{d\alpha_t}{ds}\right\|^3}=\frac{\sqrt{\left|\lambda^2\kappa^2\tau^2\left(\varepsilon_2-\varepsilon_1\lambda^2\kappa^2\right)+\varepsilon_3\left(\lambda^2\kappa^3-\varepsilon_3\left(\kappa+\kappa'\right)\right)^2\right|}}{\left(\left|\varepsilon_2\lambda^2\kappa^2-\varepsilon_1\right|\right)^{\frac{3}{2}}}$$

and

$$\tau_t=\frac{\det\left(\frac{d\alpha_t}{ds},\frac{d^2\alpha_t}{ds^2},\frac{d^3\alpha_t}{ds^3}\right)}{\left\|\frac{d\alpha_t}{ds}\times\frac{d^2\alpha_t}{ds^2}\right\|^2}=\frac{\varepsilon_2\varepsilon_3\lambda\kappa\tau\mu_2-\lambda^2\kappa^2\tau\mu_1+\left(\varepsilon_3\lambda^2\kappa^3-\kappa-\lambda\kappa'\right)\mu_3}{\left|\lambda^2\kappa^2\tau^2\left(\varepsilon_2-\varepsilon_1\lambda^2\kappa^2\right)+\varepsilon_3\left(\lambda^2\kappa^3-\varepsilon_3\left(\kappa+\kappa'\right)\right)^2\right|},$$

such that

$$\mu_1=-3\lambda\varepsilon_1\varepsilon_2\kappa\kappa'-\varepsilon_1\varepsilon_2\kappa^2,\ \mu_2=-\lambda\varepsilon_1\varepsilon_2\kappa^3+\kappa'+\lambda\kappa''-\lambda\varepsilon_1\varepsilon_2\kappa\tau^2,\ \mu_3=\kappa\tau+2\lambda\kappa'\tau+\lambda\tau'\kappa.$$

So, the proof of the theorem is completed.

Theorem 3.3. Let $\alpha_n:I\to\mathbb{R}_1^3$ be a curve at constant distance from α with the Frenet frame $\{T_n,N_n,B_n\}$ and its curvature and torsion κ_n and τ_n in Minkowski 3-space. Then, the following equations hold:

$$T_n=\frac{\left(1-\varepsilon_1\varepsilon_2\lambda\kappa\right)t+\lambda\tau b}{\sqrt{\left|2\varepsilon_2\lambda\kappa-\varepsilon_1\left(1+\lambda^2\kappa^2\right)+\varepsilon_3\lambda^2\tau^2\right|}},$$

$$N_n=\frac{\begin{array}{l}\left(\varepsilon_1\lambda^3\kappa'\tau^2-\varepsilon_1\varepsilon_3\lambda^2\tau\tau'-\varepsilon_1\lambda^3\kappa\tau\tau'\right)t+\begin{pmatrix}\varepsilon_1\varepsilon_2\lambda^2\kappa\tau^2-\varepsilon_2\varepsilon_3\lambda^3\kappa^2\tau^2+\varepsilon_3\lambda^3\tau^4-\lambda\kappa\tau\\+2\varepsilon_3\lambda^2\kappa^2\tau-\lambda^3\kappa^3\tau+\varepsilon_1\varepsilon_3\lambda^3\kappa\tau^3-\varepsilon_2\varepsilon_3\lambda^2\tau^3\end{pmatrix}n\\+\left(\lambda\tau'+2\varepsilon_3\lambda^2\kappa\tau'+\lambda^3\kappa^2\tau'-\varepsilon_3\lambda^2\kappa'\tau-\lambda^3\kappa\kappa'\tau\right)b\end{array}}{\sqrt{\left|\left(2\varepsilon_2\lambda\kappa+\varepsilon_3\lambda^2\tau^2-\varepsilon_1\left(1+\lambda^2\kappa^2\right)\right)\begin{pmatrix}\varepsilon_2\lambda^2\left(\varepsilon_3\tau'+\lambda\left(\kappa\tau'-\kappa'\tau\right)\right)^2\\-\varepsilon_1\tau^2\lambda^2\left(\varepsilon_1\kappa-\lambda\left(\varepsilon_2\kappa^2-\tau^2\right)\right)^2\\+\varepsilon_3\left(\varepsilon_3\kappa-2\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda^2\kappa\left(\varepsilon_3\kappa^2+\varepsilon_1\tau^2\right)\right)^2\end{pmatrix}\right|}},$$

