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STRONGLY n-FRACTIONAL POLYNOMIAL CONVEX FUNCTIONS AND RELATED INEQUALITIES

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Abstract. *In this paper, we introduce the idea and notion of strongly n–fractional polynomial convex functions. We investigate some algebraic properties of the newly defined class of functions. We establish certain inequalities of Hermite–Hadamard type for our novel generalization. We have also compared our newly obtained results using both Hölder and Hölder-İşcan inequalities, as well as power-mean and improved-power-mean integral inequalities. The results obtained in this work extend and improve the corresponding ones in the literature.*

Keywords: *Convex function; strongly n-fractional polynomial convex function; Hermite-Hadamard inequality; Hölder-İşcan inequality, integral inequalities.*

1. INTRODUCTION AND PRELIMINARIES

Let $\mathfrak{I} \subseteq \mathbb{R}$ be an interval. Then, a function $\Lambda: \mathfrak{I} \to \mathbb{R}$ is said to be convex if the inequality

$$\Lambda(\mu\hbar + (1-\mu)\varpi) \leq \mu\Lambda(\hbar) + (1-\mu)\Lambda(\varpi) \tag{1}$$

is valid for all $\hbar, \varpi \in \mathfrak{I}$ and $\mu \in [0,1]$. If the inequality (1) holds in the reverse direction, then Λ is said to be concave on interval $\mathfrak{I} \neq \emptyset$.

Numerous established results in the theory of inequalities can be derived utilizing the properties of functions, see [1–10] and the references therein. The Hermite–Hadamard (H–H) inequality is a well-explored and renowned result concerning convex functions, stating that if $\Lambda: \mathfrak{I} \to \mathbb{R}$ is a convex function in $\mathfrak{I}$ for all $\hbar, \varpi \in \mathfrak{I}$ with $\hbar < \varpi$ and $\Lambda \in [\hbar, \varpi]$, then the following inequality holds ([11]):

$$\Lambda\left(\frac{\hbar+\varpi}{2}\right) \leq \frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi} \Lambda(\sigma)d\sigma \leq \frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}. \tag{2}$$

Interested readers can refer to the monographs [12–20]. In [21], Polyak introduced the class of strongly convex functions as follows:

Definition 1. Let $\mathfrak{I} \subset \mathbb{R}$ be an interval and k be a positive number. A function $\Lambda: \mathfrak{I} \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus k if

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$$\Lambda(\mu\hbar+(1-\mu)\varpi)\leq\mu\Lambda(\hbar)+(1-\mu)\Lambda(\varpi)-k\mu(1-\mu)(\varpi-\hbar)^2 \tag{3}$$

for all $\hbar,\varpi\in\Im$ and $\mu\in[0,1]$.

Theorem 1. If a function $\Lambda:\Im\to\mathbb{R}$ is a strongly convex function with modulus k, then

$$\Lambda\left(\frac{\hbar+\varpi}{2}\right)+\frac{k}{12}(\varpi-\hbar)^2\leq\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\leq\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{k}{6}(\varpi-\hbar)^2 \tag{4}$$

for all $\hbar,\varpi\in\Im$ with $\hbar<\varpi$.

In [22], Varosanec introduced h-convexity as follows:

Definition 2. Let $\Im\subset\mathbb{R}$ be an interval and $h:(0,1)\to(0,\infty)$ be a given function. A function $\Lambda:\Im\to\mathbb{R}$ is called h-convex if

$$\Lambda(\mu\hbar+(1-\mu)\varpi)\leq h(\mu)\Lambda(\hbar)+(1-\mu)\Lambda(\varpi) \tag{5}$$

for all $\hbar,\varpi\in\Im$ and $\mu\in(0,1)$.

