ORIGINAL PAPER

AN EXTENSION OF BILATERAL GENERATING FUNCTIONS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE JACOBI POLYNOMIALS

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Abstract. *In this article, the author obtains a novel extension of bilateral generating functions involving biorthogonal polynomials suggested by the classical Jacobi polynomials, $K_n(\alpha,\beta,k;x)$ from the existence of quasi-bilateral generating function with the help of a group-theoretic method. Some special cases of interest are also discussed.*

Keywords: *Jacobi polynomials; Laguerre polynomials; biorthogonal polynomials; quasi-bilateral generating functions.*

1. INTRODUCTION

Over five decades ago, Konhauser ([1]; see also [2]) introduced and studied two interesting classes of polynomials $Y_n^\alpha(x;k)$ and $Z_n^\alpha(x;k)$, where $Y_n^\alpha(x;k)$ is a polynomial in x and $Z_n^\alpha(x;k)$ is a polynomial in x^k, with (in general)

$$\Re(\alpha) > -1 \text{ and } k \in \mathbb{N} := \{1,2,3,\dots.\}.$$

For $k = 1$, each of these polynomials reduces to the classical Laguerre polynomials (cf., e.g., Szegö [3, p. 101, Equations (5.1.6)]):

$$L_n^{(\alpha)}(x) := \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}. \tag{1.1}$$

The following explicit representation for the polynomials $Z_n^\alpha(x;k)$ was given by Konhauser [2, p. 304, Equations (5)]:

$$Z_n^\alpha(x;k) = \frac{\Gamma(\alpha+nk+1)}{n\,!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{x^{kj}}{\Gamma(\alpha+nk+1)}. \tag{1.2}$$

Subsequently, Carlitz [4, p. 427, Equations (9)] pointed out that

$$Y_n^\alpha(x;k) = \frac{1}{n\,!}\sum_{i=0}^{n}\frac{x^i}{i\,!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{\alpha+j+1}{k}\right)_n, \tag{1.3}$$

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where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, if\ n = 0, a \neq 0 \\ a(a+1)\dots(a+n-1), \forall\, n \in \mathbb{N}\,. \end{cases}$$

In 1982[5], Madhekar and Thakare introduced and studied another interesting pair of biorthogonal polynomials $J_n(\alpha,\beta,k;x)$ and $K_n(\alpha,\beta,k;x)$ that are suggested by the classical Jacobi polynomials, where $J_n(\alpha,\beta,k;x)$ and $K_n(\alpha,\beta,k;x)$ are respectively the polynomials of degree n in x^k and x. They gave an explicit representation for the two polynomials $J_n(\alpha,\beta,k;x)$ and $K_n(\alpha,\beta,k;x)$ respectively in the following form:

$$J_n(\alpha,\beta,k;x) = \frac{(1+\alpha)_{kn}}{n\,!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(1+\alpha+\beta+n)_{kj}}{(1+\alpha)_{kj}}\left(\frac{1-x}{2}\right)^{kj} \tag{1.4}$$

and

$$K_n(\alpha,\beta,k;x) = \sum_{r=0}^{n}\sum_{s=0}^{r}(-1)^{r+s}\binom{r}{s}\frac{(1+\beta)_n}{n\,!r!\ (1+\beta)_{n-r}}\left(\frac{s+\alpha+1}{k}\right)_n\left(\frac{x-1}{2}\right)^r\left(\frac{x+1}{2}\right)^{n-r}. \tag{1.5}$$

In [6], the quasi-bilateral generating function for two special functions is defined through the relation

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, p_n^{(\alpha)}(x)\, q_m^{(n)}(u)\, w^n, \tag{1.6}$$

where a_n are the arbitrary coefficients and $p_n^{(\alpha)}(x)$ and $q_m^{(n)}(u)$ are two special functions of order n and m and of parameters α, n respectively.