$$B_n=\frac{\left(\varepsilon_1\lambda\kappa\tau-\varepsilon_2\lambda^2\kappa^2\tau+\lambda^2\tau^3\right)t+\left(\varepsilon_3\lambda\tau'+\lambda^2\kappa\tau'-\lambda^2\kappa'\tau\right)n+\left(\varepsilon_3\kappa-\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda\kappa^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)b}{\sqrt{\left|\begin{array}{l}-\varepsilon_1\left(\varepsilon_1\lambda\kappa\tau-\varepsilon_2\lambda^2\kappa^2\tau+\lambda^2\tau^3\right)^2+\varepsilon_2\left(\varepsilon_3\lambda\tau'+\lambda^2\kappa\tau'-\lambda^2\kappa'\tau\right)^2\\+\varepsilon_3\left(\varepsilon_3\kappa-\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda\kappa^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)^2\end{array}\right|}},$$

and

$$\kappa_n=\frac{\sqrt{\left|\begin{array}{l}\varepsilon_2\left(\varepsilon_3\lambda\tau'+\lambda^2\kappa\tau'-\lambda^2\kappa'\tau\right)^2-\varepsilon_1\left(\varepsilon_1\lambda\kappa\tau-\varepsilon_2\lambda^2\kappa^2\tau+\lambda^2\tau^3\right)^2\\+\varepsilon_3\left(\varepsilon_3\kappa-\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda\kappa^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)^2\end{array}\right|}}{\left|\varepsilon_3\lambda^2\tau^2-\varepsilon_1\left(1-\varepsilon_1\varepsilon_2\lambda\kappa\right)^2\right|^{\frac{3}{2}}},$$

$$\tau_n=\frac{\eta_2\left(\varepsilon_2\varepsilon_3\lambda\tau'+\varepsilon_2\lambda^2\kappa\tau'-\varepsilon_2\lambda^2\kappa'\tau\right)-\eta_1\left(\lambda\kappa\tau-\varepsilon_1\varepsilon_2\lambda^2\kappa^2\tau+\varepsilon_1\lambda^2\tau^3\right)+\eta_3\left(\kappa-\varepsilon_3\lambda\kappa^2+\varepsilon_2\varepsilon_3\lambda\tau^2-\varepsilon_3\lambda\kappa^2-\lambda^2\kappa^3-\varepsilon_1\varepsilon_3\lambda^2\kappa\tau^2\right)}{\left|\begin{array}{l}\varepsilon_2\left(\varepsilon_3\lambda\tau'+\lambda^2\kappa\tau'-\lambda^2\kappa'\tau\right)^2-\varepsilon_1\left(\varepsilon_1\lambda\kappa\tau-\varepsilon_2\lambda^2\kappa^2\tau+\lambda^2\tau^3\right)^2\\+\varepsilon_3\left(\varepsilon_3\kappa-\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda\kappa^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)^2\end{array}\right|},$$

where

$$\eta_1=\lambda\kappa\left(\varepsilon_1\varepsilon_3\tau^2+\kappa^2\right)-\varepsilon_1\varepsilon_2\left(\lambda\kappa''+\kappa^2\right),\ \eta_2=\kappa'-3\varepsilon_2\lambda\left(\varepsilon_1\kappa\kappa'+\varepsilon_3\tau\tau'\right),\ \eta_3=\kappa-\varepsilon_2\lambda\tau\left(\varepsilon_1\kappa^2+\varepsilon_3\tau^2\right)+\lambda\tau''.$$

Proof: Let α_n be a curve with parameter s^* at constant distance from α in $\mathbb{R}_1^3$, then from Equation (3), it is known that

$$\frac{d\alpha_n}{ds^*}\frac{ds^*}{ds}=\frac{d\alpha}{ds}+\lambda\frac{dn}{ds}=\left(1-\varepsilon_1\varepsilon_2\lambda\kappa\right)t+\lambda\tau b \tag{6}$$

or

$$T_n\frac{ds^*}{ds}=\left(1-\varepsilon_1\varepsilon_2\lambda\kappa\right)t+\lambda\tau b \tag{7}$$

If we take the Lorentzian norm of this expression, then it holds that

$$\frac{ds^*}{ds}=\sqrt{\left|\varepsilon_3\lambda^2\tau^2-\varepsilon_1\left(1-\varepsilon_1\varepsilon_2\lambda\kappa\right)^2\right|}.$$