In [23], Angulo et al. introduced the concept of strongly h-convexity as follows:

Definition 3. Let $(\mathcal{F},\|\cdot\|)$denote a real normed space, $\wp$ be a convex subset of $\mathcal{F}, h:(0,1)\to(0,\infty)$ be a given function and k be a positive constant. A function $\Lambda:\wp\to\mathbb{R}$ is called strongly h-convex if

$$\Lambda(\mu\hbar+(1-\mu)\varpi)\leq h(\mu)\Lambda(\hbar)+(1-\mu)\Lambda(\varpi)-k\mu(1-\mu)\|\varpi-\hbar\|^2 \tag{6}$$

for all $\hbar,\varpi\in\wp$ and $\mu\in(0,1)$.

Theorem 2. ([24]) Let $h:(0,1)\to(0,\infty)$ be a given function. If a function $\Lambda:\Im\to\mathbb{R}$ is Lebesgue integrable and strongly h-convex with module $k>0$, then

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\left[\Lambda\left(\frac{\hbar+\varpi}{2}\right)+\frac{k}{12}(\varpi-\hbar)^2\right] &\leq \frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\\ &\leq (\Lambda(\hbar)+\Lambda(\varpi))\int_0^1 h(\mu)d\mu-\frac{k}{6}(\varpi-\hbar)^2\end{aligned}$$

for all $\hbar,\varpi\in\Im$ with $\hbar<\varpi$.

In [25], İşcan introduced the class of n-fractional polynomial convex function and related H–H type inequality as follows:

Definition 4. Let $n\in\mathbb{N}$. A non-negative function $\Lambda:\Im\subset\mathbb{R}\to\mathbb{R}$ is called an n-fractional polynomial convex if the inequality

$$\Lambda(\mu\hbar+(1-\mu)\varpi)\leq\frac{1}{n}\sum_{s=1}^{n}\mu^{\frac{1}{s}}\Lambda(\hbar)+\frac{1}{n}\sum_{s=1}^{n}(1-\mu)^{\frac{1}{s}}\Lambda(\varpi) \tag{7}$$

holds for every $\hbar,\varpi\in\Im$ and $\mu\in[0,1]$.

Theorem 3. Let $\Lambda: [\hbar, \varpi] \to \mathbb{R}$ be an n-fractional polynomial convex function. If $\hbar < \varpi$ and $\Lambda \in [\hbar, \varpi]$, then the following H–H type inequality holds:

$$\frac{n}{2\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{1/s}}\Lambda\left(\frac{\hbar+\varpi}{2}\right) \le \frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma \le \left(\frac{\Lambda(\hbar)+\Lambda(\varpi)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}. \tag{8}$$

Theorem 4. (Hölder-İşcan integral inequality [26]). Let $\mathrm{p} > 1$ and $\frac{1}{\mathrm{p}}+\frac{1}{\mathrm{q}} = 1$. If Λ and Θ are real functions defined on $[\hbar, \varpi]$ and if $|\Lambda|^{\mathrm{p}}$, $|\Theta|^{\mathrm{q}}$ are integrable functions on $[\hbar, \varpi]$, then

$$\int_{\hbar}^{\varpi}|\Lambda(\sigma)\Theta(\sigma)|d\sigma \le \frac{1}{\varpi-\hbar}\left\{\left(\int_{\hbar}^{\varpi}(\varpi-\sigma)|\Lambda(\sigma)|^{p}d\sigma\right)^{\frac{1}{p}}\left(\int_{\hbar}^{\varpi}(\varpi-\sigma)|\Theta(\sigma)|^{q}d\sigma\right)^{\frac{1}{q}}\right.$$

Theorem 5. (Improved power-mean integral inequality [27]). Let $\mathrm{q} \ge 1$. If Λ and Θ are real functions defined on $[\hbar, \varpi]$ and if $|\Lambda|$, $|\Lambda||\Theta|^{\mathrm{q}}$ are integrable functions on $[\hbar, \varpi]$ then

$$\begin{aligned}&\int_{a}^{b}|\Lambda(\sigma)\Theta(\sigma)|d\sigma\\ \le\ &\frac{1}{\varpi-\hbar}\left\{\left(\int_{\hbar}^{\varpi}(\varpi-\sigma)|\Lambda(\sigma)|d\sigma\right)^{1-\frac{1}{q}}\left(\int_{\hbar}^{\varpi}(\varpi-\sigma)|\Lambda(\sigma)||\Theta(\sigma)|^{q}d\sigma\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\hbar}^{\varpi}(\sigma-\hbar)|\Lambda(\sigma)|d\sigma\right)^{1-\frac{1}{q}}\left(\int_{\hbar}^{\varpi}(\sigma-\hbar)|\Lambda(\sigma)||\Theta(\sigma)|^{q}d\sigma\right)^{\frac{1}{q}}\right\}\end{aligned}$$

