ORIGINAL PAPER

A NEW EXTENSION OF LOG-LOGISTIC DISTRIBUTION APPLIED TO DEMOGRAPHIC AND PUBLIC HEALTH DATA IN INDIA

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Abstract. *This article introduces the SMPLog-logistic distribution (SMPLLD), a new continuous probability distribution developed as an extension of the log-logistic distribution using the SMP (Shamshad, Murtaza, Parvaiz) technique. It explores the distribution's statistical and reliability properties and utilizes various parameter estimation methods, including Maximum Likelihood, Maximum Product Spacings, Anderson-Darling, Cramer-Von Mises and Ordinary and Weighted Least Squares techniques. An extensive simulation study evaluates the performance and robustness of these estimation methods, highlighting the potential applications of SMPLLD in statistical modeling and reliability analysis.*

Keywords: *SMP transformation; order statistics; mean waiting time; maximum product of spacing estimation.*

1. INTRODUCTION

Probability distribution models provide a mathematical framework for describing and predicting outcomes in uncertain scenarios. Due to the complexity of natural phenomena, traditional distributions often struggle to accurately represent them. Therefore, generalized probability distributions have been adapted and expanded to better suit these data and improve the flexibility and precision of the distributions. One example of such distributions is the Log-logistic (LL) distribution standing out for its versatility. The LL distribution effectively models data characterized by a rapid increase followed by a gradual decline, making it particularly suitable for applications in reliability analysis, survival studies and economics. A random variable X follows an LL distribution if the logarithm of X adheres to a Logistic distribution. Over the years, numerous generalizations have enhanced the adaptability of the LL distribution. [1] introduced the transmuted LL distribution using quadratic rank transmutation, while [2] proposed the Marshall-Olkin extended LL distribution. [3] investigated the odd log-logistic Lindley-G family of distributions, [4] introduced alpha power LL distribution, [5] utilized the log-logistic tangent distribution to analyze COVID-19 data, [6] applied the Kavya-Manoharan LL model to biomedical datasets, [7] developed the cubic transmuted LL distribution. Recently,[8] introduced Kumaraswamy alpha-power log-logistic (KAPLL) distribution using the KAp-G family, [9] introduced Arcsine Log-logistic (AL-L) distribution. Building on these advancements, this study introduces a novel extension of the LL distribution using the SMP (Shamshad, Murtaza, Parvaiz) technique, as proposed by [10], to enhance the flexibility of the baseline distribution. Recently, [11] and [12] applying the SMP technique proposed SMP Kumaraswamy and the SMP Pareto distributions respectively.

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This innovative method enhances the flexibility and shape of the LL distribution, improves goodness-of-fit and broadens its range of applications across various fields.The CDF and PDF of SMP technique are given below as:

$$F_{SMP}(x)\begin{cases}\dfrac{e^{log\alpha\bar{F}(x)}-\alpha}{1-\alpha}; & \alpha \neq 1, \alpha > 0 \\ F(x); & \alpha = 1\end{cases} \quad (1)$$

$$f_{SMP}(x)=\begin{cases}\dfrac{e^{log\alpha\bar{F}(x)}log\alpha f(x)}{\alpha-1}; & \alpha \neq 1, \alpha > 0 \\ f(x); & \alpha = 1\end{cases} \quad (2)$$

where $\overline{F(x)} = 1 - F(x)$ for x ∈ $\mathbb{R}$.

By merging the attributes of this technique with those of the LL distribution, we propose a novel extension of the LL distribution, named as the SMPLog-logistic distribution (SMPLLD). We are inspired to introduce SMPLLD because

- The hazard rate function and probability density function exhibits various forms, including decreasing, increasing-decreasing, bath-tub, constant, increasing and right-skewed shapes. This flexibility enables to effectively model datasets of various type.
- It increases the adaptability and flexibilty of Log-logistic distribution enabling well defined modelling of diverse real-world instances.

The contents of this paper are organized as follows: The SMPLLD is introduced in Section 2. Some statistical properties are discussed in Section 3. The reliability analysis is covered in Section 4. In Section 5 estimation methods are discussed. Simulation results are shown in Section 6. In Section 7, applications are illustrated and Section 8 presents conclusions.

2. SMPLOG-LOGISTIC DISTRIBUTION

The two-parameter SMPLLD is a probability distribution generated using SMP technique with LL distribution as a baseline distribution. The LL distribution under consideration has respectively CDF and PDF of the form

$$F(x;\theta) = \frac{x^{\theta}}{1+x^{\theta}}; x > 0, \theta > 0 \quad (3)$$

$$f(x;\theta) = \frac{\theta x^{\theta-1}}{(1+x^{\theta})^2}; x > 0, \theta > 0 \quad (4)$$

