ORIGINAL PAPER

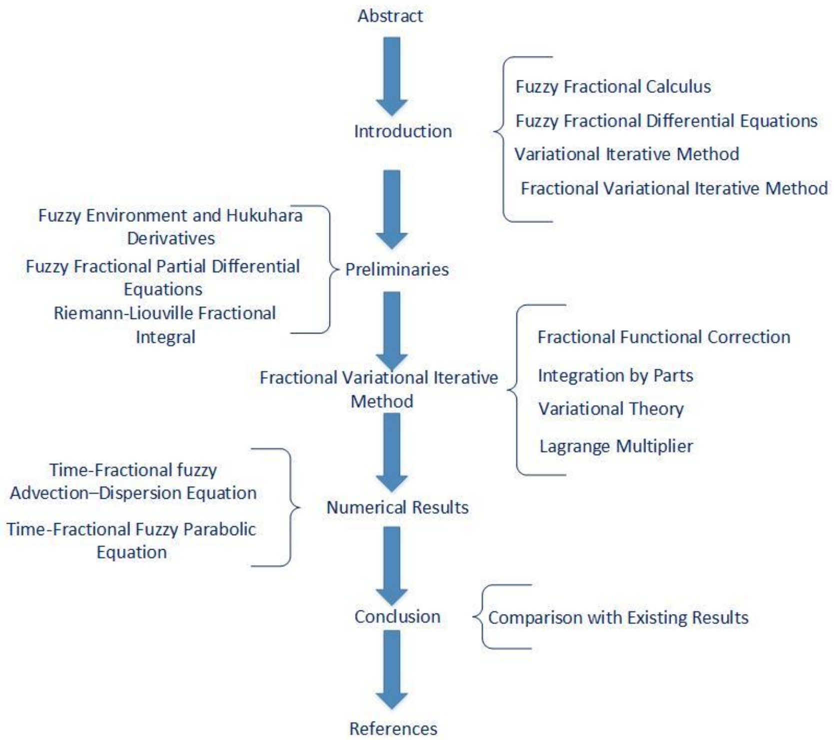
SEMI-ANALYTICAL ANALYSIS OF ADVECTION DISPERSION AND PARABOLIC TIME-FRACTIONAL FUZZY MODELS BY USING THE FRACTIONAL VARIATIONAL ITERATIVE METHOD

NAJAM UL BASAT[1], MAHMOONA ASGHAR[1]

Manuscript received: 01.04.2025; Accepted paper: 11.08.2025;
Published online: 30.05.2025.

Abstract. *Uncertainty in parameters is handled through fuzzy differential models, while their solution can elaborate on their stability. Iterative methods are powerful tools to express the approximate solution up to a finite order of iteration; therefore, here we are analyzing two renowned time-fractional fuzzy differential equation models, advection dispersion and parabolic, through the fractional variational iterative method (FVIM). In the meanwhile Riemann-Liouville fractional integral is used to evaluate the time-fractional order derivative. Exact solution of advection dispersion equation in the form of Mittag-Leffler function for integer order is obtained, and other approximate solutions for non-integer derivatives are also discussed, while the exact solution of parabolic equation is also obtained in the form of Mittag-Leffler function. All numerical results for both time-fractional fuzzy partial differential equation models are discussed and compared with existing solutions. Moreover, numerical solutions of non-integer derivatives are also obtained.*

Keywords: *Riemann-Liouville fractional integral; fractional variational iterative method; time-fractional fuzzy parabolic equation; time-fractional fuzzy advection-dispersion equation.*



Graphical abstract

[1] Grand Asian University Sialkot, Department of Mathematics and Statistics, 51310 Sialkot, Pakistan.
E-mail: najam.basat@gaus.edu.pk; moon.sky65@yahoo.com.

1. INTRODUCTION

Fuzzy set theories, fuzzy fractional calculus, and fuzzy fractional differential equations play a vital role in analyzing the uncertainty parameters. Recent publications have been investigated based on Caputo derivative [1,2], interval Atangana-Baleanu fractional derivative approach [3], Riemann-Liouville generalized fractional integral [4], Caputo Atangana–Baleanu derivatives [5], Caputo gH-fractional differentiability [6], Caputo Hadamard derivatives [7], Modified fractional Euler method [8], and many others [9-13]. Similarly, recent application of fuzzy fractional calculus have been presented in the form of: generalized nonlinear Schr¨odinger equation with local fractional derivative [14], local fractional Gardner's equation [15], the transmission of infectious diseases in a prey-predator system [16,17], edge detecting techniques based on fractional derivatives [18], modeling of hand-foot-mouth disease [19], interaction of tumor growth and the immune system [20], and fractional preypredator model [21], fractional uncertainty modeling [22].

A system of fuzzy fractional differential equations (FFDEs) is numerically investigated through lateral type Hukuhara derivative and generalized Hukuhara derivative [23]. N.V. Hoa introduced FDEs under the influence of Caputo generalized Hukuhara differentiability [24]. Long et al. [25] developed the idea of fuzzy fractional integral and Caputo gH-partial for fuzzy valued multivariable functions. Issa et al. [26] checked the validity and applicability of the Adomian decomposition method, modified Adomian decomposition method, variational iteration method, and homotopy perturbation method by investigating the fuzzy integro-differential equations. Caputo-type fractional order nonlinear differential equation with an advanced argument is solved by Asaduzzaman et al [27]. Canedo and Verdegay used the Lexicographic Method for diet and time–cost trade-off problems in fuzzy systems [28]. Runge-Kutta and biogeography-based optimization is implemented on fuzzy differential equations [29]. Osman et al. [30] presented fuzzy Adomian decomposition and fuzzy modified Laplace decomposition methods to investigate fuzzy fractional Navier–Stokes equations by fractional derivatives. In [31], Senol et al solved the fuzzy fractional partial differential equations with generalized Hukuhara derivative by using a perturbation iterative algorithm. There is also a lot of work in the literature on the solution of fuzzy differential equations and fuzzy partial differential equations [32-42].

In the early stage of the 21st century, a new iterative technique was introduced called the variational iteration method (VIM) to obtain the solution of autonomous differential equations by Ji-Huan He [43-48]. He imposed the idea of the general Lagrange multiplier, which is obtained by variational theory. The variational iterative method provides an approximate analytical solution to the exact solution, while the fractional variational iterative method (FVIM) with Riemann-Liouville derivative was proposed in [49], which also provides an approximate analytical solution. But it was a new trend in fractional calculus to obtain the fractional order differential equations; therefore, authors Wu and Lee implemented FVIM on time and space fractional order diffusion differential equations. Furthermore author [50] examined the possible implementation of FVIM is a fractal multiscale method. Recent work on FVIM can easily be obtained from the references [51-60].

In this paper, the authors would like to extend the applications of FVIM through fractional fuzzy partial differential equations and obtain the exact solutions in the form of the Mittag-Leffler function. These equations are the time-fractional fuzzy advection dispersion equation (T-FFADE) [61-64] and time-fractional fuzzy parabolic equation (T-FFPE) [65-67]. We found that FVIM methodology is still not applied to the time-fractional fuzzy partial differential equations, and it is also notable that semi-analytical analysis is still not investigated in this field, so it inspired us to utilize FVIM on time-fractional fuzzy partial differential equations.