The aim of this paper is to introduce the concept of strongly n-fractional polynomial convex functions and establishes some results connected with the right-hand side of new inequalities similar to the H–H inequality for this class of functions.

2. THE DEFINITION OF STRONGLY n-FRACTIONAL POLYNOMIAL CONVEX FUNCTIONS

In this section, we introduce a new concept, which is called strongly n-fractional polynomial convexity and we give by setting some algebraic properties for the strongly n-fractional polynomial convex functions, as follows:

Definition 5. Let $n \in \mathbb{N}$, $\mathfrak{I} \subset \mathbb{R}$ be an interval and k be a positive number. A non-negative function $\Lambda: \mathfrak{I} \subset \mathbb{R} \to \mathbb{R}$ is called strongly n-fractional polynomial convex with modulus k if the inequality

$$\Lambda(\mu\hbar+(1-\mu)\varpi) \le \frac{1}{n}\sum_{s=1}^{n}\mu^{\frac{1}{s}}\Lambda(\hbar)+\frac{1}{n}\sum_{s=1}^{n}(1-\mu)^{\frac{1}{s}}\Lambda(\varpi)-k\mu(1-\mu)(\varpi-\hbar)^{2}. \tag{9}$$

We will denote by $\mathrm{SFPC}_k(\mathfrak{I})$ the class of all strongly n-fractional polynomial convex functions with modulus k on interval $\mathfrak{I}$.

Remark 1. If Λ is a strongly n-fractional polynomial convex function with modulus k, then Λ is a non-negative function. Indeed, using the definition of strongly n-fractional polynomial convexity, one can write

$$\Lambda(\sigma) = \Lambda(\mu\sigma + (1-\mu)\sigma) \leq \frac{1}{n}\left(\sum_{s=1}^{n} \mu^{1/s} + \sum_{s=1}^{n} (1-\mu)^{1/s}\right)\Lambda(\sigma)$$

for all $\sigma \in \mathfrak{I}$ and $\mu \in [0,1]$. Therefore, one has

$$\left[\frac{1}{n}\left(\sum_{s=1}^{n} \mu^{1/s} + \sum_{s=1}^{n} (1-\mu)^{1/s}\right) - 1\right]\Lambda(\sigma) \geq 0$$

for all $\sigma \in \mathfrak{I}$ and $\mu \in [0,1]$. Since

$$\frac{1}{n}\left(\sum_{s=1}^{n} \mu^{1/s} + \sum_{s=1}^{n} (1-\mu)^{1/s}\right) - 1 \geq 0$$

for all $\mu \in [0,1]$, one obtains $\Lambda(\sigma) \geq 0$ for all $\sigma \in \mathfrak{I}$.

We note that, every strongly n-fractional polynomial convex function is a strongly h-convex function with the function $h(\mu) = \frac{1}{n}\sum_{s=1}^{n} \mu^{1/s}$. Therefore, if $\Lambda, \Theta \in \mathrm{SFPC}_k(\mathfrak{I})$, it can be easily seen that $\Lambda, \Theta \in \mathrm{SFPC}_{2k}(\mathfrak{I})$ and for $c \in \mathbb{R}$ $(c \geq 0)$, $c\Lambda \in \mathrm{SFPC}_{ck}(\mathfrak{I})$.

Remark 2. If one takes $n = 1$ in the inequality (9), then the strongly 1-polynomial convexity reduces to the classical strongly convexity.