Using (3) and (4) in (1) and (2) respectively, we obtain the CDF and PDF of the SMPLLD respectively as:

$$F_{SMPLL}(x;\alpha,\theta) = \begin{cases} \dfrac{e^{\frac{log\alpha}{1+x^{\theta}}} - \alpha}{1-\alpha}; \alpha \neq 1, \alpha > 0 \\ \dfrac{1}{1+x^{\theta}}; \alpha = 1 \end{cases} \tag{5}$$

$$f_{SMPLL}(x;\alpha,\theta) = \begin{cases} \dfrac{e^{\frac{log\alpha}{1+x^{\theta}}} log\alpha\theta x^{\theta-1}}{(\alpha-1)(1+x^{\theta})^2}\ ;\ \alpha \neq 1, \alpha > 0 \\ \dfrac{\theta x^{\theta-1}}{(1+x^{\theta})^2};\ \ \alpha = 1 \end{cases} \tag{6}$$

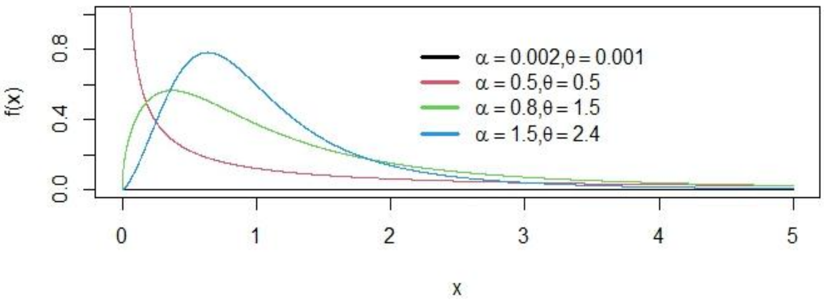

Figure 1.Plots of the pdf of the SMPLLD.

Figure 1 displays various probability density function (PDF) curves for different parameter combinations, showcasing the versatility of the SMPLL distribution. The curves demonstrate that the PDF can take on increasing-decreasing, right-skewed, constant, or decreasing shapes, highlighting the model's effectiveness in representing diverse data sets.

3. STATISTICAL PROPERTIES

This section demonstrates some important statistical properties of the SMPLog-logistic distribution (SMPLLD).

3.1. QUANTILE FUNCTION

The quantile function of a random variable $X \sim$ SMPLLD (α, θ) is given as

$$x = \left[\frac{log\alpha}{log(u(1-\alpha)+\alpha)} - 1\right]^{\frac{1}{\theta}} \tag{7}$$

Where u is a uniform random variable, $0 < u < 1$.

Proof: Let F(x; α, θ) = u

$$\frac{e^{\frac{log\alpha}{1+x^{\theta}}} - \alpha}{1-\alpha} = u$$

After calculations,

$$x = \left[\frac{log\alpha}{log(u(1-\alpha)+\alpha)} - 1\right]^{\frac{1}{\theta}}$$

Remark 1. The p^{th} quantile is given by

$$x_p = \left[\frac{log\alpha}{log(p(1-\alpha)+\alpha)} - 1\right]^{\frac{1}{\theta}}$$

3.2. MOMENTS

The r^{th} moment about origin for the SMPLLD is obtained as

$$\mu_r{}' = \int_0^\infty x^r \frac{e^{\frac{log\alpha}{1+x^\theta}} log\alpha\theta x^{\theta-1}}{(\alpha-1)(1+x^\theta)^2} dx$$

Using the expansion,

$$e^x = \sum_{i=0}^\infty \frac{x^i}{i!}$$

$$\mu_r{}' = \frac{1}{\alpha-1}\sum_{i=0}^\infty \frac{(log\alpha)^{i+1}}{i!}\int_0^\infty \frac{\theta x^{r+\theta-1}}{(1+x^\theta)^{2+i}} dx \tag{8}$$

$$\mu_r{}' = \frac{1}{\alpha-1}\sum_{i=0}^\infty \frac{(log\alpha)^{i+1}}{i!}\boldsymbol{B}\left[\frac{r+\theta}{\theta}, \left(1+i-\frac{r}{\theta}\right)\right]$$

where $\frac{r}{\theta} < (1+i)$, ensuring function is positive and $B\left[\frac{r+\theta}{\theta}, \left(1+i-\frac{r}{\theta}\right)\right]$ represents the beta function.

Mean:

$$\mu_1{}' = \frac{1}{\alpha-1}\sum_{i=0}^\infty \frac{(log\alpha)^{i+1}}{i!}\boldsymbol{B}\left[\frac{1+\theta}{\theta}, \left(1+i-\frac{1}{\theta}\right)\right]$$

$$= \frac{1}{\alpha-1}\sum_{i=0}^\infty \frac{(log\alpha)^{i+1}}{i!}\frac{\boldsymbol{\Gamma}\left(\frac{1+\theta}{\theta}\right)\boldsymbol{\Gamma}\left(1+i-\frac{1}{\theta}\right)}{\boldsymbol{\Gamma}(2+i)} \tag{9}$$