Substituting the last equation into (7), the unit tangent vector of the curve α_n is obtained as:

$$T_n=\frac{\left(1-\varepsilon_1\varepsilon_2\lambda\kappa\right)t+\lambda\tau b}{\sqrt{\left|2\varepsilon_2\lambda\kappa-\varepsilon_1\left(1+\lambda^2\kappa^2\right)+\varepsilon_3\lambda^2\tau^2\right|}}$$

for $s\in I$. By differentiating from Equation (6), we get

$$\frac{d^2\alpha_n}{ds^2}=-\varepsilon_1\varepsilon_2\lambda\kappa' t+\left(\kappa-\varepsilon_1\varepsilon_2\lambda\kappa^2-\varepsilon_2\varepsilon_3\lambda\tau^2\right)n+\lambda\tau' b$$

and

$$\frac{d^3\alpha_n}{ds^3}=\left(-\varepsilon_1\varepsilon_2\left(\lambda\kappa''+\kappa^2\right)+\lambda\kappa\left(\varepsilon_1\varepsilon_3\tau^2+\kappa^2\right)\right)t+\left(\kappa'-3\varepsilon_2\lambda\left(\varepsilon_1\kappa\kappa'+\varepsilon_3\tau\tau'\right)\right)n$$
$$+\left(\kappa\tau-\varepsilon_2\lambda\tau\left(\varepsilon_1\kappa^2+\varepsilon_3\tau^2\right)+\lambda\tau''\right)b.$$

Considering the derivative equations of the curve α_n, we can calculate the normal and binormal vectors N_n and B_n of α_n as follows:

$$B_n=\frac{\frac{d\alpha_n}{ds}\times\frac{d^2\alpha_n}{ds^2}}{\left\|\frac{d\alpha_n}{ds}\times\frac{d^2\alpha_n}{ds^2}\right\|}=\frac{\begin{array}{c}\left(\varepsilon_1\lambda\kappa\tau-\varepsilon_2\lambda^2\kappa^2\tau+\lambda^2\tau^3\right)t+\left(\varepsilon_3\lambda\tau'+\lambda^2\kappa\tau'-\lambda^2\kappa'\tau\right)n\\+\left(\varepsilon_3\kappa-\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda\kappa^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)b\end{array}}{\sqrt{\left|\begin{array}{l}\varepsilon_2\left(\varepsilon_3\lambda\tau'+\lambda^2\kappa\tau'-\lambda^2\kappa'\tau\right)^2-\varepsilon_1\left(\varepsilon_1\lambda\kappa\tau-\varepsilon_2\lambda^2\kappa^2\tau+\lambda^2\tau^3\right)^2\\+\varepsilon_3\left(\varepsilon_3\kappa-\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda\kappa^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)^2\end{array}\right|}}$$

and

$$N_n=B_n\times T_n=\frac{\begin{array}{c}\left(\varepsilon_1\lambda^3\kappa'\tau^2-\varepsilon_1\varepsilon_3\lambda^2\tau\tau'-\varepsilon_1\lambda^3\kappa\tau\tau'\right)t+\left(\begin{array}{l}\varepsilon_1\varepsilon_2\lambda^2\kappa\tau^2-\varepsilon_2\varepsilon_3\lambda^2\tau^2\left(\lambda\kappa^2+\tau\right)+\varepsilon_1\varepsilon_3\lambda^3\kappa\tau^3\\+\varepsilon_3\lambda\tau\left(\lambda^2\tau^3-\kappa+\lambda\kappa^2+\lambda\kappa^2-\lambda^2\kappa^3\right)\end{array}\right)n\\+\left(\lambda\tau'+2\varepsilon_3\lambda^2\kappa\tau'+\lambda^3\kappa^2\tau'-\varepsilon_3\lambda^2\kappa'\tau-\lambda^3\kappa\kappa'\tau\right)b\end{array}}{\sqrt{\left|\left(2\varepsilon_2\lambda\kappa+\varepsilon_3\lambda^2\tau^2-\varepsilon_1\left(1+\lambda^2\kappa^2\right)\right)\left(\begin{array}{l}\varepsilon_3\left(\varepsilon_3\kappa-\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda\kappa^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)^2\\-\varepsilon_1\left(\varepsilon_1\lambda\kappa\tau-\varepsilon_2\lambda^2\kappa^2\tau+\lambda^2\tau^3\right)^2+\varepsilon_2\left(\varepsilon_3\lambda\tau'+\lambda^2\kappa\tau'-\lambda^2\kappa'\tau\right)^2\end{array}\right)\right|}}.$$