Remark 3. Every nonnegative strongly convex function is a strongly n-fractional polynomial convex function. It is clear from the inequalities

$$\mu \leq \frac{1}{n}\sum_{s=1}^{n} \mu^{1/s} \ and \ 1-\mu \leq \frac{1}{n}\sum_{s=1}^{n} (1-\mu)^{1/s}$$

for all $\mu \in [0,1]$ and $n \in \mathbb{N}$.

3. HERMITE-HADAMARD INEQUALITY FOR STRONGLY n-FRACTIONAL POLYNOMIAL CONVEX FUNCTIONS

This section aims to establish H–H inequality for the newly defined class of functions.

Theorem 6. Let $\Lambda: [\hbar, \varpi] \to \mathbb{R}$ be a strongly n-fractional polynomial convex function with modulus k. If $\hbar < \varpi$ and $\Lambda \in [\hbar, \varpi]$, then the following H–H type inequality holds:

$$\frac{n}{2\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{\frac{1}{s}}}\left[\Lambda\left(\frac{\hbar+\varpi}{2}\right)+\frac{k}{12}(\varpi-\hbar)^2\right] \tag{10}$$
$$\leq\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\ \leq\left(\frac{\Lambda(\hbar)+\Lambda(\varpi)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}-\frac{k}{6}(\varpi-\hbar)^2$$

Proof: From the strongly n-fractional polynomial convexity of Λ, one obtains

$$\begin{aligned}
&\Lambda\left(\frac{\hbar+\varpi}{2}\right)\\
=\ &\Lambda\left(\frac{(\mu\hbar+(1-\mu)\varpi)+[(1-\mu)\hbar+\mu\varpi]}{2}\right)\\
=\ &\Lambda\left(\frac{1}{2}(\mu\hbar+(1-\mu)\varpi)+\frac{1}{2}[(1-\mu)\hbar+\mu\varpi]\right)\\
\leq\ &\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{1/s}\Lambda(\mu\hbar+(1-\mu)\varpi)+\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{1/s}\Lambda((1-\mu)\hbar+\mu\varpi)\\
&-\frac{k}{4}[(2\mu-1)\varpi-(1-2\mu)\hbar]^2\\
=\ &\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{1/s}[\Lambda(\mu\hbar+(1-\mu)\varpi)+\Lambda((1-\mu)\hbar+\mu\varpi)]-\frac{k}{4}(2\mu-1)(\varpi-\hbar)^2.
\end{aligned}$$

By taking integral in the last inequality with respect to $\mu\in[0,1]$, one gets

$$\begin{aligned}
&\Lambda\left(\frac{\hbar+\varpi}{2}\right)\\
\leq\ &\frac{1}{n}\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{1/s}\left[\int_0^1\Lambda(\mu\hbar+(1-\mu)\varpi)d\mu+\int_0^1\Lambda((1-\mu)\hbar+\mu\varpi)d\mu\right]\\
&-\frac{k}{4}(\varpi-\hbar)^2\int_0^1(2\mu-1)^2d\mu\\
=\ &\frac{2}{(\varpi-\hbar)n}\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{1/s}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma-\frac{k}{12}(\varpi-\hbar)^2,
\end{aligned}$$

which completes the left-hand side of the inequality (10). For the right-hand side of the inequality (10), changing the variable of integration as $\sigma=\mu\hbar+(1-\mu)\varpi$,and using the strongly n-fractional polynomial convexity of Λ, one obtains

$$\begin{aligned}
&\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\\
&=\int_0^1\Lambda(\mu\hbar+(1-\mu)\varpi)d\mu\\
&\leq\int_0^1\left[\frac{1}{n}\sum_{s=1}^{n}\mu^{1/s}\Lambda(\hbar)+\frac{1}{n}\sum_{s=1}^{n}(1-\mu)^{1/s}\Lambda(\varpi)-k\mu(1-\mu)(\varpi-\hbar)^2\right]d\mu
\end{aligned}$$