In a recent paper [7], the present author has proved the following bilateral generating relation:

$$\begin{aligned}&[1+kw(x+1)]^{-\frac{1+\alpha}{k}}(x+1)^{-\beta}\left[2+\frac{x-1}{[1+kw(x+1)]^{\frac{1}{k}}}\right]^{\beta} \\ &\times G\left(1+\frac{x-1}{[1+kw(x+1)]^{\frac{1}{k}}}\,,\frac{wv}{[1+kw(x+1)]}\right) \\ &= \sum_{n=0}^{\infty} w^n\sigma_n(x,v),\end{aligned} \tag{1.7}$$

where

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, K_n(\alpha,\beta,k;x)w^n$$

and

$$\sigma_n(x,v) = \sum_{p=0}^{n} a_p\,(-2k)^{n-p}\binom{n}{p}\,K_n(\alpha,\beta-n+p,k;x)\,v^p\,.$$

The main object of the present paper is to extend the generating relation (1.7) by the virtue of the existence of a quasi-bilateral generating relation as defined in (1.6). The main result of this paper is stated in the form of the following theorem:

Theorem 1. If there exists a quasi-bilateral generating function involving biorthogonal polynomials $K_n(\alpha,\beta,k;x)$ suggested by the classical Jacobi polynomials and $Y_n^\alpha(u;l)$ suggested by the classical Laguerre polynomials of the form:

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n\, K_n(\alpha,\beta,k;x)\, Y_m^n(u;l)\, w^n, \tag{1.8}$$

then

$$\begin{aligned}&[1+kw(x+1)z]^{-\frac{1+\alpha}{k}}(x+1)^{-\beta}\left\{2+\frac{x-1}{[1+k(x+1)z]^{\frac{1}{k}}}\right\}^{\beta}\exp(-\mathrm{w})\\ &\quad\times G\left(1+\frac{x-1}{[1+k(x+1)z]^{\frac{1}{k}}},\mathrm{u}+\mathrm{w},\frac{-2wzt}{1+kw(x+1)z}\right)\\ &=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n\frac{w^{n+p+q}}{p!\,q!}\,k^q(-2z)^{n+q}(n+1)_q\,(-1)^p K_{n+q}(\alpha,\beta\\ &\qquad -q,k;x)\,Y_m^{n+p}(u;l)t^n.\end{aligned} \tag{1.9}$$

It is of interest to mention that Theorem 1, in special cases, yields some interesting results (Theorem 2, Theorem 3, and Theorem 5 of this paper) on quasi-bilateral generating functions involving biorthogonal polynomials $Y_n^\alpha(x;k)$, Jacobi polynomials and Laguerre polynomials.

2. PROOF OF THE THEOREM

For the biorthogonal polynomials $K_n(\alpha,\beta,k;x)$ suggested by the classical Jacobi polynomials and $Y_n^\alpha(x;k)$ suggested by the classical Laguerre polynomials, we consider the following linear partial differential operators R_1 and R_2:

$$R_1=(1-x^2)y^{-1}z\frac{\partial}{\partial \mathrm{x}}+(1-\mathrm{x})\mathrm{z}\frac{\partial}{\partial \mathrm{y}}-\mathrm{k}(1+\mathrm{x})y^{-1}z^2\frac{\partial}{\partial \mathrm{z}}-(1+\alpha)(1+\mathrm{x})\mathrm{y}^{-1}\mathrm{z}$$

and

$$R_2=v\frac{\partial}{\partial \mathrm{u}}-\mathrm{v},$$

such that

$$R_1\left(K_n(\alpha,\beta,k;x)\,y^\beta z^n\right)=-2k(n+1)K_{n+1}(\alpha,\beta-1,k;x)\,y^{\beta-1}z^{n+1} \tag{2.1}$$

and

$$R_2(Y_m^n(u;l)\,v^n)=-Y_m^{n+1}(u;l)\,v^{n+1} \tag{2.2}$$

and also

$$\mathrm{e}^{\mathrm{wR}_1}\mathrm{f}(\mathrm{x},\mathrm{y},\mathrm{z})=\ (1+\mathrm{kw}(\mathrm{x}+1)\mathrm{y}^{-1}\mathrm{z})^{-\frac{1+\alpha}{\mathrm{k}}}\,\mathrm{f}(\mathrm{x}^*,\mathrm{y}^*,\mathrm{z}^*), \tag{2.3}$$

where

$$x^* = 1 + \frac{x-1}{[1 + kw(x+1)zy^{-1}]^{\frac{1}{k}}},$$
$$y^* = \frac{y}{x+1}\left\{2 + \frac{x-1}{[1 + kw(x+1)zy^{-1}]^{\frac{1}{k}}}\right\},$$
$$z^* = \frac{z}{1 + kw(x+1)zy^{-1}}$$

and

$$\mathrm{e}^{\mathrm{wR_2}} \mathrm{f(u,v)} = \exp(-\mathrm{wv})\ \mathrm{f(u + vw, v)}. \tag{2.4}$$