This paper is partitioned into the following sections: some basic and important preliminaries about fuzzy theory are imported into section 2, detail discussion has been made on FVIM in the section 3, numerical results and discussion on the solution of our two investigated problems are mentioned in section 4, and final conclusion by comparing our results and other literature results has been made in section 5.

2. PRELIMINARIES

2.1. FUZZY CALCULUS

In this section, we are presenting some definitions that are useful to understand the definition of Hukuhara derivatives.

Definition 2.1.1. A fuzzy environment depends on fuzzy numbers, which are completely obtained by a pair $y(x) = \left(\bar{y}(x), \underline{y}(x)\right)$, where $\bar{y}, \underline{y}: [0,1] \to \mathbb{R}$ fulfil the following conditions [68]:
i. $\bar{y}$ is a decreasing left and right-hand continuous function for $\xi = 0$.
ii. $\underline{y}$ is an increasing left and right-hand continuous function for $\xi = 0$.
iii. $\forall \xi \in (0,1]: \overline{y} \leq \underline{y}$.

Definition 2.1.2. A function $s: (a, b) \to \acute{E}$ is called Hukuhara differentiable at $\hat{\eta} \in (a, b)$ if, for $h > 0$ approximately small, there exist the H-differences $s(\hat{\eta} + h) - s(\hat{\eta}), s(\hat{\eta}) - s(\hat{\eta} - h)$ and an element $\acute{s}(\hat{\eta}) \in \acute{E}$ such that

$$\lim_{h \to 0^+} \left(\frac{s(\hat{\eta} + h) - s(\hat{\eta})}{h}, \acute{s}(\hat{\eta}) = \lim_{h \to 0^+} \left(\frac{s(\hat{\eta}) - s(\hat{\eta} - h)}{h}, \acute{s}(\hat{\eta}) \right) \right) = 0.$$

Then $\acute{s}(\hat{\eta})$ is called the fuzzy derivative of s at $\hat{\eta}$.

Definition 2.1.3. A fuzzy number y is a fuzzy subset of the real line with a normal, convex, and upper semi-continuous membership function of bounded support [68].

Definition 2.1.4. Let I be a real interval. A mapping $y: I \to E$ is called fuzzy process and denoted by the set

$$[y(t)]_\beta = \left[\bar{y}(t, \rho), \underline{y}(t, \rho)\right].$$

Definition 2.1.5. (α-Level Set). Let M be a member function and be defined as $M: X \to [0,1]$, where X is a fuzzy subset of $X \times [0,1]$ and it is further defined as $\{(a, M(a)): a \in [0,1]\}$. Then the α-level set can be defined as

$$[M]_\alpha = \begin{cases} \{a \in \mathbb{R}: M(a) \geq \alpha\} \in E_i \\ [M_\alpha^-, M_\alpha^+] \; for \; each \; \alpha \in [0,1] \end{cases}.$$

According to Kaleva [69], the membership function is bounded, monotonically increasing, and

i) left continuous $= \begin{cases} M^-(\alpha)\ for\ [0,1) \\ M^+(\alpha)\ for\ (0,1] \end{cases}$
ii) right continuous $= \begin{cases} M^-(\alpha)\ for\ \alpha = 0 \\ M^+(\alpha)\ for\ \alpha = 0 \end{cases}$
iii) $M^-(\alpha) \leq M^+(\alpha)$.

Definition 2.1.6. (Fuzzy Function). The α-level set for fuzzy function, $F_z: G \to E$, can also be defined as $F_z(g,\alpha) = [F_z^-, F_z^+]$ for all $\alpha \in [0,1]$. Fuzzy functions also have a domain and range, so mapping of E to E is a fuzzy function [35].

Definition 2.1.7. (Generalized Hukuhara difference). The generalized Hukuhara difference (gh-difference) is evaluated at two fuzzy numbers, say E_1, E_2, and denoted by $E_1 \ominus_{gH} E_2$ and defined as

$$E_1 \ominus_{gH} E_2 = E_3 \Leftrightarrow \begin{cases} E_1 = E_2 + E_3 \\ or \\ E_2 = E_2(1) + (-1)E_3, \end{cases}$$

where, E_1, E_2 , and $E_3 \in E_i$.

If $g = [0, a_0] \times [0, b_0] \subset \mathbb{R}_2^2$ then a mapping from g to E is a fuzzy function at (a_0, b_0) [70].

Definition 2.1.8. (Generalized Hukuhara Partial Derivative). Generalized Hukuhara partial derivatives (gH-partial derivative) are defined as [71]

$$F_x(x_0,y_0) = \lim_{a\to 0} \frac{F(x_0 + a, y_0) \ominus_{gH} F(x_0,y_0)}{a},$$
$$F_y(x_0,y_0) = \lim_{b\to 0} \frac{F(x_0, y_0 + b) \ominus_{gH} F(x_0,y_0)}{b}.$$

Definition 2.1.9 (Generalized Hukuhara Differentiability). The gH-differentiability with respect x and y for the fuzzy function F can be defined in two types [72]:

Type-I.

$$[F_x(x_0,y_0)]^\alpha = \left[\frac{\partial F_\alpha^-}{\partial x}, \frac{\partial F_\alpha^+}{\partial x}\right] \forall \alpha \in [0,1],$$

Type-II.

$$\left[F_y(x_0,y_0)\right]^\alpha = \left[\frac{\partial F_\alpha^-}{\partial y}, \frac{\partial F_\alpha^+}{\partial y}\right] \forall \alpha \in [0,1].$$

2.2. FUZZY FRACTIONAL CALCULUS

Definition 2.2.1. Let $f \in C_{\mathbb{R}}$ be a real valued space, then the left-sided Riemann-Liouville fractional integral is defined on f of $\alpha \geq 0$ order in such a way

$$J_{t_0}^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{\alpha} (t-\xi)^{\alpha-1} f(\xi,t) d\xi. \quad (1)$$

Definition 2.2.2. The modified definition of the Riemann-Liouville definition is called Caputo's fractional derivative or modified Riemann-Liouville derivative, and it is defined as

$$D_\alpha^x f(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_0^x (t-\xi)^{m-\alpha}\big(f(t)-f(0)\big)d\xi, \tag{2}$$

where, $0 \le x \le 1, \mathrm{m}-1 \le \alpha \le \mathrm{m}$, and $m \ge 1$.