$$= \frac{\Lambda(\hbar)}{n}\int_0^1 \sum_{s=1}^{n} \mu^{1/s} d\mu + \frac{\Lambda(\varpi)}{n}\int_0^1 \sum_{s=1}^{n} (1-\mu)^{1/s} d\mu$$
$$-k(\varpi-\hbar)^2 \int_0^1 \mu(1-\mu)d\mu$$
$$= \frac{\Lambda(\hbar)}{n}\sum_{s=1}^{n} \int_0^1 \mu^{1/s} d\mu + \frac{\Lambda(\varpi)}{n}\sum_{s=1}^{n} \int_0^1 (1-\mu)^{1/s} d\mu - k(\varpi-\hbar)^2 \int_0^1 \mu(1-\mu)d\mu$$
$$= \left[\frac{\Lambda(\hbar)+\Lambda(\varpi)}{n}\right]\sum_{s=1}^{n} \frac{s}{s+1} - \frac{k}{6}(\varpi-\hbar)^2,$$

where

$$\int_0^1 \mu^{1/s} d\mu = \int_0^1 (1-\mu)^{1/s} d\mu = \frac{s}{s+1},$$
$$\int_0^1 \mu(1-\mu)d\mu = \frac{1}{6}.$$

This completes the proof of theorem.

Remark 4. For $n = 1$ and $k = 0$, the inequality (10) coincides with the inequality (2).

Remark 5. For $n = 1$, the inequality (10) coincides with the inequality (4).

Remark 6. For $k = 0$, the inequality (10) coincides with the inequality (8).

4. NEW INEQUALITIES OF H–H TYPE FOR STRONGLY n-FRACTIONAL POLYNOMIAL CONVEX FUNCTIONS

The aim of this section is to establish novel refinements of the H–H inequality for functions whose first derivatives in absolute value at certain power are strongly n-fractional polynomial convex. Let us recall the following crucial lemma that we will use in the sequel:

Lemma 1 ([2]). Let $\Lambda: \mathfrak{J}^\circ \to \mathbb{R}$ be a differentiable mapping on $\mathfrak{J}^\circ, \hbar, \varpi \in \mathfrak{J}^\circ$ with $\hbar < \varpi$. If $\Lambda' \in L[\hbar, \varpi]$, then the following identity holds:

$$\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2} - \frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi} \Lambda(\sigma)d\sigma = \frac{\varpi-\hbar}{2}\int_0^1 (1-2\mu)\Lambda'(\mu\hbar+(1-\mu)\varpi)d\mu.$$

Theorem 7. Let $\Lambda: \mathfrak{J} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathfrak{J}^\circ, \hbar, \varpi \in \mathfrak{J}^\circ$ with $\hbar < \varpi$ and assume that $\Lambda' \in L[\hbar, \varpi]$. If $|\Lambda'|$ is a strongly n-fractional polynomial convex function with modulus k on interval $[\hbar, \varpi]$, then the following inequality holds:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2} - \frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi} \Lambda(\sigma)d\sigma\right|$$
$$\le \frac{\varpi-\hbar}{n}\sum_{s=1}^{n} \left[\frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}\right] A(|\Lambda'(\hbar)|, |\Lambda'(\varpi)|) - \frac{k}{12}(\varpi-\hbar)^3, \tag{11}$$

where A is the arithmetic mean.

Proof: Using Lemma 1 and the inequality

$$|\Lambda'(\mu\hbar+(1-\mu)\varpi)| \leq \frac{1}{n}\sum_{s=1}^{n} \mu^{1/s}|\Lambda'(\hbar)| + \frac{1}{n}\sum_{s=1}^{n} (1-\mu)^{1/s}|\Lambda'(\varpi)| - k\mu(1-\mu)(\varpi-\hbar)^2,$$