Let us now consider the formula

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, K_n(\alpha,\beta,k;x)\, Y_m^n(u;l)\, w^n. \tag{2.5}$$

Replacing w by $(-2wzvt)$ and multiplying both sides of (2.5) by y^{β}, we get

$$y^{\beta} G(x,u,-2wzvt) = \sum_{n=0}^{\infty} a_n\big(K_n(\alpha,\beta,k;x) y^{\beta} z^n\big)(Y_m^n(u;l)v^n)(-2wt)^n. \tag{2.6}$$

Operating both sides of (2.6) by $\mathrm{e}^{\mathrm{wR_1}}\mathrm{e}^{\mathrm{wR_2}}$, we obtain

$$\begin{aligned} &\mathrm{e}^{\mathrm{wR_1}}\mathrm{e}^{\mathrm{wR_2}}\big[y^{\beta} G(x,u,-2wzvt)\big] \\ &= \mathrm{e}^{\mathrm{wR_1}}\mathrm{e}^{\mathrm{wR_2}}\left[\sum_{n=0}^{\infty} a_n\big(K_n(\alpha,\beta,k;x) y^{\beta} z^n\big)(Y_m^n(u;l)v^n)(-2wt)^n\right] \end{aligned} \tag{2.7}$$

The left-hand side of (2.7), with the help of (2.3) and (2.4), becomes

$$\begin{aligned} &[1 + kw(x+1)y^{-1}z]^{-\frac{1+\alpha}{k}}(x+1)^{-\beta}\left\{2 + \frac{x-1}{[1 + kw(x+1)y^{-1}z]^{\frac{1}{k}}}\right\}^{\beta} \exp(-\mathrm{wv})\, \mathrm{y}^{\boldsymbol{\beta}} \\ &\times G\left(1 + \frac{x-1}{[1 + kw(x+1)y^{-1}z]^{\frac{1}{k}}}, \mathrm{u + vw}, \frac{-2wvzt}{1 + kw(x+1)y^{-1}z}\right) \end{aligned} \tag{2.8}$$

whereas the right-hand side of (2.7), with the help of (2.1) and (2.2), becomes

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\,q!} (-2k)^q (n+1)_q\, (-1)^p\, y^{\beta-q} z^{n+q} v^{n+p} \\ &\times K_{n+q}(\alpha,\beta-q,k;x)\, Y_m^{n+p}(u;l)(-2t)^n \end{aligned} \tag{2.9}$$

Equating (2.8) and (2.9) and then putting $y = v = 1$, we get

$$[1 + kw(x+1)z]^{-\frac{1+\alpha}{k}}(x+1)^{-\beta}\left\{2 + \frac{x-1}{[1 + k(x+1)z]^{\frac{1}{k}}}\right\}^{\beta} \exp(-\mathrm{w}) \tag{2.10}$$

$$\times G\left(1+\frac{x-1}{[1+k(x+1)z]^{\frac{1}{k}}}, \mathrm{u}+\mathrm{w}, \frac{-2wzt}{1+kw(x+1)z}\right)$$
$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\,q!} k^q (-2z)^{n+q} (n+1)_q (-1)^p K_{n+q}(\alpha,\beta - q,k;x)\, Y_m^{n+p}(u;l) t^n.$$

Thus, the theorem is completely proved.