Lemma 1. Suppose a random variable $X \sim SMPLLD$ (α, θ) with pdf by (6) and let $I_r(t) = \int_0^t x^r f_{SMPLL}(x;\alpha,\theta)dx$ denotes the r^{th} incomplete moment, then we have

$$I_r(t) = \frac{1}{\alpha - 1}\sum_{i=0}^{\infty}\frac{(log\alpha)^{i+1}}{i!}\boldsymbol{B}\left[t^{\lambda};\left(\frac{r}{\theta}+1\right),\left(i+1-\frac{r}{\theta}\right)\right] \tag{10}$$

Proof:

$$I_r(t) = \int_0^t x^r \frac{e^{\frac{log\alpha}{1+x^{\theta}}} log\alpha\theta x^{\theta-1}}{(\alpha-1)(1+x^{\theta})^2}dx$$

$$I_r(t) = \frac{1}{\alpha - 1}\sum_{i=0}^{\infty}\frac{(log\alpha)^{i+1}}{i!}\boldsymbol{B}\left[t^{\lambda};\left(\frac{r}{\theta}+1\right),\left(i+1-\frac{r}{\theta}\right)\right]$$

Where $B[z{:}\,p,q] = \int_0^t x^{q-1}(1-x)^{r-1}dx$ is incomplete beta function. By substituting r=1 in (10) first incomplete moment is obtained as given by

$$I_1(t) = \frac{1}{\alpha - 1}\sum_{i=0}^{\infty}\frac{(log\alpha)^{i+1}}{i!}\boldsymbol{B}\left[t^{\lambda};\left(\frac{1}{\theta}+1\right),\left(i+1-\frac{1}{\theta}\right)\right]$$

4. RELIABILITY ANALYSIS

This section focuses on reliability analysis of the SMPLLD.

4.1. SURVIVAL FUNCTION

The survival function for the SMPLLD is given as

$$R(x;\alpha,\theta) = \frac{e^{\frac{log\alpha}{1+x^{\theta}}} - \alpha}{1-\alpha};\ \alpha \neq 1$$

4.2. HAZARD RATE

The hazard rate for the SMPLLD is given by

$$h(x;\alpha,\theta) = \frac{e^{\frac{log\alpha}{1+x^{\theta}}} log\alpha\theta x^{\theta-1}}{(1+x^{\theta})^2 e^{\frac{log\alpha}{1+x^{\theta}}} - 1};\ \alpha \neq 1$$

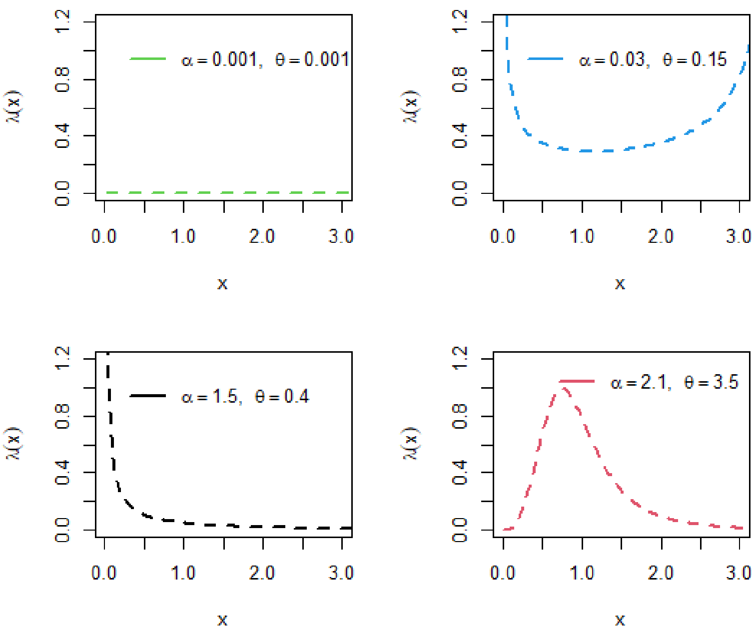

Figure 2. Plots of the hazard rate of SMPLLD.

Fig. 2 reveals that the SMPLLD exhibits diverse shapes of hazard rate, including constant, decreasing and bathtub forms, depending on parameter values. This flexibility makes the proposed model well-suited for modeling datasets with such hazard rate patterns.

4.3. REVERSE HAZARD FUNCTION

The reverse hazard rate for SMPLLD is given as

$$h_r(x;\alpha,\theta) = \frac{e^{\frac{log\alpha}{1+x^\theta}} log\alpha\theta x^{\theta-1}}{(1+x^\theta)^2\left(\alpha - e^{\frac{log\alpha}{1+x^\theta}}\right)};\ \alpha \neq 1$$

4.4. MEAN RESIDUAL LIFE

Mean residual life for SMPLLD is given by

$$\mu(x) = \frac{\sum_{i=0}^{\infty}\frac{(log\alpha)^{i+1}}{i!}\left\{\boldsymbol{B}\left[\frac{1+\theta}{\theta}, 1+i-\frac{1}{\theta}\right] - \boldsymbol{B}\left[t^\lambda;\left(\frac{1}{\theta}+1\right),\left(1+i-\frac{1}{\theta}\right)\right]\right\}}{e^{\frac{log\alpha}{1+x^\theta}}-1} - x$$