one gets

$$\begin{aligned}
&\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2} - \frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi} \Lambda(\sigma)d\sigma\right| \\
\leq\ & \left|\frac{\varpi-\hbar}{2}\int_0^1 (1-2\mu)\Lambda'(\mu\hbar+(1-\mu)\varpi)d\mu\right| \\
\leq\ & \frac{\varpi-\hbar}{2}\int_0^1 |1-2\mu|\left(\frac{1}{n}\sum_{s=1}^{n} \mu^{1/s}|\Lambda'(\hbar)| + \frac{1}{n}\sum_{s=1}^{n} (1-\mu)^{1/s}|\Lambda'(\varpi)| - k\mu(1-\mu)(\varpi-\hbar)^2\right)d\mu \\
\leq\ & \frac{\varpi-\hbar}{2n}\Bigg(|\Lambda'(\hbar)|\int_0^1 |1-2\mu|\sum_{s=1}^{n} \mu^{1/s}d\mu + |\Lambda'(\varpi)|\int_0^1 |1-2\mu|\sum_{s=1}^{n} (1-\mu)^{1/s}d\mu \\
& -k(\varpi-\hbar)^2\int_0^1 \mu(1-\mu)d\mu\Bigg) \\
=\ & \frac{\varpi-\hbar}{2n}\left(|\Lambda'(\hbar)|\sum_{s=1}^{n}\int_0^1 |1-2\mu|\mu^{1/s}d\mu + |\Lambda'(\varpi)|\sum_{s=1}^{n}\int_0^1 |1-2\mu|(1-\mu)^{1/s}d\mu - \frac{k}{6}(\varpi-\hbar)^2\right. \\
=\ & \frac{\varpi-\hbar}{2n}\left(|\Lambda'(\hbar)|\sum_{s=1}^{n}\left[\frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}\right] + |\Lambda'(\varpi)|\sum_{s=1}^{n}\left[\frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}\right] - \frac{k}{6}(\varpi-\hbar)^2\right) \\
=\ & \frac{\varpi-\hbar}{n}\sum_{s=1}^{n}\left[\frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}\right]A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|) - \frac{k}{12}(\varpi-\hbar)^3
\end{aligned}$$

where

$$\int_0^1 \mu(1-\mu)d\mu = \frac{1}{6},$$

$$\int_0^1 |1-2\mu|\mu^{1/s}d\mu = \int_0^1 |1-2\mu|(1-\mu)^{1/s}d\mu = \frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}.$$

This completes the proof of theorem.

Corollary 1. If one takes $n=1$ and $k=0$ in the inequality (11), then one has the following inequality:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2} - \frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi} \Lambda(\sigma)d\sigma\right| \leq \frac{\varpi-\hbar}{4}A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|).$$

This inequality coincides with the inequality in [2].

Corollary 2. If one takes $k=0$ in the inequality (11), then one has the following inequality:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\leq\frac{\varpi-\hbar}{n}\sum_{s=1}^{n}\left[\frac{s\left(s+2^{\frac{1}{s}}\right)}{2^{\frac{1}{s}}(s+1)(2s+1)}\right]A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|).$$

This inequality coincides with the inequality in [25].

Corollary 3. If one takes $\mathrm{n}=1$ in the inequality (10), then one has the following inequality:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\leq\frac{\varpi-\hbar}{4}A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|)-\frac{k}{12}(\varpi-\hbar)^{3}.$$

Theorem 8. Let $\Lambda:\mathfrak{I}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathfrak{I}^{\circ}$, $\hbar,\varpi\in\mathfrak{I}^{\circ}$ *with* $\hbar<\varpi$, $q>1, \frac{1}{p}+\frac{1}{q}=1$ and assume that $\Lambda'\in L[\hbar,\varpi]$. If $|\Lambda'|^{q}$ is a strongly n-fractional polynomial convex function with modulus k on interval $[\hbar,\varpi]$, then the following inequality holds:

$$\begin{aligned}&\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\\ &\leq\frac{\varpi-\hbar}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}A(|\Lambda'(\hbar)|^{q},|\Lambda'(\varpi)|^{q})-\frac{k}{6}(\varpi-\hbar)^{2}\right)^{\frac{1}{q}},\end{aligned}\tag{12}$$

where A is the arithmetic mean.