Corollary 1. If we put $m=0$, we notice that $G(x,u,w)$ becomes $G(x,w)$, since $Y_0^{n+p}(u;l)=1$. Hence, from (2.10), we get

$$[1+kw(x+1)z]^{-\frac{1+\alpha}{k}}(x+1)^{-\beta}\left\{2+\frac{x-1}{[1+k(x+1)z]^{\frac{1}{k}}}\right\}^{\beta}\exp(-\mathrm{w})$$
$$\times G\left(1+\frac{x-1}{[1+k(x+1)z]^{\frac{1}{k}}}, \frac{-2wzt}{1+kw(x+1)z}\right)$$
$$=\sum_{p=0}^{\infty}\frac{(-w)^p}{p!}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+q}}{q!} k^q (-2z)^{n+q}(n+1)_q K_{n+q}(\alpha,\beta-q,k;x) t^n \tag{2.11}$$
$$=\exp(-\mathrm{w})\sum_{n=0}^{\infty}(wz)^n\,\sigma_n(x,t),$$

where

$$\sigma_n(x,t)=\sum_{q=0}^{n} a_q\,(-2k)^{n-q}\binom{n}{q} K_n(\alpha,\beta-n+q,k;x)(-2t)^q.$$

Now putting $z=1$ and replacing $(-2t)$ by v in (2.11), we get

$$[1+kw(x+1)]^{-\frac{1+\alpha}{k}}(x+1)^{-\beta}\left\{2+\frac{x-1}{[1+k(x+1)]^{\frac{1}{k}}}\right\}^{\beta}$$
$$\times G\left(1+\frac{x-1}{[1+k(x+1)]^{\frac{1}{k}}}, \frac{wv}{1+kw(x+1)}\right) \tag{2.12}$$
$$=\sum_{n=0}^{\infty} w^n \sigma_n(x,v),$$

where

$$\sigma_n(x,v)=\sum_{q=0}^{n} a_q\,(-2k)^{n-q}\binom{n}{q} K_n(\alpha,\beta-n+q,k;x)\, v^q,$$

which is (1.7) and is found derived in [7].

3. SOME SPECIAL CASES

We now discuss some special cases of our Theorem 1.

3.1. SPECIAL CASE 1

Putting $l = k$ and replacing x by $(1 - \frac{2x}{\beta})$ and then taking the limit as $\beta \to \infty$ in our Theorem 1 and finally using the relation [5, p. 419, Equations (12)]:

$$\lim_{\beta \to \infty} K_n\left(\alpha, \beta, k; 1 - \frac{2x}{\beta}\right) = Y_n^{\alpha}(x; k),$$

we get the following theorem on quasi-bilinear generating function involving biorthogonal polynomials $Y_n^{\alpha}(u; k)$ suggested by the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ of the form:

Theorem 2. If there exists, the following quasi-bilinear generating function for Laguerre polynomials

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha}(x; k)\, Y_m^{n}(u; k)\, w^n, \tag{3.1}$$

then

$$\begin{aligned}[1 + 2kwz]^{-\frac{1+\alpha}{k}} \exp\left\{\mathrm{x} - \mathrm{x}(1 + 2kwz)^{-\frac{1}{\mathrm{k}}} - \mathrm{w}\right\} G\left(\mathrm{x}(1 + 2kwz)^{-\frac{1}{\mathrm{k}}}, \mathrm{u}\right.\\ \left. + \mathrm{w}, \frac{-2wzt}{1 + 2kwz}\right)\\ = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n\ k^q \frac{w^{n+p+q}}{p!\, q!}\ (n+1)_q (-2z)^{n+q}\ (-1)^p Y_{n+q}^{\alpha}(x; k)\ Y_m^{n+p}(u; k) t^n,\end{aligned} \tag{3.2}$$

which is noteworthy.

Corollary 2. If we put $k = 1$, then $Y_n^{\alpha}(x; k)$ reduce to the classical Laguerre polynomials $L_n^{(\alpha)}(x)$. Thus putting $k = 1$ in our Theorem 2, we get the following theorem on quasi-bilinear generating function involving Laguerre polynomials:

Theorem 3. If

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)\, L_m^{(n)}(u)\, w^n, \tag{3.3}$$

then

$$\begin{aligned}[1 + 2wz]^{-(1+\alpha)} \exp\{\mathrm{x} - \mathrm{x}(1 + 2wz)^{-1} - \mathrm{w}\} G\left(\mathrm{x}(1 + 2wz)^{-1}, \mathrm{u} + \mathrm{w}, \frac{-2wzt}{1 + 2wz}\right)\\ = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\, q!}\ (n+1)_q (-2z)^{n+q}\ (-1)^p L_{n+q}^{(\alpha)}(x)\ L_m^{(n+p)}(u) t^n,\end{aligned} \tag{3.4}$$

which is noteworthy.