On the other hand, the curvature and torsion functions of α_n are obtained as:

$$\kappa_n=\frac{\sqrt{\left|\begin{array}{l}\varepsilon_3\left(\varepsilon_3\kappa-\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\lambda\kappa^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)^2-\varepsilon_1\lambda^2\tau^2\left(\varepsilon_1\kappa-\varepsilon_2\lambda\kappa^2+\lambda\tau^2\right)^2\\+\varepsilon_2\lambda^2\left(\varepsilon_3\tau'+\lambda\kappa\tau'-\lambda\kappa'\tau\right)^2\end{array}\right|}}{\left|\varepsilon_3\lambda^2\tau^2-\varepsilon_1\left(1-\varepsilon_1\varepsilon_2\lambda\kappa\right)^2\right|^{\frac{3}{2}}}$$

and

$$\tau_n=\frac{\begin{array}{c}\eta_2\varepsilon_2\lambda\left(\varepsilon_3\tau'+\lambda\kappa\tau'-\lambda\kappa'\tau\right)-\eta_1\tau\lambda\left(\kappa-\varepsilon_1\varepsilon_2\lambda\kappa^2+\varepsilon_1\lambda\tau^2\right)\\+\eta_3\left(\kappa-2\varepsilon_3\lambda\kappa^2+\varepsilon_3\lambda\tau^2\left(\varepsilon_2-\varepsilon_1\lambda\kappa\right)-\lambda^2\kappa^3\right)\end{array}}{\left|\begin{array}{l}\varepsilon_3\left(\varepsilon_3\kappa-2\lambda\kappa^2+\varepsilon_2\lambda\tau^2-\varepsilon_3\lambda^2\kappa^3-\varepsilon_1\lambda^2\kappa\tau^2\right)^2-\varepsilon_1\left(\varepsilon_1\lambda\kappa\tau-\varepsilon_2\lambda^2\kappa^2\tau+\lambda^2\tau^3\right)^2\\+\varepsilon_2\left(\varepsilon_3\lambda\tau'+\lambda^2\kappa\tau'-\lambda^2\kappa'\tau\right)^2\end{array}\right|}$$

such that

$$\eta_1=\lambda\kappa\left(\varepsilon_1\varepsilon_3\tau^2+\kappa^2\right)-\varepsilon_1\varepsilon_2\left(\lambda\kappa''+\kappa^2\right),\ \eta_2=\kappa'-3\varepsilon_2\lambda\left(\varepsilon_1\kappa\kappa'+\varepsilon_3\tau\tau'\right),\ \eta_3=\kappa\tau+\lambda\tau''-\varepsilon_2\lambda\tau\left(\varepsilon_1\kappa^2+\varepsilon_3\tau^2\right).$$

Theorem 3.4. Let $\alpha_b : I \to \mathbb{R}_1^3$ be a curve at constant distance from α with the Frenet frame $\{T_b, N_b, B_b\}$ and its curvature and torsion κ_b and τ_b in Minkowski 3-space. Then, the following equations hold:

$$T_b = \frac{t - \varepsilon_2\varepsilon_3\lambda\tau n}{\sqrt{\left|\varepsilon_2\lambda^2\tau^2 - \varepsilon_1\right|}},$$

$$N_b = \frac{\begin{array}{c}\left(-\varepsilon_1\varepsilon_2\lambda\kappa\tau + \varepsilon_1\varepsilon_3\lambda^2\tau + \lambda^3\kappa\tau^3\right)t + \left(\kappa + \varepsilon_2\varepsilon_3\lambda + \varepsilon_3\lambda^2\kappa\tau^2\right)n \\ + \left(\varepsilon_2\varepsilon_3\lambda\tau^2(s) - \varepsilon_3\lambda^3\tau^4(s)\right)b\end{array}}{\sqrt{\left|\left(\varepsilon_2\left(-\varepsilon_2\lambda\tau^2\right)^2 - \varepsilon_1\left(-\varepsilon_1\lambda^2\tau^3\right)^2 + \varepsilon_3\left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)^2\right)\left(\varepsilon_2\lambda^2\tau^2 - \varepsilon_1\right)\right|}},$$

$$B_b = \frac{-\varepsilon_1\lambda^2\tau^3 t - \varepsilon_2\lambda\tau^2 n + \left(\varepsilon_2\lambda + \lambda^2\kappa\tau^2 - \varepsilon_3\kappa\right)b}{\sqrt{\left|\varepsilon_2\left(-\varepsilon_2\lambda\tau^2\right)^2 - \varepsilon_1\left(-\varepsilon_1\lambda^2\tau^3\right)^2 + \varepsilon_3\left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)^2\right|}},$$

and

$$\kappa_b = \frac{\sqrt{\left|\varepsilon_2\left(-\varepsilon_2\lambda\tau^2\right)^2 - \varepsilon_1\left(-\varepsilon_1\lambda^2\tau^3\right)^2 + \varepsilon_3\left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)^2\right|}}{\left|\varepsilon_2\lambda^2\tau^2 - \varepsilon_1\right|^{\frac{3}{2}}},$$

