ORIGINAL PAPER

COMMON TERMS *k*-GENERALIZED FIBONACCI AND LUCAS SEQUENCES

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Abstract. *Let* $(F_n^{(k)})$ *and* (L_n) *be the k-generalized Fibonacci and Lucas sequences. In this study, we find k-generalized Fibonacci numbers which are Lucas numbers. Namely, we tackle the Diophantine equation*

$$F_n^{(k)} = L_m,$$

in non-negative integers k, n, m *with* $k \geq 3$. *Solutions to this equation can be expressed as*

$$F_3^{(k)} = L_0 = 2, F_1^{(k)} = L_1 = 1, F_4^{(k)} = L_3 = 4, F_5^{(3)} = L_4 = 7, \text{and } F_7^{(4)} = L_7 = 29.$$

Keywords: *k-generalized Fibonacci numbers; Lucas numbers; exponential Diophantine equations; Baker's method.*

1. INTRODUCTION

The sequence of Lucas numbers (L_n) defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Let $k \geq 2$ be an integer. The k-generalized Fibonacci sequence $F_n^{(k)}$ is given by $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$ for all $n \geq 2$, with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. When $k = 2$ and $k = 3$, this sequence is called the Fibonacci and Tribonacci sequence, respectively.

Let (u_n) and (v_m) be two linear recurrent sequences. In [1-2], the investigators showed that these two sequences have only finitely many solutions in integers m and n under some assumptions. After, mathematicians found common terms of two different integer sequences. More details can be seen in [3-8].

Motivated by these works, we dealt with the intersection of k-generalized Fibonacci and Lucas sequences. Namely, we solved the Diophantine equation

$$F_n^{(k)} = L_m, \tag{1}$$

in non-negative integers k, n, m with $k \geq 3$.

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2. MATERIALS AND METHODS

2.1. LINEAR FORM IN LOGARITHMS

Here, we recall some basic notions from algebraic number theory. Let γ be an algebraic number of degree d with a minimal polynomial

$$u_0x^d + u_1x^{d-1} + \cdots + u_d = u_0 \prod_{i=1}^{d}\left(x - \gamma^{(i)}\right) \in \mathbb{Z}[x],$$

where the u_i's are relatively prime integers with $u_0 > 0$ and the $\gamma^{(i)}$'s are conjugates of γ. Then, $h(\gamma)$ is called the logarithmic height of γ and

$$h(\gamma) = \frac{1}{d}\left(\log u_0 + \sum_{i=1}^{d} \log\left(max\left\{\left|\gamma^{(i)}\right|, 1\right\}\right)\right).$$

The following lemma is given in [9].

Lemma 2.1.1. Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \gamma_2, \ldots, \gamma_s$ be positive real algebraic numbers of $\mathbb{K}$, and $b_1, b_2, \ldots, b_s$ are rational integers. Put

$$\Lambda \coloneqq {\gamma_1}^{b_1} \cdot {\gamma_2}^{b_2} \cdots {\gamma_s}^{b_s} - 1$$

$$B \geq max\{|b_1|, |b_2|, \ldots, |b_s|\} \quad \text{and} \quad A_i \geq \max\{D \cdot h(\gamma_i), |\log \gamma_i|, (0.16)\}$$

for all $i = 1,2, \ldots, s$. If Λ is not zero, then we have

$$|\Lambda| > exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdot A_2 \cdots A_s).$$

2.2. LUCAS AND k-FİBONACCİ SEQUENCES

The Binet formula for Lucas numbers is

$$L_n = \emptyset^n + \left(\frac{-1}{\emptyset}\right)^n \text{ for all } n \geq 0, \tag{2}$$

where $\emptyset$ is the golden ratio. The inequality

$$\emptyset^{n-1} \leq L_n < 2\emptyset^n \text{ for all } n \geq 0 \tag{3}$$

holds. The characteristic polynomial of the sequence $\left(F_n^{(k)}\right)_{n\geq 2-k}$ is

$$\psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

which is irreducible over $\mathbb{Q}[x]$ and has just one zero outside the unit circle. Put $\alpha =: \alpha(k)$. α is called the dominant root of $\psi_k(x)$, denote the single real root. The following inequality

$$2(1-2^{-k}) < \alpha < 2$$

can be found in [10]. Let

$$f_k(z) = \frac{z-1}{2+(k+1)(z-2)} \text{ for all } k \geq 2 \text{ and } z \in \mathbb{C}.$$

Fundamental properties of k-Fibonacci sequences are compiled in the following lemma.