Definition 2.2.3. Let $G = g(x)$ be a function then α order derivative is defined as

$$dG = g(x)(dx)^\alpha, \qquad x \ge 0, \qquad G(0) = 0, \qquad \alpha \in (0,1). \tag{3}$$

Lemma 2.2.1. Suppose G = g(x) is a continuous function, then its integral w.r.t $(dx)^\alpha$ is the solution of the fractional differential equation, Equation (3), is written as

$$G = \int_0^x g(\xi)\,(d\xi)^\alpha = \alpha\int_0^x (x-\xi)^{\alpha-1} g(\xi)d\xi, \qquad \alpha \in (0,1] \tag{4}$$

Similarly, the integration by parts formula can be written as

$$\int_a^b \big[f^{(\alpha)}(x)g(x)\big]\,(dx)^\alpha = \alpha!\,[f(x)g(x)]_{x=a}^{x=b} - \int_a^b f(x)g^{(\alpha)}(x)\,(dx)^\alpha. \tag{5}$$

Corollary 2.2.1. Suppose $\alpha \in (0,1]$ and $\beta > 0$ then the fractional derivative can also be defined for a variable t such as

$$D^\alpha t^\alpha = \Gamma[1+\beta]\Gamma^{-1}[1+\beta-\alpha]t^{\beta-\alpha}. \tag{6}$$

Definition 2.2.4 (fuzzy Riemann–Liouville integral). Suppose $F_z(x) \in C[a,b] \cap L[a,b]$ be a fuzzy valued function, then the fuzzy Riemann-Liouville integral of F_z is defined as

$$J_{t_0}^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^\alpha (t-\xi)^{\alpha-1} f(\xi,t)d\xi, \tag{7}$$

where, $C[a,b]$ is a space of all fuzzy-valued functions and $L[a,b]$ is a space of Lebesgue integrable fuzzy valued functions. While $0 < \alpha \le 1$ and $t > 0$.

From Equation (4), we can get

$$\int_0^x (x-\xi)^{\alpha-1} g(\xi)d\xi = \frac{1}{\alpha}\int_0^x g(\xi)(d\xi)^\alpha. \tag{8}$$

A similar type of result can also be obtained as

$$\int_0^t \xi^\beta d\xi^\alpha = \Gamma[1+\alpha]\Gamma[1+\beta]\Gamma^{-1}[1+\alpha+\beta]t^{\alpha+\beta}. \tag{9}$$

Therefore, Equation 7 is modified as

$$J_{t_0}^\alpha F(x,t) = \frac{1}{\Gamma(1+\alpha)}\int_0^t F(\xi,t)(d\xi)^\alpha. \tag{10}$$

Now, according to Definition (2.1.4), $F_z(x,t) = [F_z^-, F_z^+]$ therefore,

$$J_{t_0}^{\alpha} F_z(x,t) = \left[\frac{1}{\Gamma(1+\alpha)}\int_0^t F_z^-(\xi,t)(d\xi)^\alpha, \frac{1}{\Gamma(1+\alpha)}\int_0^t F_z^+(\xi,t)(d\xi)^\alpha\right]. \tag{11}$$

Definition 2.2.5 (Fuzzy Caputo's gH-differentiable).

Suppose $F_z(x) \in C[a,b] \cap L[a,b]$ be a fuzzy valued function, then fuzzy Caputo's gH-differentiable of F_z is defined as

$$D_{x_0}^{\alpha} F_z(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_{x_0}^{\alpha} (t-\xi)^{-\alpha} F_z(\xi,t)d\xi, \qquad \alpha \in (0,1]. \tag{12}$$

By using a similar definition, $F_z(x,t) = [F_z^-, F_z^+]$, the $D_{x_0}^{\alpha} F_z(x,t)$ is defined as

$$D_{x_0}^{\alpha} F_z(x,t) = \left[\frac{1}{\Gamma(1-\alpha)}\int_{x_0}^{x} (x-\xi)^{-\alpha} F_z^-(\xi,t)d\xi, \frac{1}{\Gamma(1-\alpha)}\int_{x_0}^{x} (x-\xi)^{-\alpha} F_z^+(\xi,t)d\xi\right] \tag{13}$$

3. METHOD ANALYSIS

Here we are elaborating the fractional variational iterative method (FVIM) for α order of fuzzy function w.r.t. time. Suppose we have a time-fractional differential equation,

$$D_t^{\mu} y(x,t) + P(x)y(x,t) + Q(x) = 0, \tag{14}$$

then, according to FVIM, the correction function can be defined as

$$y_{i+1}(x,t) = y_i(x,t) + J^{\mu}\left[\lambda(x,\xi)\left(\frac{\partial^{\mu} y_i}{\partial \xi^{\mu}} + P(x)y_i(x,\xi) + Q(x)\right)\right], \tag{15}$$

from Equation (10)

$$y_{i+1}(x,t) = y_i(x,t) + \frac{1}{\Gamma(1+\mu)}\int_0^t \left[\lambda(x,\xi)\left(\frac{\partial^{\mu} y_i}{\partial \xi^{\mu}} + P(x)y_i(x,\xi) + Q(x)\right)\right](d\xi)^{\mu}, \tag{16}$$

where $\lambda(x,\xi)$ is general Lagrange multiplier which can be evaluated by variational theory. Therefore Equation 16 is modified under the variation δ and restricted variation $\delta\tilde{y} = 0$:

$$\delta y_{i+1}(x,t) = \delta y_i(x,t) + \frac{1}{\Gamma(1+\mu)}\int_0^t \lambda(x,\xi)\delta {y_i}^{(\mu)}(d\xi)^{\mu},$$

by using Equation (5)

$$= (1+\lambda(x,t))\delta y_i(x,t) - \frac{1}{\Gamma(1+\mu)}\int_0^t \lambda^{(\mu)}(x,\xi)\delta y_i(d\xi)^{\mu}.$$

Now comparing the i^{th} coefficient of δy_i and we can get

$$1 + \lambda(x,t) = 0. \tag{17}$$

For $\lambda(x,t) = -1$, Equation (16) in finalized form is

$$y_{i+1}(x,t) = y_i(x,t) - \frac{1}{\Gamma(1+\mu)}\int_{t_0}^{t}\left[\frac{\partial^\mu y_i}{\partial \xi^\mu} - P(x)y_i(x,\xi) - Q(x)\right](d\xi)^\mu. \tag{18}$$

Thus, the final fuzzy fractional variational iteration formula can be written as

$$F_{z\ i+1}^{-}(x,t) = F_{z\ i}^{-}(x,t) - \frac{1}{\Gamma(1+\mu)}\int_{t_0}^{t}\left[\frac{\partial^\mu F_{z\ i}^{-}}{\partial \xi^\mu} - P(x)F_{z\ i}^{-}(x,\xi) - Q(x)\right](d\xi)^\mu, \tag{19}$$

$$F_{z\ i+1}^{+}(x,t) = F_{z\ i}^{+}(x,t) - \frac{1}{\Gamma(1+\mu)}\int_{0}^{t}\left[\frac{\partial^\mu F_{z\ i}^{+}}{\partial \xi^\mu} - P(x)F_{z\ i}^{+}(x,\xi) - Q(x)\right](d\xi)^\mu. \tag{20}$$

4. RESULTS AND DISCUSSION

4.1. TIME-FRACTIONAL FUZZY ADVECTION DISPERSION EQUATION

Consider a time-fractional fuzzy advection dispersion equation (T-FFADE)

$$\frac{\partial^\mu g(x,t)}{\partial t^\mu} + \frac{\partial g(x,t)}{\partial x} = \tau \frac{\partial^2 g(x,t)}{\partial x^2}$$
$$x \in \mathbb{R}, \quad t > 0, \quad 0 < x < 1, \quad 0 < \mu \leq 1, \tag{21}$$

With initial conditions

$$g(x,0) = re^{-x},$$

where r is α-level dependent fuzzy number.