$$\tau_b = \frac{\sigma_1\lambda^2\tau^3 - \sigma_2\lambda\tau^2 + \sigma_3\left(-\kappa + \varepsilon_2\varepsilon_3\lambda + \varepsilon_3\lambda^2\kappa\tau^2\right)}{\left|\varepsilon_2\left(-\varepsilon_2\lambda\tau^2\right)^2 - \varepsilon_1\left(-\varepsilon_1\lambda^2\tau^3\right)^2 + \varepsilon_3\left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)^2\right|},$$

where

$$\sigma_1 = \varepsilon_1\left(\varepsilon_3\lambda\kappa'\tau - \varepsilon_2\kappa^2\right) + \varepsilon_1\varepsilon_3\left(\tau'\kappa + \lambda\kappa\right),\ \sigma_2 = \varepsilon_1\varepsilon_3\lambda\kappa^2\tau + \kappa' + \lambda\tau^3,\ \sigma_3 = \kappa\tau - \varepsilon_2\varepsilon_3\lambda\tau\left(1 + 2\tau'\right).$$

Proof: Let α_b be a curve with parameter s^* at constant distance from α in $\mathbb{R}_1^3$, then from Equation (3), it is known that

$$\frac{d\alpha_b}{ds^*}\frac{ds^*}{ds} = \frac{d\alpha}{ds} + \lambda\frac{db}{ds} = t - \varepsilon_2\varepsilon_3\lambda\tau n \tag{8}$$

or

$$T_b\frac{ds^*}{ds} = t - \varepsilon_2\varepsilon_3\lambda\tau n. \tag{9}$$

If we take the Lorentzian norm of this expression, then it holds that

$$\frac{ds^*}{ds} = \sqrt{\left|-\varepsilon_1 + \varepsilon_2\lambda^2\tau^2\right|}.$$

Substituting the last Equation into (9), the unit tangent vector of the curve α_b is obtained as

$$T_n = \frac{t - \varepsilon_2\varepsilon_3\lambda\tau n}{\sqrt{\left|\varepsilon_2\lambda^2\tau^2 - \varepsilon_1\right|}}$$

for $s \in I$. By differentiating from Equation (8), we get

$$\frac{d^2\alpha_b}{ds^2} = \varepsilon_1\varepsilon_3\lambda\kappa\tau t + \left(\kappa - \varepsilon_2\varepsilon_3\lambda\right)n - \varepsilon_2\varepsilon_3\lambda\tau^2 b$$

and

$$\frac{d^3\alpha_b}{ds^3} = \left(\varepsilon_1\varepsilon_3\lambda\kappa'\tau(s) + \varepsilon_1\varepsilon_3\tau'\kappa - \varepsilon_1\varepsilon_2\kappa^2 + \varepsilon_1\varepsilon_3\lambda\kappa\right)t + \left(\varepsilon_1\varepsilon_3\lambda\kappa^2\tau + \kappa' + \lambda\tau^3\right)n$$
$$+\left(\kappa\tau - \varepsilon_2\varepsilon_3\lambda\tau - 2\varepsilon_2\varepsilon_3\lambda\tau\tau'\right)b.$$

Considering the derivative equations $\frac{d\alpha_b}{ds}$, $\frac{d^2\alpha_b}{ds^2}$, and $\frac{d^3\alpha_b}{ds^3}$ of the curve α_b, the normal and binormal vectors N_b and B_b of α_b are found as follows:

$$B_b = \frac{\frac{d\alpha_b}{ds} \times \frac{d^2\alpha_b}{ds^2}}{\left\|\frac{d\alpha_b}{ds} \times \frac{d^2\alpha_b}{ds^2}\right\|} = \frac{-\varepsilon_1\lambda^2\tau^3 t - \varepsilon_2\lambda\tau^2 n + \left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)b}{\sqrt{\left|\varepsilon_2\left(-\varepsilon_2\lambda\tau^2\right)^2 - \varepsilon_1\left(-\varepsilon_1\lambda^2\tau^3\right)^2 + \varepsilon_3\left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)^2\right|}}$$

and

$$N_b = B_b \times T_b = \frac{\left(-\varepsilon_1\varepsilon_2\lambda\kappa\tau + \varepsilon_1\varepsilon_3\lambda^2\tau + \lambda^3\kappa\tau^3\right)t + \left(\kappa + \varepsilon_2\varepsilon_3\lambda + \varepsilon_3\lambda^2\kappa\tau^2\right)n + \left(\varepsilon_2\varepsilon_3\lambda\tau^2 - \varepsilon_3\lambda^3\tau^4\right)b}{\sqrt{\left|\left(\varepsilon_2\left(-\varepsilon_2\lambda\tau^2\right)^2 - \varepsilon_1\left(-\varepsilon_1\lambda^2\tau^3\right)^2 + \varepsilon_3\left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)^2\right)\left(\varepsilon_2\lambda^2\tau^2 - \varepsilon_1\right)\right|}}.$$