Lemma 2.2.1. The followings hold:
(i) $1/2 < f_k(\alpha) < \frac{3}{4}$ and $\left|f_k\left(\alpha^{(i)}\right)\right| < 1$ for $2 \leq i \leq k$
(ii) $h\left(f_k(\alpha)\right) < 3\log k$ for $k \geq 2$
(iii) $F_n^{(k)} = \sum_{i=1}^{k} f_k\left(\alpha^{(i)}\right)\alpha^{(i)^{(n-1)}}$ and $\left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2}$ for all $n \geq 2-k$
(iv) $\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}$ for all $n \geq 1$ and $k \geq 2$
(v) $F_n^{(k)} = 2^{n-2}$ for all $2 \leq n \leq k+1$.

Proof: (i) and (ii) are from [11], for others, see [12], [13], and [14], respectively.

2.3. REDUCTION METHOD

We present the following lemma, which is given in [11].

Lemma 2.3.1. Let M be a positive integer, let $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let the function $\|\cdot\|$ denote the distance from x to the nearest integer. If $\varepsilon := \|\mu q\| - M\|\gamma q\| > 0$. Then, there exists no solution to the inequality

$$0 < |u\gamma - v + \mu| < A \cdot B^{-w}$$

in positive integers u, v and w with $u \leq M$ and $w \geq \frac{\log\left(Aq/\varepsilon\right)}{\log B}$.

2.4. USEFUL LEMMAS

The following two lemmas can be found [15-16].

Lemma 2.4.1. Let $p, q \in \mathrm{Z}$ and ρ be a real number. Put $\rho = [a_0; a_1, a_2, a_3, \dots]$. If $\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is a convergent of the continued fraction of ρ. Moreover, if M and n are non-negative integers such that $q_n > M$, then $\left|x - \frac{p}{q}\right| > \frac{1}{(b+2)q^2}$, where $b := \max\{a_i : i = 0,1,2, \dots, n\}$.

Lemma 2.4.2. If $m \geq 1, T > (4m^2)^m$ and $T > \frac{z}{(\log z)^m}$, then $z < 2^m \cdot T \cdot (\log T)^m$.

The following lemma can be easily checked.

Lemma 2.4.3. If the real numbers x and K satisfy $|e^x - 1| < K < 3/4$, then $|x| < 2K$.

Thanks to the following lemma, we find the relationship between the variables m and n in the equation (1).

Lemma 2.4.4. Let $m \geq 2$. Then, we have the following.
(i) $n < m + 4$
(ii) $m < 2n - 1$

Proof: From Lemma 2.2.1(iv) and the inequality (3), we write

$$\alpha^{n-2} \leq F_n^{(k)} = L_m \leq 2\emptyset^m < \emptyset^{m+2} \leq \alpha^{m+2}$$
$$\emptyset^{m-1} \leq L_m = F_n^{(k)} \leq \alpha^{n-1} < \emptyset^{2(n-1)}.$$

These inequalities imply that $n < m + 4$ and $m < 2n - 1$, respectively.

3. MAIN RESULT

Before giving the main theorem, we point out that when $k = 2$, the Fibonacci and Lucas sequences have common elements. However, since this case has already been addressed in [17], we will consider only $k \geq 3$ in our theorem.

Theorem 3.1. The only nontrivial solutions to the Diophantine equation (1) in non-negative integers k, n, m with $k \geq 3$ are given by

$$(k, n, m) \in \{(k, 3,0), (k, 1,1), (k, 2,1), (k, 4,3), (3,5,4), (4,7,7)\}.$$

Proof: Assume that equation (1) holds. If $m = 0$, then we have $(k, n, m) = (k, 3,0)$. If $m = 1$, then we get the trivial solutions as $(k, n, m) \in \{(k, 1,1), (k, 2,1)\}$. From now on, we take $m \geq 2$. If $2 \leq n \leq k + 1$, then we have $2^{n-2} = L_m$ by Lemma 2.2.1(v). In this case, the only solution to this equation is $(k, n, m) = (k, 4,3)$ for $m \geq 2$ by [9]. We assume that $n \geq k + 2$. Moreover, we can take $n \geq 5$ as $k \geq 3$. We can write the equation (1) as

$$f_k(\alpha)\alpha^{n-1} - \emptyset^m = (-1)^m \cdot \emptyset^{-m} - e_k(n)$$

by Lemma 2.2.1(iii) and the equality (2). Thus, we have

$$|\alpha^{n-1} f_k(\alpha)\emptyset^{-m} - 1| < \frac{1.5}{\emptyset^m}. \tag{4}$$

Let's consider the inequality (4). In order to use Lemma 2.1.1, we take

$$(\gamma_1, \gamma_2, \gamma_3) := (\alpha, f_k(\alpha), \emptyset) \text{ and } (b_1, b_2, b_3) := (n - 1,1, -m).$$

Here, γ_1, γ_2, and γ_3 are positive real numbers and belong to $\mathbb{Q}(\alpha)$, and so $D = k$. Put
$$\Lambda_1 =: \alpha^{n-1} f_k(\alpha) \emptyset^{-m} - 1.$$