4.5. MEAN WAITING TIME

Mean waiting time for SMPLLD is given as

$$\bar{\mu}(x) = \frac{\left(e^{\frac{\log\alpha}{1+x^{\theta}}} - \alpha\right)x + \sum_{i=0}^{\infty}\frac{(\log\alpha)^{i+1}}{i!}\boldsymbol{B}\left[t^{\lambda};\left(\frac{1}{\theta}+1\right),\left(1+i-\frac{1}{\theta}\right)\right]}{e^{\frac{\log\alpha}{1+x^{\theta}}} - \alpha}$$

4.6. ORDER STATISTICS

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample of size n and let $X_{(r:n)}$ denote the r^{th} order statistic, then the PDF of r^{th} order statistics is given by

$$f_{(r:n)}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)$$

Using $[1-F(x)]^{\,n-r}$ as

$$[1-F(x)]^{\,n-r} = \sum_{p=0}^{n-r}(-1)^p\binom{n-r}{p}[F(x)]^p$$

Thus, we obtain

$$f_{(r:n)}(x) = \frac{1}{B(r,n+1-r)}\sum_{p=0}^{n-r}(-1)^p\binom{n-r}{p}\left(\frac{e^{\frac{\log\alpha}{1+x^{\theta}}}-\alpha}{1-\alpha}\right)^{p+r-1}\frac{e^{\frac{\log\alpha}{1+x^{\theta}}}\log\alpha\,\theta x^{\theta-1}}{(\alpha-1)(1+x^{\theta})^2} \tag{11}$$

CDF of r th order statistics is given by

$$F_{(r:n)}(x) = \sum_{k=r}^{n}\binom{n}{k}[F(x)]^k[1-F(x)]^{n-k}$$
$$F_{(r:n)}(x) = \sum_{k=r}^{n}\sum_{p=0}^{n-k}\binom{n-k}{j}\binom{n}{k}[F(x)]^{k+j} \tag{12}$$
$$F_{(r:n)}(x) = \sum_{k=r}^{n}\sum_{p=0}^{n-k}\binom{n-k}{j}\binom{n}{k}\left[\frac{e^{\frac{\log\alpha}{1+x^{\theta}}}-\alpha}{1-\alpha}\right]^{k+j}$$

The expressions for PDF and CDF of minimum order statistics $X_{(1)}$ and maximum order statistics $X_{(n)}$ of SMPLLD are respectively obtained by setting r=1 and r=n in (11) and (12).

5. PARAMETER ESTIMATION OF SMPLLD

In this section, we employ different methods of estimation to estimate the parameters of the SMPLLD.

5.1. MAXIMUM LIKELIHOOD ESTIMATION

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample drawn from the SMPLLD with parameters α, θ. Then the log-likelihood function is given as:

$$l = \frac{log\alpha}{1 + {x_i}^{\theta}} + \log(log\alpha) + log\theta + (\theta - 1)logx_i - \log(\alpha - 1) - 2\log(1 + {x^{\theta}}_i)$$

By solving the below normal equations, we obtain the MLEs

$$\frac{\partial l}{\partial \alpha} = \frac{1}{\alpha(1 + {x^{\theta}}_i)} + \frac{1}{\alpha log\alpha} - \frac{1}{\alpha - 1} = 0$$

$$\frac{\partial l}{\partial \theta} = -log\alpha \frac{{x^{\theta}}_i logx_i}{((1 + {x^{\theta}}_i))^2} + \frac{1}{\theta} + logx_i - \frac{2{x^{\theta}}_i logx_i}{1 + {x^{\theta}}_i} = 0$$

The above normal equations cannot be solved using exact algebraic methods. Therefore, we use numerical iteration procedures using R software.

5.2. ADERSON DARLING ESTIMATION

The Anderson and Darling estimate for SMPLLD is given by

$$AD(\alpha, \theta) = -n - \frac{1}{n}\sum_{k=1}^{n}(2k - 1)\left\{ln\left(\frac{e^{\frac{log\alpha}{1+{x^{\theta}}_k}} - \alpha}{1 - \alpha}\right) + ln\left(\frac{1}{1 + {x_k}^{\theta}}\right)\right\}$$

5.3. CRAMER VON MISES ESTIMATION

The Cramer-von-Mises estimate for SMPLLD is

$$C(\alpha, \theta) = \frac{1}{12n} + \sum_{k=1}^{n}\left\{\left(\frac{e^{\frac{log\alpha}{1+{x^{\theta}}_k}} - \alpha}{1 - \alpha}\right) - \frac{2k - 1}{2n}\right\}^2$$