On the other hand, the curvature and torsion functions of α_b are obtained as:

$$\kappa_b = \frac{\left\|\frac{d\alpha_b}{ds} \times \frac{d^2\alpha_b}{ds^2}\right\|}{\left\|\frac{d\alpha_b}{ds}\right\|^3} = \frac{\sqrt{\left|\varepsilon_2\left(-\varepsilon_2\lambda\tau^2\right)^2 - \varepsilon_1\left(-\varepsilon_1\lambda^2\tau^3\right)^2 + \varepsilon_3\left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)^2\right|}}{\left|\varepsilon_2\lambda^2\tau^2 - \varepsilon_1\right|^{\frac{3}{2}}}$$

and

$$\tau_b = \frac{\det\left(\frac{d\alpha_b}{ds}, \frac{d^2\alpha_b}{ds^2}, \frac{d^3\alpha_b}{ds^3}\right)}{\left\|\frac{d\alpha_b}{ds} \times \frac{d^2\alpha_b}{ds^2}\right\|^2} = \frac{\sigma_1\lambda^2\tau^3 - \sigma_2\lambda\tau^2 + \sigma_3\left(-\kappa + \varepsilon_2\varepsilon_3\lambda + \varepsilon_3\lambda^2\kappa\tau^2\right)}{\left|\varepsilon_2\left(-\varepsilon_2\lambda\tau^2\right)^2 - \varepsilon_1\left(-\varepsilon_1\lambda^2\tau^3\right)^2 + \varepsilon_3\left(-\varepsilon_3\kappa + \varepsilon_2\lambda + \lambda^2\kappa\tau^2\right)^2\right|},$$

such that

$$\sigma_1 = \varepsilon_1\left(\varepsilon_3\lambda\kappa'\tau - \varepsilon_2\kappa^2\right) + \varepsilon_1\varepsilon_3\kappa\left(\tau' + \lambda\right),\ \sigma_2 = \varepsilon_1\varepsilon_3\lambda\kappa^2\tau + \kappa' + \lambda\tau^3,\ \sigma_3 = \kappa\tau - \varepsilon_2\varepsilon_3\lambda\tau\left(1 + 2\tau'\right).$$

Corollary 3.5. Let the curve α be a circular helix (κ and τ are non-zero constants) in Minkowski 3-space, then the curves α_t, α_n, and α_b are circular helix.

Proof: Let α_t, α_n, α_b be curves at constant distance from α, if the curvature κ and torsion τ of the curve α are non-zero constants;

- from Theorem 3.2, it can be said that $\frac{\kappa_t}{\tau_t}$ is constant,
- from Theorem 3.3, it can be said that $\frac{\kappa_n}{\tau_n}$ is constant,
- from Theorem 3.4, it can be said that $\frac{\kappa_b}{\tau_b}$ is constant.

4. RULED SURFACES GENERATED BY THE CURVES OBTAINED AT A CONSTANT DISTANCE FROM THE NON-NULL CURVE.

In this section, ruled surfaces whose basic curves are the curves obtained from a space curve at a constant distance and whose generators are the tangents of these curves are examined. The developability of these ruled surfaces is investigated by using the distribution parameters.

Definition 4.1. A ruled surface in Minkowski 3-space by using generator vectors T_t, T_n, and T_b can be presented as

$$X_t(s,v) = \alpha_t(s) + vT_t(s),$$
$$X_n(s,v) = \alpha_n(s) + vT_n(s),$$
$$X_b(s,v) = \alpha_b(s) + vT_b(s),$$

where the base curves α_t, α_n, and α_b of the surfaces are defined by (3).

Theorem 4.2. Let α_t be a curve obtained at a constant distance from the curve α and T_t be a tangent vector of α_t. Then, the ruled surface $X_t(s,v) = \alpha_t(s) + vT_t(s)$ is developable.

Proof: Let $X_t(s,v)$ be a ruled surface with the base curve α_t and the generator vector T_t. Using Equation (2) from Theorem 2.1, we get

$$\det\left(\alpha_t'(s), T_t(s), T_t'(s)\right) = 0 .$$

So, we can say that $X_t(s,v)$ is developable.

Theorem 4.3. Let α_n be a curve obtained at a constant distance from the curve α and T_n be a tangent vector of α_n. Then, the ruled surface $X_n(s,v) = \alpha_n(s) + vT_n(s)$ is developable if the curve α is planar.