We can say $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, then we have that $f_k(\alpha) = \alpha^{-(n-1)} \emptyset^m$. From this, we get $f_k(\alpha)$ would be an algebraic integer, which is not possible. Moreover, we can choose $A_1 := 0.7$, $A_2 := 3k\log k$, and $A_3 := k\frac{\log\emptyset}{2}$ since $h(\gamma_1) = h(\alpha) = \frac{0.7}{k}$, $h(\gamma_2) = h\big(f_k(\alpha)\big) = 3\log k$ (from Lemma 2.2.1(ii)) and $h(\gamma_3) = h(\emptyset) = \frac{\log\emptyset}{2}$. Considering Lemma 2.4.4(i), we can say $B := m + 4$. Let $S = -1.4 \cdot 30^6 \cdot 3^{4.5} k^2 (1 + \log k)$. Thus, Lemma 2.1.1 and the inequality (4) together with a straightforward calculation give

$$\frac{1.5}{\emptyset^m} > |\Lambda_1| > exp(S \cdot (1 + \log(m + 4)) \cdot 0.7 \cdot 3k\log k \cdot k\frac{\log\emptyset}{2}),$$

i.e.,

$$m < 2.26 \times 10^{12} k^4 (\log k)^2 logm,$$

where we used that $1 + \log k < 3\log k$ for all $k \geq 3$ and $1 + \log(m + 4) < 5\log m$ for all $m \geq 2$. Using Lemma 2.4.2, we obtain

$$m < 4.52 \times 10^{12} \mathrm{k}^4 (\log k)^2 \log(2.26 \times 10^{12} \mathrm{k}^4 (\log k)^2) < 4.52 \times 10^{12} \mathrm{k}^4 (\log k)^2 (31 \log k)$$

or

$$n < m + 4 < 1.41 \times 10^{14} k^4 (\log k)^3. \tag{5}$$

Since the bound on n is too large, we will reduce this bound.

The Case $3 \leq k \leq 707$: Here, we treat the case when $k \in [3,707]$. Put

$$x_1 := (n - 1)\log\alpha - m\log\emptyset + \log\big(f_k(\alpha)\big).$$

We observe that the inequality (4) can be rewritten

$$|e^{x_1} - 1| < \frac{1.5}{\emptyset^m} < \frac{3}{4}$$

for all $m \geq 2$. According to Lemma 2.4.3, we get

$$0 < \left|(n - 1)\log\alpha - m\log\emptyset + \log\big(f_k(\alpha)\big)\right| < 3 \times \emptyset^{-m}$$

and so

$$\left|(n - 1)\frac{\log\alpha}{\log\emptyset} - m + \frac{\log\big(f_k(\alpha)\big)}{\log\emptyset}\right| < 6.24 \times \emptyset^{-m}. \tag{6}$$

Using the inequality (6), we can choose

$$\gamma := \frac{\log\alpha}{\log\emptyset},\ \mu := \frac{\log\big(f_k(\alpha)\big)}{\log\emptyset}, M := \lceil 1.41 \times 10^{14} k^4 (\log k)^3 \rceil,\ A := 6.24,\ B := \emptyset,$$

and $w := m$ in Lemma 2.3.1. If the first convergence does not satisfy the condition $\varepsilon(k) > 0$ such that $q_t(k) > 6M$, we proceed to the next convergence and continue this process until we find the smallest t for which $\varepsilon(k) > 0$ holds for all $k \in [3,707]$. According to Lemma 2.3.1, we can say that

$$m < \frac{\log\left(\frac{Aq_{234}}{\varepsilon}\right)}{\log B} < 1009.3,$$

which leads to $n < m + 4 \leq 1013$ by Lemma 2.4.4 (i). With a computer program, solutions of the equation (1) can be represented as $F_5^{(3)} = L_4 = 7$ and $F_7^{(4)} = L_7 = 29$ for $2 \leq m \leq 1009$, $5 \leq n \leq 1012$ and $3 \leq k \leq 707$.

The Case $k \geq 708$: The inequality

$$n < m + 4 < 1.41 \times 10^{14} k^4 (\log k)^3 < 2^{k/2}$$

holds for $k \geq 168$.

The following lemma is given in [8].

Lemma 3.2. If $n < 2^{k/2}$, then the following estimate holds:

$$F_n^{(k)} = 2^{n-2}(1 + \zeta(n,k)),$$

where $|\zeta(n,k)| < 2^{-\frac{k}{2}}$.