Corollary 3. Putting $m = 0$ and then taking $z = -\frac{1}{2}$ in our Theorem 2, we get the following result on bilateral generating relation involving biorthogonal polynomials $Y_n^{\alpha}(x;k)$ suggested by Laguerre polynomials $L_n^{(\alpha)}(x)$:

Theorem 4. If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha}(x;k)\, w^n, \tag{3.5}$$

then

$$[1+2kwz]^{-\frac{1+\alpha}{k}} \exp\left\{x - x(1+2kwz)^{-\frac{1}{k}}\right\} G\left(x(1+2kwz)^{-\frac{1}{k}}, \frac{-2wzt}{1+2kwz}\right) = \sum_{n=0}^{\infty} \sigma_n(t)\, Y_n^{\alpha}(x;k)\, w^n, \tag{3.6}$$

where

$$\sigma_n(t) = \sum_{q=0}^{n} a_q\, k^{n-q} \binom{n}{q}\, t^q \tag{3.7}$$

which is found derived in [8].

Corollary 4. Putting $m = 0$ and then taking $k = 1$ and $z = -\frac{1}{2}$ in our Theorem 2, we get the theorem found derived in [9-11].

3.2. SPECIAL CASE 2

If we put $k = l = 1$, then $K_n(\alpha,\beta,k;x)$ reduces to the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $Y_n^{\alpha}(x;k)$ reduces to the classical Laguerre polynomials $L_n^{(\alpha)}(x)$. Thus putting $k = l = 1$ in our Theorem 1, we get the following theorem.

Theorem 5. If there exists a quasi-bilateral generating function involving Jacobi and Laguerre polynomials of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\, P_n^{(\alpha,\beta)}(x)\, L_m^{(n)}(u)\, w^n, \tag{3.8}$$

then

$$\begin{aligned}[1+w(x+1)z]^{-(1+\alpha)}(x+1)^{-\beta}\left\{2+\frac{x-1}{1+w(x+1)z}\right\}^{\beta} \exp(-w) \\ \times G\left(1+\frac{x-1}{1+w(x+1)z}, u+w, \frac{-2wzt}{1+w(x+1)z}\right) \\ = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!\,q!}(-2z)^{n+q}(n+1)_q\,(-1)^p P_{n+q}^{(\alpha,\beta-q)}(x)\, L_m^{(n+p)}(u) t^n.\end{aligned} \tag{3.9}$$

Corollary 5. Putting $m = 0$, $z = 1$ and replacing $(-2t)$ by v in our Theorem 5, we get the following result on the bilateral generating relation of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$:

Theorem 6. If

$$G(x,w)=\sum_{n=0}^{\infty}a_n\,P_n^{(\alpha,\beta)}(x)\,w^n, \tag{3.10}$$

then

$$[1+w(x+1)]^{-(1+\alpha)}(x+1)^{-\beta}\left\{2+\frac{x-1}{1+w(x+1)}\right\}^{\beta}G\left(\frac{x+w(x+1)}{1+w(x+1)},\frac{wv}{1+w(x+1)}\right)=\sum_{n=0}^{\infty}w^n\sigma_n(x,v)\,, \tag{3.11}$$

where

$$\sigma_n(x,v)=\sum_{q=0}^{n}a_q\,(-2)^{n-q}\binom{n}{q}\,P_n^{(\alpha,\beta-n+q)}(x)\,v^q, \tag{3.12}$$

which is found derived from [12].

4. CONCLUSION

From the above discussion, we may conclude, under the existence of a quasi-bilateral generating function, that the result found derived in [7], may be extended to the more general generating function stated in Theorem 1.

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