5.4. MAXIMUM PRODUCT OF SPACING ESTIMATION

The MPS estimators can be obtained by optimizing the following function

$$M(\alpha, \theta) = \frac{1}{n + 1}\sum_{k=1}^{n+1} logD_k$$

$$M(\alpha,\theta)=\frac{1}{n+1}\sum_{k=1}^{n+1}log\left[\left(\frac{e^{\frac{log\alpha}{1+x^{\theta}(k)}}-\alpha}{1-\alpha}\right)-\left(\frac{e^{\frac{log\alpha}{1+x^{\theta}(k-1)}}-\alpha}{1-\alpha}\right)\right]$$

5.5. ORDINARY AND WEIGHTED LEAST SQUARE ESTIMATION

In these estimation methods the unknown parameters are estimated by minimizing the below given function with respect to parameters.

$$S(\alpha,\theta)=\sum_{k=1}^{n}n_k\left(Fx_{(k)}-\frac{k}{n+1}\right)^2$$

$$S(\alpha,\theta)=\sum_{k=1}^{n}n_k\left\{\left(\frac{e^{\frac{log\alpha}{1+x^{\theta}(k)}}-\alpha}{1-\alpha}\right)-\frac{k}{n+1}\right\}^2$$

By Substituting n_k = 1, the least square estimators of the unknown parameter are obtained and for $n_k=\frac{(n+1)^2(n+2)}{k(n-k+1)}$, the weighted least square estimators are obtained.

6. SIMULATION

A simulation study was conducted using R software to evaluate and compare performance of aforementioned estimation methods. Random samples of sizes 20, 50, 125 and 500 were generated using the quantile function from Section 3 for parameter combinations (α = 0.5, θ = 1.7) & (α = 1.4, θ = 0.2), with each scenario repeated 1000 times. Estimates of parameters were computed along with their bias and mean square errors (MSEs). The results are summarized in Tables 1-3, reveal that as sample size increases, parameter estimates stabilize and converge to true values, with both bias and MSE decreasing, thus satisfying consistency criteria. Among the methods compared, the maximum product of spacing estimation consistently outperformed others in minimizing bias and MSE for each parameter combination taken into account.

Table 1. For α = 0.5, θ = 1.7, Average estimate (AVE), Bias and MSE.

n	Estimate	Parameter	MLE	ADE	CVME	MPSE	OLSE	WLSE
20	AVE	$\hat{\alpha}$	0.67631	0.69865	0.64452	0.69998	0.66890	0.65602
		$\hat{\theta}$	1.80980	1.72152	1.78480	1.55927	1.69077	1.69614
	Bias	$\hat{\alpha}$	0.40027[5]	0.41334[6]	0.38600[4]	0.37678[3]	0.37344[2]	0.36879[1]
		$\hat{\theta}$	0.27786[2]	0.26771[1]	0.30453[5]	0.28200[4]	0.31298[6]	0.28168[3]
	MSE	$\hat{\alpha}$	0.45149[4]	0.58301[6]	0.51140[5]	0.41704[3]	0.37704[2]	0.35790[1]
		$\hat{\theta}$	0.13343[3]	0.11857[2]	0.16844[6]	0.11578[1]	0.16634[5]	0.13847[4]
	Sum of Ranks		14[3]	15[4.5]	20[6]	11[2]	15[4.5]	9[1]
50	AVE	$\hat{\alpha}$	0.57089	0.57241	0.54918	0.57356	0.54167	0.56284
		$\hat{\theta}$	1.73537	1.71519	1.74800	1.63964	1.69503	1.71035
	Bias	$\hat{\alpha}$	0.23106[6]	0.22349[5]	0.21826[4]	0.21582[3]	0.20839[1]	0.21570[2]
		$\hat{\theta}$	0.17103[2]	0.17059[1]	0.19015[6]	0.17364[3]	0.18653[5]	0.17822[4]

n	Estimate	Parameter	MLE	ADE	CVME	MPSE	OLSE	WLSE
	MSE	$\hat{\alpha}$	0.11207[6]	0.10364[5]	0.09215[2]	0.09806[4]	0.08725[1]	0.09740[3]
		$\hat{\theta}$	0.04898[3]	0.04628[2]	0.06153[6]	0.04526[1]	0.05657[5]	0.05052[4]
	Sum of Ranks		17[5]	13[3.5]	18[6]	11[1]	12[2]	13[3.5]
125	AVE	$\hat{\alpha}$	0.51194	0.52485	0.53137	0.53047	0.52345	0.52254
		$\hat{\theta}$	1.71559	1.70860	1.71679	1.65141	1.69465	1.70147
	Bias	$\hat{\alpha}$	0.12627[3]	0.13035[5]	0.13554[6]	0.12931[4]	0.12595[2]	0.12513[1]
		$\hat{\theta}$	0.10308[1]	0.10876[3]	0.11949[6]	0.10836[2]	0.11645[5]	0.11250[4]
	MSE	$\hat{\alpha}$	0.02532[1]	0.02870[5]	0.03201[6]	0.02790[3]	0.02674[2]	0.02812[4]
		$\hat{\theta}$	0.01709[1]	0.01902[3]	0.02252[6]	0.018075[2]	0.02131[5]	0.02002[4]
	Sum of Ranks		6[1]	16[5]	24[6]	11[2]	14[4]	13[3]
500	AVE	$\hat{\alpha}$	0.50363	0.51131	0.50293	0.50743	0.50216	0.50607
		$\hat{\theta}$	1.70185	1.70293	1.70059	1.68783	1.69719	1.70157
	Bias	$\hat{\alpha}$	0.06073[1]	0.06410[6]	0.06399[5]	0.06168[2]	0.06359[4]	0.06182[3]
		$\hat{\theta}$	0.05308[3]	0.05350[4]	0.06028[6]	0.05081[1]	0.05819[5]	0.05179[2]
	MSE	$\hat{\alpha}$	0.00600[1]	0.00636[4]	0.00653[6]	0.00640[5]	0.00632[3]	0.00616[2]
		$\hat{\theta}$	0.00444[4]	0.00442[3]	0.00570[6]	0.00401[1]	0.000534[5]	0.00417[2]
	Sum of Ranks		9[2]	17[4.5]	23[6]	9[2]	17[4.5]	9[2]