According to Lemma 3.2, we get

$$\left|2^{n-2} - F_n^{(k)}\right| < 2^{n-2} \cdot 2^{-\frac{k}{2}}.$$

From the inequalities (1) and (2), we conclude that

$$\left|F_n^{(k)} - \emptyset^m\right| = \emptyset^{-m}.$$

Considering the above two inequalities, we deduce that

$$\left|1 - 2^{-(n-2)} \cdot \emptyset^m\right| < \frac{1}{2^{\frac{k}{2}}} + \frac{1}{2^{n-2} \cdot \emptyset^m} < \frac{1}{\emptyset^{\frac{k}{2}}} + \frac{1}{\emptyset^{n-2} \cdot \emptyset^{n-4}} < \frac{2}{\emptyset^{\frac{k}{2}}}. \tag{7}$$

Here, we consider that $\frac{1}{\emptyset^{n-2}} \leq \frac{1}{\emptyset^k} < \frac{1}{\emptyset^{k/2}}$ for $n \geq k+2$. If we apply Lemma 2.1.1 to inequality (7) with similar arguments as we have applied to inequality (4), we get $= 2$, $A_1 \coloneqq 2\log 2$, $A_2 \coloneqq \log\emptyset$, and $B = 2n - 1$ (from Lemma 2.4.4(ii)). Thus, we have

$$2 \cdot \emptyset^{-\frac{k}{2}} > \exp(1.4 \cdot 30^5 \cdot 2^{4.5} \cdot 2^2 \cdot (1 + \log 2) \cdot (1 + \log(2n-1)) \cdot \log 4 \cdot \log\emptyset).$$

This inequality leads to

$$k < 2.9 \times 10^{10} \log n, \tag{8}$$

where we have used that $1+\log(2n-1)<2\log n$ for $n\geq k+2\geq 710$. By (5) and (8), we obtain

$$k<2.9\times 10^{10}\log(1.41\times 10^{14}k^4(\log k)^3)<2.9\times 10^{10}(10\log k).$$

Therefore, we obtain $k<8.64\times 10^{12}$. This upper bound and the inequalitiy (5) considered together, we have that

$$n<1.41\times 10^{14}(8.64\times 10^{12})^4\cdot(\log(8.64\times 10^{12}))^3<2.08\times 10^{70}. \tag{9}$$

Now, we will apply the reduction method once again. Let

$$x_2 := m\log\emptyset-(n-2)\log 2.$$

From (7), we can say that

$$|e^{x_2}-1|<2\emptyset^{-\frac{k}{2}}<\frac{3}{4}$$

for $k\geq 708$. Using Lemma 2.4.3, we get

$$|m\log\emptyset-(n-2)\log 2|<4\emptyset^{-\frac{k}{2}}$$

or

$$\left|\frac{m}{n-2}-\frac{\log 2}{\log\emptyset}\right|<\frac{8.32}{(n-2)}\cdot\emptyset^{-\frac{k}{2}}. \tag{10}$$

With a simple calculation, the inequality

$$\frac{\emptyset^{\frac{k}{2}}}{16.64}>2.08\times 10^{70}>n-2,$$

holds for $k\geq 708$. From this, the inequality (10) convert to

$$\left|\frac{m}{n-2}-\frac{\log 2}{\log\emptyset}\right|<\frac{1}{2\cdot(n-2)^2}.$$

From Lemma 2.4.1, we know that $\left(\frac{m}{n-2}\right)$ is a convergent to $\gamma=\frac{\log 2}{\log\emptyset}$. Put $\frac{p_r}{q_r}$ be γ's r-th convergent and $\frac{m}{n-2}=\frac{p_s}{q_s}$ for some s. Then, we have $q_{142}>6.9\times 10^{70}>n-2$. Hereby, $s\in\{0,1,2,\ldots,141\}$. Here, $[a_0;a_1,a_2,\ldots]$ is the continued fraction expansion of γ. Furthermore, we can say that $b=\max\{a_i : i=0,1,\ldots,141\}=134$. Therefore, we obtain

$$\left|\gamma-\frac{p_t}{q_t}\right|>\frac{1}{136\cdot(n-2)^2} \tag{11}$$

from Lemma 2.4.1. The inequalities (10) and (11) tell us

$$\frac{8.32}{\emptyset^{\frac{k}{2}}}>\frac{1}{136\cdot(n-2)}>\frac{1}{136\times 6.9\times 10^{70}},$$

which shows that $k \leq 707$, a contradiction. Thus, the proof ends.

Corollary 3.3. Intersections of Tribonacci and Lucas numbers expressed as

$$T_1 = T_2 = L_1 = 1, T_3 = L_0 = 2, T_4 = L_3 = 4, \text{and } T_5 = L_4 = 7.$$

4. CONCLUSION

In recent years, the problem of identifying common elements between distinct integer sequences has garnered considerable interest within the mathematical community. In this work, we analyze whether terms from the k-generalized Fibonacci sequence also appear in the classical Lucas sequence, restricting our attention to non-negative integers and parameters $k \geq 3$. As a special case, we identified common elements between the Tribonacci and Lucas sequences for $k = 3$.

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