As g is a fuzzy function; therefore initial condition can be defined as,

$$g(x,0) = [g^-(x,0), g^+(x,0)] = [r^-e^{-x}, r^+e^{-x}], \tag{22}$$

Now, according to Equation (16), the lower fuzzy correction function for Equation (21) can be written as

$$g_{i+1}^-(x,t) = g_i^-(x,t) + \frac{1}{\Gamma(1+\mu)}\int_0^t \lambda\left[\frac{\partial^\mu g_i^-(x,t)}{\partial t^\mu} + \frac{\partial g_i^-(x,t)}{\partial x} - \tau\frac{\partial^2 g_i^-(x,t)}{\partial x^2}\right](d\xi)^\mu.$$

By applying the variational parameter δ on both sides of above equation and $\delta \tilde{g}^- = 0$, authors can set $\lambda = -1$

$$g_{i+1}^-(x,t) = g_i^-(x,t) - \frac{1}{\Gamma(1+\mu)}\int_0^t \left[\frac{\partial^\mu g_i^-(x,t)}{\partial t^\mu} + \frac{\partial g_i^-(x,t)}{\partial x} - \tau\frac{\partial^2 g_i^-(x,t)}{\partial x^2}\right](d\xi)^\mu. \tag{23}$$

Consider the initial condition as the zeroth iteration $g^-(x,0) = g_0^- = r^-e^{-x}$ therefore, further iterated results are obtained as:
$i = 0$:

$$g_1^-(x,t) = g_0^-(x,t) - \frac{1}{\Gamma(1+\mu)}\int_0^t \left[\frac{\partial^\mu g_0^-(x,t)}{\partial t^\mu} + \frac{\partial g_0^-(x,t)}{\partial x} - \tau\frac{\partial^2 g_0^-(x,t)}{\partial x^2}\right](d\xi)^\mu,$$

by using

$$\frac{\partial^\mu g_0^-}{\partial \xi^\mu} = 0, \qquad \frac{\partial g_0^-}{\partial x} = -r^- e^{-x}, \qquad \frac{\partial^2 g_0^-}{\partial x^2} = r^- e^{-x}$$

and get

$$g_1^-(x,t) = r^- e^{-x} + \frac{(1+\tau) r^- e^{-x}}{\Gamma(1+\mu)} t^\mu. \tag{21}$$

$i = 1$:

$$g_2^-(x,t) = g_1^-(x,t) - \frac{1}{\Gamma(1+\mu)} \int_0^t \left[\frac{\partial^\mu g_1^-(x,t)}{\partial t^\mu} + \frac{\partial g_1^-(x,t)}{\partial x} - \tau \frac{\partial^2 g_1^-(x,t)}{\partial x^2} \right] (d\xi)^\mu,$$

by using

$$\frac{\partial^\mu g_1^-}{\partial \xi^\mu} = (1+\tau) r^- e^{-x}, \qquad \frac{\partial g_1^-}{\partial x} = -r^- e^{-x} - \frac{(1+\tau) r^- e^{-x}}{\Gamma(1+\mu)} \xi^\mu,$$

$$\frac{\partial^2 g_1^-}{\partial x^2} = r^- e^{-x} + \frac{(1+\tau) r^- e^{-x}}{\Gamma(1+\mu)} \xi^\mu,$$

and get

$$g_2^-(x,t) = r^- e^{-x} + \frac{(1+\tau) r^- e^{-x}}{\Gamma(1+\mu)} t^\mu + \frac{(1+\tau)^2 r^- e^{-x}}{\Gamma(1+2\mu)} t^{2\mu}. \tag{25}$$

Similarly for the k^{th} iteration:

$$g_k^-(x,t) = r^- e^{-x} \sum_{j=0}^{k} \frac{(1+\tau)^j}{\Gamma(1+j\mu)} t^{j\mu} = r^- e^{-x} E_\mu[(1+\tau) t^\mu], \tag{26}$$

where $E_\mu[(1+\tau)t^\mu]$ is the Mittag-Leffler function. Now, the approximate lower fuzzy function solution can be written as

$$g^-(x,t) = r^- e^{-x} + \frac{(1+\tau) r^- e^{-x}}{\Gamma(1+\mu)} t^\mu + \frac{(1+\tau)^2 r^- e^{-x}}{\Gamma(1+2\mu)} t^{2\mu} + \cdots \tag{27}$$

and the exact solution of the lower fuzzy function can be obtained as

$$g^-(x,t) = \lim_{k\to\infty} g_k^-(x,t) = r^- e^{-x} E_\mu[(1+\tau) t^\mu]. \tag{28}$$

Similarly,

$$g_k^+(x,t) = r^+ e^{-x} \sum_{j=0}^{k} \frac{(1+\tau)^j}{\Gamma(1+j\mu)} t^{j\mu} = r^+ e^{-x} E_\mu[(1+\tau) t^\mu], \tag{29}$$

and the exact solution of the upper fuzzy function

$$g^+(x,t) = \lim_{k\to\infty} g_k^+(x,t) = r^+ e^{-x} E_\mu[(1+\tau) t^\mu]. \tag{30}$$

From Equation (26), we can obtain the required result up to any order by using computational software. Lower and upper fuzzy values are tabulated in Table 1, where the author has made a comparison for $\mu = 1$ with the exact solution obtained by Mehmet Senol and his team [61]. We used Mathematica 11.0 to obtain further results up to 5^{th} iteration and then compare the FVIM result with the exact solution, and then zero absolute error is obtained. While other FVIM results against different orders of μ are also presented in the table. There are also graphical results based on Table 1 that are plotted in Figs. 1-4. While

Table 2 is based on the base error analysis for different values of μ. We can easily investigate from the error table that as the error is getting lower for the increment in μ values, which shows that our results are getting aligned to the exact solution for the $\mu = 1$. While fractional values of μ produce more approximation complexity, which leads to increasing error, therefore these fractional values era generate the connection between numerical observation and theoretical expectations.

Table 1. Numerical results of T-FFADE via fractional variational iterative method and comparison for μ=1 with exact solution [61].