Proof: Using Lemma 1, Hölder's integral inequality and the strongly n-fractional polynomial convexity of $|\Lambda'|^{q}$, one gets

$$\begin{aligned}&\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\\ \leq\;&\frac{\varpi-\hbar}{2}\left(\int_{0}^{1}|1-2\mu|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_{0}^{1}|\Lambda'(\mu\hbar+(1-\mu)\varpi)|^{q}d\mu\right)^{\frac{1}{q}}\\ \leq\;&\frac{\varpi-\hbar}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^{q}}{n}\sum_{s=1}^{n}\int_{0}^{1}\mu^{1/s}d\mu+\frac{|\Lambda'(\varpi)|^{q}}{n}\sum_{k=1}^{n}\int_{0}^{1}(1-\mu)^{1/s}d\mu-c(\varpi-\hbar)^{2}\int_{0}^{1}\mu(1-\mu)d\mu\right)^{\frac{1}{q}}\\ =\;&\frac{\varpi-\hbar}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}A(|\Lambda'(\hbar)|^{q},|\Lambda'(\varpi)|^{q})-\frac{k}{6}(\varpi-\hbar)^{2}\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\int_{0}^{1}|1-2\mu|^{p}d\mu=\frac{1}{p+1},$$
$$\int_{0}^{1}\mu^{1/s}d\mu=\int_{0}^{1}(1-\mu)^{1/s}d\mu=\frac{s}{s+1},$$
$$\int_{0}^{1}\mu(1-\mu)d\mu=\frac{1}{6}.$$

So, the proof is completed.

Corollary 4. If one takes $n=1$ and $k=0$ in the inequality (12),then one gets the following inequality:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\leq\frac{\varpi-\hbar}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}A^{\frac{1}{q}}(|\Lambda'(\hbar)|^{q},|\Lambda'(\varpi)|^{q}).$$

This inequality coincides with the inequality in [2].

Corollary 5. If one takes $k=0$ in (12), then

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|$$
$$\leq\frac{\varpi-\hbar}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|\Lambda'(\hbar)|^{q},|\Lambda'(\varpi)|^{q}).$$

This inequality coincides with the inequality in [25].

Corollary 6. If one takes $n=1$ in (12), then

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\leq\frac{\varpi-\hbar}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(A(|\Lambda'(\hbar)|^{q},|\Lambda'(\varpi)|^{q})-\frac{k}{6}(\varpi-\hbar)^{2}\right)^{\frac{1}{q}}.$$

Theorem 9. Let $\Lambda:\Im\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\Im^{\circ}$, $\hbar,\varpi\in\Im^{\circ}$ with $\hbar<\varpi, q\geq 1$ and assume that $\Lambda'\in L[\hbar,\varpi]$. If $|\Lambda'|^{q}$ is a strongly n-fractional polynomial convex function with modulus k on $[\hbar,\varpi]$, then the following inequality holds:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|$$
$$\leq\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{s\left(s+2^{\frac{1}{s}}\right)}{2^{\frac{1}{s}}(s+1)(2s+1)}A(|\Lambda'(\hbar)|^{q},|\Lambda'(\varpi)|^{q})-\frac{k}{16}(\varpi-\hbar)^{2}\right)^{\frac{1}{q}}, \tag{13}$$

where A is the arithmetic mean.

Proof: From Lemma 1, power-mean integral inequality and the strongly n-fractional polynomial convexity of $|\Lambda'|^{q}$, one obtains

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|$$
$$\leq\frac{\varpi-\hbar}{2}\left(\int_{0}^{1}|1-2\mu|d\mu\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}|1-2\mu||\Lambda'(\mu\hbar+(1-\mu)\varpi)|^{q}d\mu\right)^{\frac{1}{q}}$$

$$\leq \frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\int_0^1 |1-2\mu|\left(\frac{1}{n}\sum_{s=1}^{n} \mu^{1/s}|\Lambda'(\hbar)|^q+\frac{1}{n}\sum_{s=1}^{n} (1-\mu)^{1/s}|\Lambda'(\varpi)|^q - k(\varpi-\hbar)^2\mu(1-\mu)\right)\right]^{\frac{1}{q}} d\mu$$