Proof: Let $X_n(s,v)$ be a ruled surface with the base curve α_n and the generator vector T_n. From Theorem 2.1, we get

$$\det\left(\alpha_n'(s),T_n(s),T_n'(s)\right)=(\varphi_1\kappa-\varepsilon_2\varepsilon_3\varphi_2\tau)\left(\varepsilon_2\varphi_2(\varepsilon_2-\varepsilon_1\lambda)-\varphi_1\lambda\tau\right)$$

such that

$$\varphi_1=\frac{(1-\varepsilon_1\varepsilon_2\lambda\kappa)}{\sqrt{\left|2\varepsilon_2\lambda\kappa+\varepsilon_3\lambda^2\tau^2-\varepsilon_1-\varepsilon_1\lambda^2\kappa^2\right|}} \text{ and } \varphi_2=\frac{\lambda\tau}{\sqrt{\left|2\varepsilon_2\lambda\kappa+\varepsilon_3\lambda^2\tau^2-\varepsilon_1-\varepsilon_1\lambda^2\kappa^2\right|}}.$$

So, since $\tau=0$, there exists

$$(\varphi_1\kappa-\varepsilon_2\varepsilon_3\varphi_2\tau)\left(\varepsilon_2\varphi_2(\varepsilon_2-\varepsilon_1\lambda)-\varphi_1\lambda\tau\right)=0.$$

Therefore, we get $\det\left(\alpha_n'(s),T_n(s),T_n'(s)\right)=0$. We can say that $X_t(s,v)$ is developable.

Theorem 4.4. Let α_b be a curve obtained at a constant distance from the curve α and T_b be a tangent vector of α_b. Then, the ruled surface $X_b(s,v)=\alpha_b(s)+vT_b(s)$ is developable.

Proof: Let $X_b(s,v)$ be a ruled surface with the base curve α_b and the generator vector T_b. From Theorem 2.1, we get

$$\det\left(\alpha_b'(s),T_b(s),T_b'(s)\right)=\left\langle\left(\alpha_b'\times T_b\right).T_b'\right\rangle=-\varepsilon_3\phi_2\tau\left(\varepsilon_3\phi_2+\varepsilon_2\phi_1\lambda\tau\right),$$

where

$$\alpha_b'=t-\varepsilon_2\varepsilon_3\lambda\tau n,\ T_b'=\left(\phi_1'-\varepsilon_1\varepsilon_2\phi_2\kappa\right)t+\left(\phi_1\kappa+\phi_2'\right)n+\phi_2\tau b,\ T_b=\frac{t-\varepsilon_2\varepsilon_3\lambda\tau n}{\sqrt{\left|\varepsilon_2\lambda^2\tau^2-\varepsilon_1\right|}},$$

and

$$\phi_1=\frac{1}{\sqrt{\left|\varepsilon_2\lambda^2\tau^2-\varepsilon_1\right|}},\ \phi_2=\frac{-\varepsilon_2\varepsilon_3\lambda\tau}{\sqrt{\left|\varepsilon_2\lambda^2\tau^2-\varepsilon_1\right|}}.$$

So, since $\tau=0$, there exists

$$\varepsilon_3\phi_2\tau\left(\varepsilon_3\phi_2+\varepsilon_2\phi_1\lambda\tau\right),=0.$$

Therefore, we can say that $X_b(s,v)$ is developable.

4. APPLICATIONS

Example 4.1. Let us consider a timelike curve parameterized as

$$\alpha(s)=\left(\frac{-5}{9}\sinh 3s,\frac{-5}{9}\cosh 3s,\frac{4}{3}s\right).$$

Then, the Frenet vectors of the timelike curve α are given by

$$T(s)=\left(-\frac{5}{3}\cosh 3s,-\frac{5}{3}\sinh 3s,\frac{4}{3}\right),$$
$$N(s)=\left(-\sinh 3s,-\cosh 3s,0\right),$$
$$B(s)=\left(\frac{-4}{3}\cosh 3s,\frac{-4}{3}\sinh 3s,\frac{5}{3}\right).$$

Thus, for $\lambda=1$, the curves at a constant distance from the timelike curve are

$$\alpha_t(s)=\left(-\frac{5}{9}\left(3\cosh 3s+\sinh 3s\right),-\frac{5}{9}\left(\cosh 3s+3\sinh 3s\right),\frac{4(1+s)}{3}\right),$$
$$\alpha_n(s)=\left(-\frac{14}{9}\sinh 3s,-\frac{14}{9}\cosh 3s,\frac{4s}{3}\right),$$
$$\alpha_b(s)=\left(-\frac{4}{3}\cosh 3s-\frac{5}{9}\sinh 3s,-\frac{5}{9}\cosh 3s-\frac{4}{3}\sinh 3s,\frac{(5+4s)}{3}\right).$$