Fuzzy Lower Function: $g^{-}(0.5,0.25)$						
α	μ				Comparison for $\mu = 1$	
	0.25	0.5	0.75	1	Exact	Absolute Error
0	1.74531	0.961552	0.711654	0.598887	0.598887	0.00E+00
0.2	1.86167	1.025660	0.759098	0.638813	0.638813	0.00E+00
0.4	1.97802	1.089760	0.806541	0.678739	0.678739	0.00E+00
0.6	2.09438	1.153860	0.853985	0.718665	0.718665	0.00E+00
0.8	2.21073	1.217970	0.901429	0.758590	0.758590	0.00E+00
1	2.32709	1.282070	0.948872	0.798516	0.798516	0.00E+00
Fuzzy Upper Function: $g^{+}(0.5,0.25)$						
α	μ				Comparison for $\mu = 1$	
	0.25	0.5	0.75	1	Exact	Absolute Error
0	2.90886	1.60259	1.186090	0.998145	0.998145	0.00E+00
0.2	2.79250	1.53848	1.138650	0.958219	0.958219	0.00E+00
0.4	2.67615	1.47438	1.091200	0.918294	0.918294	0.00E+00
0.6	2.55979	1.41028	1.043760	0.878368	0.878368	0.00E+00
0.8	2.44344	1.34617	0.996316	0.838442	0.838442	0.00E+00
1	2.32709	1.28207	0.948872	0.798516	0.798516	0.00E+00

Table 2. Absolute error results of T-FFADE via the fractional variational iterative method with different values of μ.

Fuzzy Lower Function: $y^{-}(0.5,0.25)$				
α	μ			
	0.25	0.5	0.75	1
0	7.60E-01	1.18E-01	9.10E-03	0.00E+00
0.2	8.77E-01	1.82E-01	5.65E-02	0.00E+00
0.4	9.93E-01	2.47E-01	1.04E-01	0.00E+00
0.6	1.11E+00	3.11E-01	1.51E-01	0.00E+00
0.8	1.23E+00	3.75E-01	1.99E-01	0.00E+00
1	1.34E+00	4.39E-01	2.46E-01	0.00E+00
Fuzzy Upper Function: $y^{+}(0.5,0.25)$				
α	μ			
	0.25	0.5	0.75	1
0	1.92E+00	7.59E-01	4.84E-01	0.00E+00
0.2	1.81E+00	6.95E-01	4.36E-01	0.00E+00
0.4	1.69E+00	6.31E-01	3.89E-01	0.00E+00
0.6	1.57E+00	5.67E-01	3.41E-01	0.00E+00
0.8	1.46E+00	5.03E-01	2.94E-01	0.00E+00
1	1.34E+00	4.39E-01	2.46E-01	0.00E+00

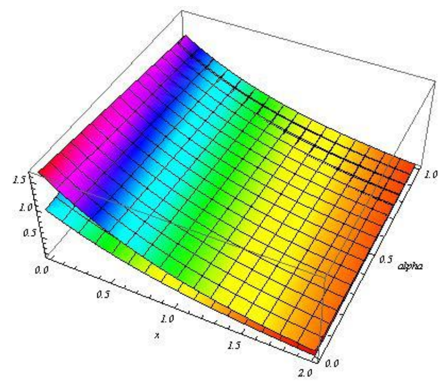


Figure 1. T-FFADE: FVIM results of lower and upper functions at $\mu = 1$, x= 0.5, $t = 0.25$

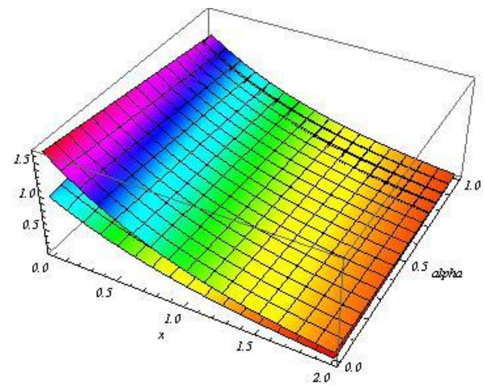


Figure 2. T-FFADE: Exact solution of lower and upper functions at $\mu = 1$, x= 0.5, $t = 0.25$

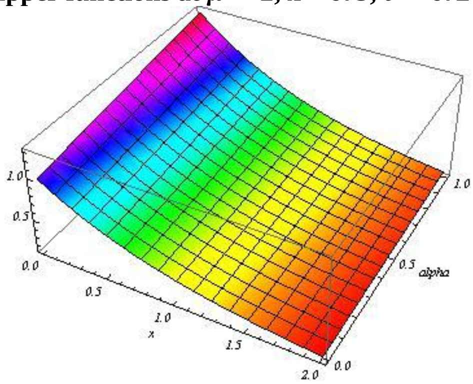


Figure 3. T-FFADE: FVIM results of lower functions at $\mu = 1$, x= 0.5, $t = 0.25$

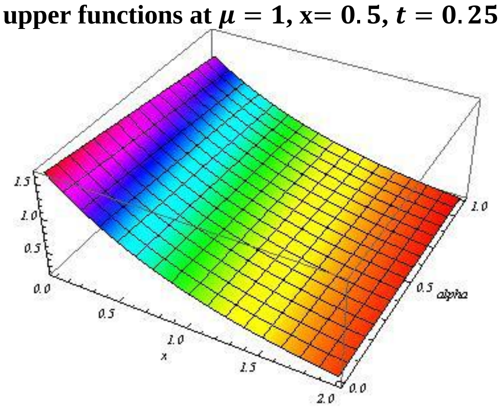


Figure 4. T-FFADE: FVIM results of upper functions at $\mu = 1$, x= 0.5, $t = 0.25$

4.2. TIME-FRACTIONAL FUZZY PARABOLIC EQUATION

Consider a time-fractional fuzzy parabolic equation (T-FFPE)

$$\frac{\partial^\mu y(x,t)}{\partial t^\mu} = \frac{x^2}{2}\frac{\partial^2 y(x,t)}{\partial x^2}, \qquad x \in \mathbb{R}, \quad t > 0, \quad 0 < x < 1, \quad 0 < \mu \le 1, \tag{30}$$

with initial conditions

$$y(x,0) = hx^2, \tag{32}$$

where h is α-level dependent fuzzy number.

As y is a fuzzy function; therefore, the initial condition can be defined as,

$$y(x,0) = [y^-(x,0), y^+(x,0)] = [h^-x^2, h^+x^2]. \tag{33}$$

For $\mu = 1$, exact solution of Equation (31) has been presented in literature [66] and can be written as

$$y^-(x,t;\alpha) = h^-x^2e^t,$$
$$y^+(x,t;\alpha) = h^+x^2e^t,$$

where $h[\alpha] = [h^-, h^+] = [\alpha - 1, 1 - \alpha]$.