Table 2. For α = 1.4, θ = 0.2, Average estimate (AVE), Bias and MSE.

n	Estimate	Parameter	MLE	ADE	CVME	MPSE	OLSE	WLSE
20	AVE	$\hat{\alpha}$	1.96502	1.96502	1.96502	1.96502	1.96502	1.96502
		$\hat{\theta}$	0.21380	0.20308	0.21232	0.18621	0.19602	0.19779
	Bias	$\hat{\alpha}$	1.15778[3]	1.20146[5]	1.24470[6]	0.99602[1]	1.20076[4]	1.14295[2]
		$\hat{\theta}$	0.03412[4]	0.03280[3]	0.03844[6]	0.03079[1]	0.03643[5]	0.03262[2]
	MSE	$\hat{\alpha}$	4.55679[2]	5.75105[5]	5.37646[4]	2.64840[1]	16.96997[6]	4.89217[3]
		$\hat{\theta}$	0.00206[4]	0.00187[3]	0.00277[6]	0.00142[1]	0.00217[5]	0.00171[2]
	Sum of Ranks		13[3]	16[4]	22[6]	4[1]	20[5]	9[2]
50	AVE	$\hat{\alpha}$	1.57019	1.61775	1.56704	1.53999	1.51869	1.61611
		$\hat{\theta}$	0.20474	0.20176	0.20431	0.19148	0.19897	0.20109
	Bias	$\hat{\alpha}$	0.59292[3]	0.61601[5]	0.61358[4]	0.54513[1]	0.56520[2]	0.62869[6]
		$\hat{\theta}$	0.01933[1]	0.01986[2]	0.02250[6]	0.01987[3]	0.02171[5]	0.02063[4]
	MSE	$\hat{\alpha}$	0.71174[3]	0.96924[6]	0.74361[4]	0.60447[1]	0.65818[2]	0.94131[5]
		$\hat{\theta}$	0.00062[2]	0.00063[3]	0.00084[6]	0.00061[1]	0.00076[5]	0.00068[4]
	Sum of Ranks		9[2]	16[4]	20[6]	6[1]	14[3]	19[5]
125	AVE	$\hat{\alpha}$	1.47674	1.46067	1.47838	1.45907	1.46856	1.48774
		$\hat{\theta}$	0.20177	0.20076	0.20141	0.19532	0.19985	0.20025
	Bias	$\hat{\alpha}$	0.36674[6]	0.35302[3]	0.35480[4]	0.34741[1]	0.35037[2]	0.36357[5]
		$\hat{\theta}$	0.01192[1]	0.0124[2]	0.01378[5]	0.01245[3.5]	0.01410[6]	0.01245[3.5]
	MSE	$\hat{\alpha}$	0.24308[6]	0.23013[3]	0.23387[5]	0.21662[1]	0.21795[2]	0.23373[4]
		$\hat{\theta}$	0.00023[1]	0.00025[3.5]	0.00031[5]	0.00024[2]	0.00032[6]	0.00025[3.5]
	Sum of Ranks		14[3]	11.5[2]	19[6]	7.5[1]	16[4.5]	16[4.5]
500	AVE	$\hat{\alpha}$	1.41237	1.40984	1.40694	1.41228	1.40936	1.41545
		$\hat{\theta}$	0.20051	0.20009	0.20075	0.19872	0.19983	0.20043
	Bias	$\hat{\alpha}$	0.16814[2]	0.18087[6]	0.17701[3]	0.15696[1]	0.17744[4]	0.17855[5]
		$\hat{\theta}$	0.00586[1]	0.00647[4]	0.00692[6]	0.00616[2]	0.00672[5]	0.00623[3]
	MSE	$\hat{\alpha}$	0.04554[2]	0.05106[5]	0.05004[3]	0.04415[1]	0.05174[6]	0.05027[4]
		$\hat{\theta}$	0.00005[1]	0.00006[3]	0.00008[6]	0.00006[3]	0.00007[5]	0.00006[3]
	Sum of Ranks		6[1]	18[4.5]	18[4.5]	7[2]	20[6]	15[3]