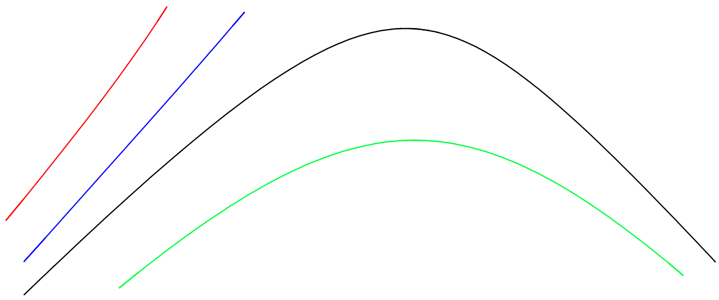

Figure 1. The curves α_t (red), α_n (green), and α_b (blue) at a constant distance from the timelike curve α (black) with $s=\left(\frac{-\pi}{6},\frac{\pi}{6}\right)$.

On the other hand, we have the ruled surfaces constructed by the curves obtained at a constant distance from the curve as follows (Figures 2-4):

$$X_t(s,v)=\frac{-1}{72}\begin{pmatrix}40\left(3\cosh 3s+\sinh 3s\right)+5v\left(\cosh 3s+3\sinh 3s\right),\\ 40\left(\cosh 3s+3\sinh 3s\right)+5v\left(3\cosh 3s+\sinh 3s\right),-4\left(24+24s+v\right)\end{pmatrix},$$
$$X_n(s,v)=-\frac{2}{180}\left(7\left(3v\cosh 3s+20\sinh 3s\right),7\left(20\cosh 3s+3v\sinh 3s\right),-6\left(20s+v\right)\right),$$
$$X_b(s,v)=\frac{-1}{135}\begin{pmatrix}\left(15\left(12\cosh 3s+5\sinh 3s\right)+3v\left(5\cosh 3s+12\sinh 3s\right)\right),\\ \left(15\left(5\cosh 3s+12\sinh 3s\right)+3v\left(12\cosh 3s+5\sinh 3s\right)\right),-3\left(15\left(5+4s\right)+4v\right)\end{pmatrix}.$$

Figure 2. The ruled surface X_t constructed by the curve α_t obtained at a constant distance from the timelike curve with $s=\left(\frac{-\pi}{6},\frac{\pi}{8}\right)$ and $v=\left(-10,10\right)$.

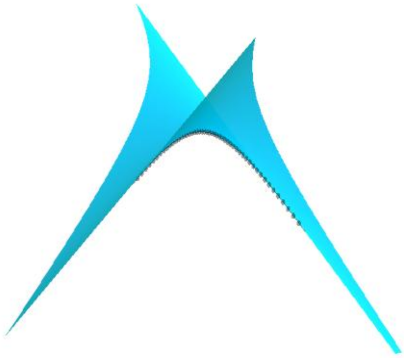

Figure 3. The ruled surface X_n constructed by the curve α_n obtained at a constant distance from the timelike curve with $s=\left(\frac{-\pi}{6},\frac{\pi}{8}\right)$ and $v=\left(-10,10\right)$.

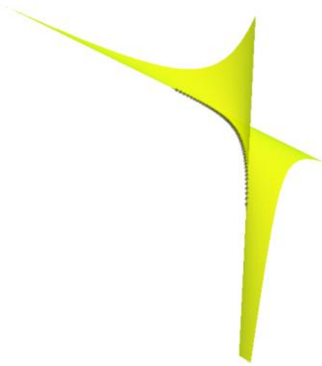

Figure 4. The ruled surface X_b constructed by the curve α_b obtained at a constant distance from the timelike curve with $s=\left(\frac{-\pi}{6},\frac{\pi}{8}\right)$ and $v=\left(-10,10\right)$.

5. CONCLUSIONS

In this study, the constant distance curves were obtained using tangent, normal, and binormal vectors of a unit speed space curve in Minkowski 3-space. Then, the Frenet vectors of these curves, curvature, and torsion functions were calculated by using the curvature and torsion of the given unit speed curve. Additionally, the developability of ruled surfaces generated using these curves and their tangents were investigated. Finally, an example containing these curves and surfaces was given, and their graphs are drawn. A similar study can be conducted to investigate the developability of ruled surfaces generated by the normal and binormal vectors of a space curve. Furthermore, such a study can be extended to examine the minimality of these surfaces.

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