Now, according to Equation (16), the lower fuzzy correction function for Equation (31) can be written as

$$y_{i+1}^{-}(x,t)=y_i^{-}(x,t)+\frac{1}{\Gamma(1+\mu)}\int_0^t \lambda\left[\frac{\partial^\mu y_i^{-}(x,\xi)}{\partial t^\mu}-\frac{x^2}{2}\frac{\partial^2 y_i^{-}(x,\xi)}{\partial x^2}\right](d\xi)^\mu.$$

Similarly, the author can set $\lambda=-1$,

$$y_{i+1}^{-}(x,t)=y_i^{-}(x,t)-\frac{1}{\Gamma(1+\mu)}\int_0^t \left[\frac{\partial^\mu y_i^{-}(x,\xi)}{\partial t^\mu}-\frac{x^2}{2}\frac{\partial^2 y_i^{-}(x,\xi)}{\partial x^2}\right](d\xi)^\mu. \tag{34}$$

Similarly, the upper fuzzy function is expressed as

$$y_{i+1}^{+}(x,t)=y_i^{+}(x,t)-\frac{1}{\Gamma(1+\mu)}\int_0^t \left[\frac{\partial^\mu y_i^{+}(x,\xi)}{\partial t^\mu}-\frac{x^2}{2}\frac{\partial^2 y_i^{+}(x,\xi)}{\partial x^2}\right](d\xi)^\mu. \tag{35}$$

Consider the initial condition as the zeroth iteration $y^{-}(x,0)=y_0^{-}=h^{-}x^2$ therefore, further iterated results are obtained as:
$i=0$:

$$y_1^{-}(x,t)=y_0^{-}(x,t)-\frac{1}{\Gamma(1+\mu)}\int_0^t \left[\frac{\partial^\mu y_0^{-}(x,\xi)}{\partial t^\mu}-\frac{x^2}{2}\frac{\partial^2 y_0^{-}(x,\xi)}{\partial x^2}\right](d\xi)^\mu,$$

by using

$$\frac{\partial^\mu y_0^{-}}{\partial \xi^\mu}=0,\quad \frac{\partial^2 y_0^{-}}{\partial x^2}=2h^{-}$$

and get

$$y_1^{-}(x,t)=h^{-}x^2+\frac{h^{-}x^2}{\Gamma(1+\mu)}t^\mu. \tag{36}$$

$i=1$:

$$y_2^{-}(x,t)=y_1^{-}(x,t)-\frac{1}{\Gamma(1+\mu)}\int_0^t \left[\frac{\partial^\mu y_1^{-}(x,\xi)}{\partial t^\mu}-\frac{x^2}{2}\frac{\partial^2 y_1^{-}(x,\xi)}{\partial x^2}\right](d\xi)^\mu,$$

by using

$$\frac{\partial^\mu y_1^{-}}{\partial \xi^\mu}=h^{-}x^2,\quad \frac{\partial^2 y_1^{-}}{\partial x^2}=2h^{-}+\frac{2h^{-}x^2}{\Gamma(1+\mu)}t^\mu$$

and get

$$y_2^{-}(x,t)=h^{-}x^2+\frac{h^{-}x^2}{\Gamma(1+\mu)}t^\mu+\frac{2h^{-}x^2}{\Gamma(1+2\mu)}t^{2\mu}. \tag{37}$$

Similarly for the k^{th} iteration:

$$y_k^{-}(x,t)=h^{-}x^2\sum_{j=0}^{k}\frac{t^{j\mu}}{\Gamma(1+j\mu)}=h^{-}x^2E_\mu[t^\mu], \tag{38}$$

where $E_\mu[t^\mu]$ is the Mittag-Leffler function. Now, the approximate lower fuzzy function solution can be written as

$$y^{-}(x,t)=h^{-}x^2+\frac{h^{-}x^2}{\Gamma(1+\mu)}t^\mu+\frac{h^{-}x^2}{\Gamma(1+2\mu)}t^{2\mu}+\cdots \tag{39}$$

and the approximate exact solution of the lower fuzzy function can be obtained as

$$y^{-}(x,t)=\lim_{k\to\infty}y_k^{-}(x,t)=h^{-}x^2E_\mu[t^\mu]. \tag{40}$$

Similarly,

$$y_k^+(x,t) = h^+x^2 \sum_{j=0}^{k} \frac{1}{\Gamma(1+j\mu)} t^{j\mu} = h^- x^2 E_\mu[t^\mu], \tag{41}$$

and the approximate exact solution of the upper fuzzy function

$$y^+(x,t) = \lim_{k\to\infty} y_k^+(x,t) = h^+x^2E_\mu[t^\mu]. \tag{42}$$

From Equations (38) and (41), anyone can obtain the required results up to any order by using computational software. Lower and upper fuzzy values are tabulated in Table 2, where the author has made a comparison for $\mu = 1$ with the exact solution obtained by Allahviranloo and Taheri [66]. We used Mathematica 11.0 to obtain further results up to the 5th iteration and then compare the FVIM result with the exact solution, and thence E-07 to E-08 decimal places absolute error obtained. While other FVIM results against different orders of μ are also presented in the table. From the absolute error column, it is easy to see that the error goes down by increasing α. Further graphical illustration of Table 3 in Figs. 5-8 are plotted. While Table 4 is also scattered based on error analysis for different values of μ. We can easily investigate from the error table that as the error is getting lower for the increment in μ values, which shows that our results are getting aligned to the exact solution for the $\mu = 1$. While fractional values of μ produce more approximation complexity, which leads to increasing error, therefore these fractional values era generate the connection between numerical observation and theoretical expectations.

Table 3. Numerical results of T-FFPE via fractional variational iterative method and comparison for $\mu = 1$ with exact solution [66].

Fuzzy Lower Function: $y^-(0.5,0.25)$						
α	μ				Comparison for $\mu = 1$	
	0.25	0.5	0.75	1	Exact	Absolute Error
0	-1.12741	-0.690718	-0.505388	-0.41218	-0.41218	4.1316E-07
0.2	-0.90193	-0.552574	-0.404311	-0.329744	-0.329744	3.3053E-07
0.4	-0.67645	-0.414431	-0.303233	-0.247308	-0.247308	2.479E-07
0.6	-0.45096	-0.276287	-0.202155	-0.164872	-0.164872	1.6526E-07
0.8	-0.22548	-0.138144	-0.101078	-0.082436	-0.082436	8.2632E-08
1	0	0	0	0	0	0
Fuzzy Upper Function: $y^+(0.5,0.25)$						
α	μ				Comparison for $\mu = 1$	
	0.25	0.5	0.75	1	Exact	Absolute Error
0	1.12741	0.690718	0.505388	0.41218	0.41218	4.1316E-07
0.2	0.901927	0.552574	0.404311	0.329744	0.329744	3.3053E-07
0.4	0.676445	0.414431	0.303233	0.247308	0.247308	2.479E-07
0.6	0.450964	0.276287	0.202155	0.164872	0.164872	1.6526E-07
0.8	0.225482	0.138144	0.101078	0.082436	0.082436	8.2632E-08
1	0	0	0	0	0	0

Table 4. Absolute error results of T-FFPE via fractional variational iterative method with different values of μ.