Table 3. Partial and overall rankings of all estimation procedures.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	WLSE
α =0.5, $\theta = 1.7$	20	3	4.5	6	2	4.5	1
	50	5	3.5	6	1	2	3.5
	125	1	5	6	2	4	3
	500	2	4.5	6	2	4.5	2
α =0.5, $\theta = 1.7$	20	3	4	6	1	5	2
	50	2	4	6	1	3	5
	125	3	2	6	1	4.5	4.5
	500	1	4.5	4.5	2	6	3
Sum of Ranks		20	32	46.5	12	33.5	24
Overall Rank		2	4	6	1	5	3

7. APPLICATIONS

This section examines the real life applications of SMPLLD by introducing two real life data sets.

Dataset 1. 1.822184002, 1.808446421, 1.776767874, 1.724269032, 1.67281087, 1.604129169, 1.52430796, 1.46637175, 1.422391509, 1.39119493, 1.377595812, 1.361588085, 1.33219205, 1.297548835, 1.240362184, 1.18779532, 1.185046229, 1.155624491, 1.087527722, 1.025310772, 0.955220858, 0.797216094, 0.680372581, 0.805014301. The data observations are annual population growth percentage in India from year 2000 to 2023 [13] (and was fetched from https://data.worldbank.org /indicator/SP.POP.GROW on 24 November 2024).

Dataset 2. 4.2, 4.6, 0.7, 2.6, 1.7, 6.8, 0.9, 1.2, 5.9, 2.5, 1.2, 2.7, 3.8, 6.8, 1.7, 2, 8.7, 4.9, 3.7, 1.1, 2.1, 3, 2, 0.9, 0.5, 2.8, 7.6, 3.4, 1.3, 2.2, 4.1, 5.8, 3.6, 1.3, 2.4, 5.3. The data points consist of state-wise percentage of women in the age group 15-49 years who reported Goitre or any other Thyroid disorder in India from 2019-2021 [14]. Source: Indiastat, “State-wise Percentage of Women in the Age group 15-49 Years who reported Goitre or any other Thyroid disorder in India from 2019-2021", 2024, Accessed on: 28 November 2024.

For the analysis of the aforementioned datasets and how well the SMPLLD performs in terms of goodness-of-fit (GoF) the following distributions are taken into consideration for comparison ; Exponentiated Log-logistic distribution (ELL) [15], Transmuted Log-logistic distribution (TLL) [1] and Log-Logistic distribution(LL). The performance and potentiality of the SMPLLD is assessed using following goodness of fit measures (GoF) : Akaike information criterion (AIC), Akaike information criterion corrected (AICc) , Bayesian information criterion (BIC), Cramer-von Mises (W), Anderson–Darling (A), Kolmogorov-Smirnov (K-S) and associated p-value. The value of the estimated parameters using estimation methods mentioned in section 3 and the goodness-of-fit measures for datasets are given in Tables 4-6 respectively. Also, MLEs of the parameters along with goodness-of-fit measures (GoF) for SMPLLD and competitive models for data set 1 and data set 2 are given in table 5 and table 7 respectively. The relative histograms for two data sets are presented in Figure 3 and 4. The distribution with the smallest value of above mentioned GoF statistic along with largest p-value is considered to be the best fit.

Table 4. Parameter estimation and GoF statistics for Data Set 1.

	$\hat{\alpha}$	$\hat{\theta}$	A	W	K-S	p-value
MLE	0.03578	5.61841	0.46709	0.06338	0.10858	0.91092
ADE	0.03371	5.46346	0.45826	0.06189	0.11990	0.84051
CVME	0.02156	5.71992	0.46262	0.06259	0.11568	0.86888
MPSE	0.05000	5.15645	0.45339	0.06110	0.12820	0.77899
OLSE	0.02917	5.40794	0.45264	0.06094	0.12789	0.78140
WLSE	0.02940	5.63371	0.46379	0.06281	0.11290	0.88620

Table 5. GoF statistics for SMPLLD and competitive models for Data set 1.

Model	MLE	AIC	AICC	BIC	A	W	K-S	p-value
SMPLLD	$\hat{\alpha}$ =0.03578 $\hat{\theta}$ = 5.61841	21.56937	22.1408	23.92548	0.46709	0.06338	0.10858	0.91092
ELL	$\hat{\alpha}$ =4.74139 $\hat{\theta}$ =2.3875	23.36116	23.93259	25.71727	0.69259	0.10207	0.12760	0.78366
TLL	$\hat{\alpha}$ =−1.0000 $\hat{\theta}$ =4.69324	24.06687	24.6383	26.42298	0.66697	0.09765	0.19020	0.30974
LL	$\hat{\alpha}$ =4.54416	35.16973	35.35155	36.34778	0.57975	0.08271	0.40865	0.00038

Table 6. Parameter estimation and GoF statistics for Data Set 2.