Fuzzy Lower Function: $y^{-}(0.5,0.25)$				
α	μ			
	0.25	0.5	0.75	1
0	3.86E-01	8.24E-02	9.25E-03	4.13E-07
0.2	1.60E-01	5.58E-02	9.18E-02	3.31E-07
0.4	6.51E-02	1.94E-01	1.93E-01	2.48E-07
0.6	2.91E-01	3.32E-01	2.94E-01	1.65E-07
0.8	5.16E-01	4.70E-01	3.95E-01	8.26E-08
1	7.42E-01	6.08E-01	4.96E-01	0.00E+00
Fuzzy Upper Function: $y^{+}(0.5,0.25)$				
α	μ			
	0.25	0.5	0.75	1
0	3.86E-01	8.24E-02	9.25E-03	4.13E-07
0.2	1.60E-01	5.58E-02	9.18E-02	3.31E-07
0.4	6.51E-02	1.94E-01	1.93E-01	2.48E-07
0.6	2.91E-01	3.32E-01	2.94E-01	1.65E-07
0.8	5.16E-01	4.70E-01	3.95E-01	8.26E-08
1	7.42E-01	6.08E-01	4.96E-01	0.00E+00

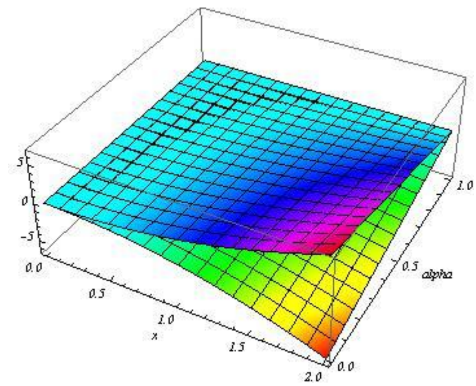


Figure 5. T-FFPE: FVIM results of lower and upper functions at $\mu = 1$, x= 0.5, $t = 0.25$.

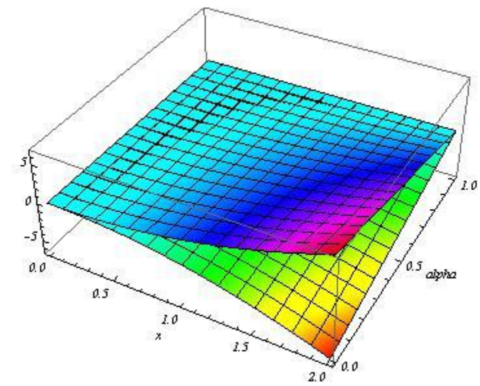


Figure 6. T-FFPE: Exact solution of lower and upper functions at $\mu = 1$, x= 0.5, $t = 0.25$.

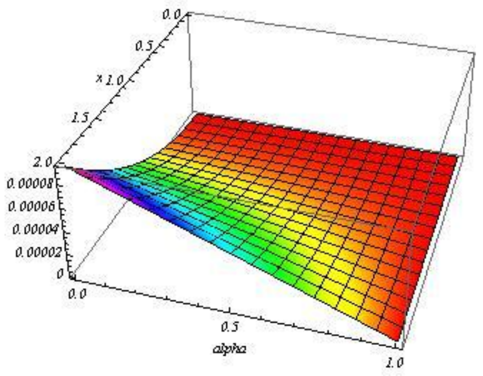


Figure 7. T-FFPE: absolute error of FVIM result and exact solution of lower functions at $\mu = 1$, x= 0.5, $t = 0.25$.

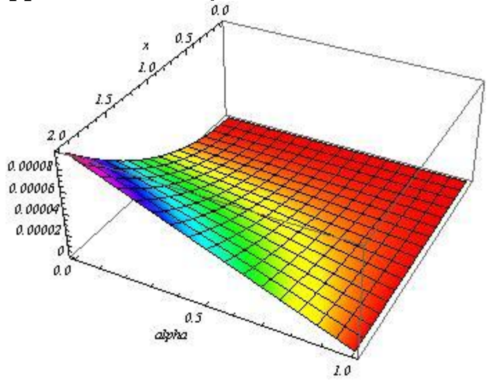


Figure 8. T-FFPE: absolute error of FVIM result and exact solution of upper functions at $\mu = 1$, x= 0.5, $t = 0.25$.

5. CONCLUSIONS

In this research article, we implemented the fractional variational iterative method (FVIM) for fixed values of $x = 0.5$ and t= 0.25 on two well-known fuzzy fractional order partial differential equations and obtained the following important outcomes:

1. We presented FVIM results up to three iterations for both problems in the lower fuzzy function form and then express the lower and upper fuzzy functions into the Mittag-Leffler function form.
2. We obtained the exact solution of the time-fractional fuzzy advection dispersion equation (T-FFADE) for $\mu = 1$ and compared it with [61] and obtained 0.00E+00 absolute error. While for the time-fractional fuzzy parabolic equation (T-FFPE), we compared FVIM results with the exact solution for $\mu = 1$ and obtained E-08 to E-07 decimal places absolute error.
3. Other numerical results for different orders; $\mu = 0.25, 0.5, 0.75$, for both problems are also discussed.

In future, we planned to apply FVIM to approximate the system of time-fractional differential equations.

REFERENCES

[1] Allahviranloo, T., Armand, A., Gouyandeh, Z., *Journal of Intelligent & Fuzzy Systems*, **26**(3), 1481, 2014.
[2] Alinezhad, M., Allahviranloo, T., *Information Sciences*, **421**, 218, 2017.
[3] Allahviranloo, T., Ghanbari, B., *Chaos, Solitons and Fractals*, **130**, 109397, 2020.
[4] An, T.V., Van Hoa, N., *Fuzzy Sets and Systems*, **429**, 74, 2022.
[5] Dong, N. P., Long, H. V., Giang, N. L., *Fuzzy Sets and Systems*, **429**, 28, 2022.
[6] Van Hoa, N., Lupulescu, V., O'Regan, D., *Fuzzy Sets and Systems*, **309**, 1, 2017.
[7] Liu, Y., Zhu, Y., Lu, Z., *Chaos, Solitons & Fractals*, **146**, 110894, 2021.
[8] Mazandarani, M., Kamyad, A.V., *Communications in Nonlinear Science and Numerical Simulation*, **18**(1), 12, 2013.
[9] Agarwal, R. P., Arshad, S., O'Regan, D., Lupulescu, V., *Fractional Calculus and Applied Analysis*, **15**, 572, 2012.
[10] Long, H.V., Kim Son, N.T., Thanh Tam, H.T., Yao, J.C., *Acta Mathematica Vietnamica*, **42**, 675, 2017.
[11] Dai, R., Chen, M., *Fuzzy Sets and Systems*, **400**, 134, 2020.
[12] Mazandarani, M., Najariyan, M., *Communications in Nonlinear Science and Numerical Simulation*, **19**(7), 2354, 2014.
[13] Rivaz, A., Fard, O. S., Bidgoli, T. A., *SeMA Journal*, **73**, 149, 2016.
[14] Ghanbari, B., *Mathematical Methods in the Applied Sciences*, **44**(11), 8759, 2021.
[15] Ghanbari, B., *Mathematical Methods in the Applied Sciences*, **44**(6), 4673, 2021.
[16] Ghanbari, B., *Mathematical Methods in the Applied Sciences*, **46**(7), 8106, 2023.
[17] Ghanbari, B., *Mathematical Methods in the Applied Sciences*, **44**(13), 9998, 2021.
[18] Ghanbari, B., Atangana, A., *Advances in Difference Equations*, **2020**(1), 435, 2020.
[19] Ghanbari, B., *Advances in Difference Equations*, **2020**(1), 536, 2020.
[20] Ghanbari, B., *Advances in Difference Equations*, **2020**(1), 585, 2020.
[21] Ghanbari, B, *Advances in Difference Equations*, **2020**(1), 679, 2020.
[22] Shah, S. M. A., Nie, Y., Alkhazzan, A., Cemil, T., Din, A., *Journal of Taibah University for Science*, **18**(1), 2414523, 2024.
[23] Alijani, Z., Baleanu, D., Shiri, B., Wu, G. C., *Chaos, Solitons and Fractals*, **131**, 109510 2020.
[24] Van Hoa, N., *Communications in Nonlinear Science and Numerical Simulation*, **22**(1-3), 1134, 2015.
[25] Long, H. V., Son, N. T. K., Tam, H. T. T., *Fuzzy Sets and Systems*, **309**, 35, 2017.
[26] Issa, M. B., Hamoud, A., Ghadle, K., *Journal of Mathematics and Computer Science*, **23**(1), 67, 2021.

[27] Asaduzzaman, M., Kilicman, A., Ali, M., *Journal of Mathematics and Computer Science,* **23**(3), 230, 2021.

[28] P´erez-Canedo, B., Verdegay, J. L., *Journal of Computational and Cognitive Engineering,* **2**(1), 47, 2023.

[29] Garg, H., *Journal of Intelligent and Fuzzy Systems,* **30**(4), 2417, 2016.

[30] Osman, M., Xia, Y., Omer, O. A., Hamoud, A., *Mathematics*, **10**(13), 2295, 2022.

[31] Salahshour, S., Allahviranloo, T., Abbasbandy, S., *Communications in Nonlinear Science and Numerical Simulation*, **17**(3), 1372, 2012.

[32] Salah, A., Khan, M., Gondal, M. A., *Neural Computing and Applications*, **23**, 269, 2013.

[33] Sekar, S., Sakthivel, A., *IOSR Journal of Mathematics*, **11**(5), 33, 2015.

[34] Singh, J., Swroop, R., Kumar, D., *Ain Shams Engineering Journal*, **9**(4), 1019, 2018.

[35] Jafari, R., Yu, W., Li, X., *Ieee Access,* **4**, 9428, 2016.

[36] Asghar, M., Ahmad, I., Ilyas, H., Naz, S., ul Basat, N., Raja, M. A. Z., *International Journal of Modern Physics B,* 2350254, 2023.

[37] Sindu Devi, S., Ganesan, K., *Journal of Physics: Conference Series,* **1377** (1), 012024, 2019.

[38] Osman, M., Xia, Y., *Alexandria Engineering Journal*, **69**, 529, 2023.

[39] Shah, S. M. A., Nie, Y., Din, A., Alkhazzan, A., *Mathematics*, **12**(11), 1645, 2024.

[40] Shah, S. M. A., Nie, Y., Din, A., Alkhazzan, A., Younas, B., *Chinese Physics B*, **33**(11), 110203, 2024.

[41] Alkhazzan, A., Wang, J., Nie, Y., Shah, S. M. A., Almutairi, D. K., Khan, H., Alzabut, J., Alexandria Engineering Journal, **118**, 337-353, 2025.

[42] Alkhazzan, A., Loqman, I. G. H., Shah, S. M. A., Alkhazzan, A., *Sana'a University Journals,* **3**(2), 690-697, 2025.

[43] He, J. H., *Applied mathematics and computation,* **114**(2-3), 115, 2000.

[44] He, J. H., *Chaos, Solitons and Fractals*, **19**(4), 847, 2004.

[45] He, J. H., *Chaos, Solitons and Fractals,* **34**(5), 1430, 2007.

[46] He, J. H., *International journal of Modern physics B*, **20**(10), 1141, 2006.

[47] He, J. H., *Journal of computational and applied mathematics*, **207**(1), 3, 2007.

[48] He, J. H., Wu, X. H., *Computers and Mathematics with Applications*, **54**(7-8), 881, 2007.

[49] Wu, G. C., Lee, E. W. M., *Physics Letters A*, **374**(25), 2506, 2010.

[50] Wu, G. C., *Communications in Fractional Calculus*, **2**(2), 59, 2011.

[51] Elbeleze, A. A., Kılıçman, A., Taib, B. M., *Mathematical Problems in Engineering,* **2013**(1), 543848, 2013.

[52] Prakash, A., Goyal, M., Gupta, S., *Nonlinear Engineering,* **8**(1), 164, 2019.

[53] Jafari, H., Jassim, H. K., *Applications and Applied Mathematics: An International Journal (AAM),* **10**(2), 29, 2015.

[54] Hichem, D., Kacem, B., *Communications in Mathematics and Applications*, **12**(2), 241, 2021.

[55] Singh, B. K., Kumar, P., *International Journal of Differential Equations,* **2017**(1), 5206380, 2017.

[56] Si-yuan, B., *Mathematical Problems in Engineering,* **2014**(1), 925016, 2014.

[57] Hong, B., Lu, D., *Scientific World Journal,* **2014**(1), 964643, 2014.

[58] He, J. H., *Physics Letters A,* **375**(38), 3362, 2011.

[59] Nama, A. Y., Fadhel, F. Subhi., *Journal of Physics: Conference Series*, **2322**(1), 012011, 2022.

[60] Monzon, G., *Journal of Fractional Calculus and Applications,* **15**(1), 1-15, 2024.

[61] Senol, M., Atpinar, S., Zararsiz, Z., Salahshour, S., Ahmadian, A., *Computational and Applied Mathematics*, **38**, 1, 2019.
[62] Hashemi, H. Ezzati, R. Mikaeilvand, N. M., Nazari, M., *Iranian Journal of Fuzzy Systems*, **21**(5) 105-119, 2024.
[63] Zureigat, H. Ismail, I., Sathasivam, S., *Mathematical Methods in the Applied Sciences*, **44**(10), 7956-7968, 2021.
[64] Kumar, S., *Numerical Methods for Partial Differential Equations*, **38**(3), 490-508, 2022.
[65] Al-Khateeb, A., *Axioms*, **13**(4) 221, 2024
[66] Allahviranloo, T., Taheri, N., *International Journal of Contemporary Mathematical Sciences*, **4**(3), 105, 2009.
[67] Jameel, A., *International Journal of Applied Physics and Mathematics*, **4**(6), 371, 2014.
[68] Ghazanfari, B., Shakerami, A., *Fuzzy sets and Systems*, **189**(1), 74, 2012.
[69] Kaleva, O., *Fuzzy Sets and Systems*, **24**(3), 301, 1987.
[70] Majumder, P., Mondal, S. P., Bera, U. K., Maiti, M., *Operations Research Perspectives*, **3**, 77, 2016.
[71] Ghaffari, M., Allahviranloo, T., Abbasbandy, S., Azhini, M., *Iranian Journal of Fuzzy Systems*, **18**(3), 51, 2021.
[72] Stefanini, L., Bede, B., *Nonlinear Analysis: Theory, Methods and Applications*, **71**(3-4), 1311, 2009.