	$\hat{\alpha}$	$\hat{\theta}$	A	W	K-S	p-value
MLE	0.00986	1.92242	0.41285	0.06068	0.08325	0.96426
ADE	0.01111	1.84339	0.39483	0.05768	0.07952	0.97672
CVME	0.01092	1.81031	0.38786	0.05651	0.08535	0.95567
MPSE	0.03788	1.67987	0.35424	0.05125	0.13219	0.55536
OLSE	0.01366	1.75202	0.37382	0.05423	0.09175	0.92233
WLSE	0.01206	1.84000	0.39338	0.05746	0.07874	0.97889

Table 7. GoF statistics for SMPLLD and competitive models for Data set 2.

Model	MLE	AIC	AICC	BIC	A	W	K-S	p-value
SMPLLD	$\hat{\alpha}$ =0.00986 $\hat{\theta}$ =1.92242	152.7396	153.1032	155.1032	0.41285	0.06068	0.08325	0.96426
ELLD	$\hat{\alpha}$ =1.70223 $\hat{\theta}$ =3.41963	153.4730	153.8366	156.6400	0.51692	0.07974	0.09441	0.90531
TLLD	$\hat{\alpha}$ =−1.0000 $\hat{\theta}$ =1.54030	161.5184	161.8821	164.6855	0.42320	0.06377	0.24820	0.02369
LLD	$\hat{\theta}$ =1.40349	186.843	186.9606	188.4265	0.4712	0.06195	0.42803	0.00037

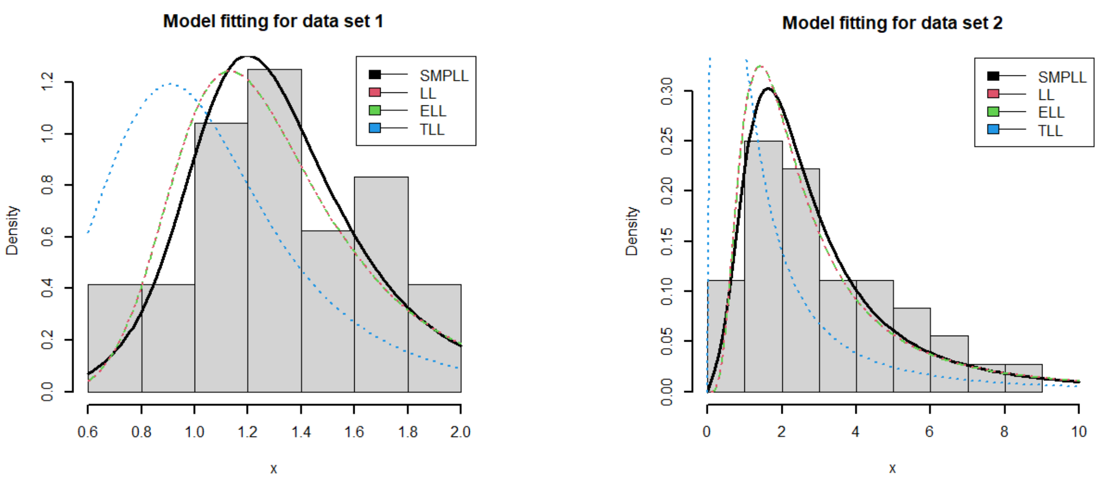

Figure 3. Density plots of SMPLLD & other comparative models for data sets.

Data application results. The results in Tables 4-7 show that the SMPLLD outperforms other comparative models by achieving the lowest values of AIC, BIC, AICC, Cramer-von Mises (W), Anderson–Darling (A) and Kolmogorov-Smirnov (K-S) statistics, along with the largest associated p-values. This indicates a superior fit to the data compared to alternative models. Additionally, histograms confirm that the SMPLLD aligns closely with the actual data distributions. Therefore, the SMPLLD is identified as the optimal choice for fitting the data sets under consideration.

8. CONCLUSIONS

In this paper, we put forward a new generalization of the Log-logistic distribution using the SMP technique and referred to it as the SMPLog-logistic distribution (SMPLLD). The statistical and reliability characteristics of the proposed distribution were extensively analyzed, revealing its ability to exhibit diverse probability density function shapes, including decreasing, constant, and increasing-decreasing trends.

The hazard function demonstrates varying forms such as constant, decreasing, increasing-decreasing, and bathtub shapes, enhancing its flexibility for modeling data with diverse failure rates. Parameters were estimated using multiple estimation method and a simulation study showed that the maximum product of spacing estimation method outperformed others.

The study also confirmed that larger sample sizes reduce bias and mean square error across all methods. To highlight the practical use of the SMPLLD, two real-life datasets were analyzed, where the SMPLLD model demonstrated superior performance compared to competing models. These findings highlight the potential applicability of the SMPLLD for researchers and practitioners in various fields.

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