$$=\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n} \int_0^1 |1-2\mu|\mu^{1/s}d\mu+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n} \int_0^1 |1-2\mu|(1-\mu)^{1/s}d\mu -k(\varpi-\hbar)^2\int_0^1 |1-2\mu|\mu(1-\mu)d\mu\right]^{\frac{1}{q}}$$

$$=\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n} \frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}A(|\Lambda'(\hbar)|^q,|\Lambda'(\varpi)|^q)-\frac{k}{16}(\varpi-\hbar)^2\right)^{\frac{1}{q}}.$$

Thus, the proof is completed.

Corollary 7. Under the assumption of Theorem 9 with $q=1$, one gets the conclusion of Theorem 7.

Corollary 8. If one takes $n=1$, $k=0$ *and* $q=1$ in the inequality (13), then one gets the following inequality:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi} \Lambda(\sigma)d\sigma\right| \leq \frac{\varpi-\hbar}{4}A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|).$$

This inequality coincides with the inequality in [2, Theorem 1].

Corollary 9. If one takes $k=0$ in the inequality (13), then one gets

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi} \Lambda(\sigma)d\sigma\right| \leq \frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n} \frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|\Lambda'(\hbar)|^q,|\Lambda'(\varpi)|^q).$$

This inequality coincides with the inequality in [25]. Also, if one takes $q=1$ in the above inequality, then one has

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi} \Lambda(\sigma)d\sigma\right| \leq \frac{\varpi-\hbar}{n}\sum_{s=1}^{n} \frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|).$$

This inequality coincides with the inequality in [25].

Corollary 10. If one takes $n = 1$ in (13), then one has

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right| \leq \frac{\varpi-\hbar}{4}\left(A(|\Lambda'(\hbar)|^q,|\Lambda'(\varpi)|^q)-\frac{k}{8}(\varpi-\hbar)^2\right)^{\frac{1}{q}}.$$

Also, if one takes $q = 1$ in the above inequality, then one has

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right| \leq \frac{\varpi-\hbar}{4}\left(A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|)-\frac{k}{8}(\varpi-\hbar)^2\right).$$

Now, let us prove the Theorem 8 using Hölder-İşcan integral inequality and demonstrate that the obtained result in this theorem gives a better approach than that obtained in the Theorem 8.

Theorem 10. Let $\Lambda: \mathfrak{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathfrak{I}^\circ$, $\hbar, \varpi \in \mathfrak{I}^\circ$ *with* $\hbar < \varpi$, $q > 1, \frac{1}{p}+\frac{1}{q} = 1$ and assume that $\Lambda' \in L[\hbar, \varpi]$. If $|\Lambda'|^q$ is a strongly n-fractional polynomial convex function with modulus k on interval $[\hbar, \varpi]$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right| \\
\leq\ & \frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s^2}{(s+1)(2s+1)}+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s}{2s+1}-\frac{k}{12}(\varpi-\hbar)^2\right)^{\frac{1}{q}} \\
&+\frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s}{2s+1}+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s^2}{(s+1)(2s+1)}-\frac{k}{12}(\varpi-\hbar)^2\right)^{\frac{1}{q}}
\end{aligned} \tag{14}$$

where A is the arithmetic mean.

Proof: From Lemma 1, Hölder-İşcan integral inequality and the strongly n-fractional polynomial convexity of $|\Lambda'|^q$, one obtains

$$\begin{aligned}
&\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right| \\
&\leq \frac{\varpi-\hbar}{2}\left(\int_0^1(1-\mu)|1-2\mu|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1(1-\mu)|\Lambda'(\mu\hbar+(1-\mu)\varpi)|^q d\mu\right)^{\frac{1}{q}} \\
&\quad+\frac{\varpi-\hbar}{2}\left(\int_0^1\mu|1-2\mu|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^1\mu|\Lambda'(\mu\hbar+(1-\mu)\varpi)|^q d\mu\right)^{\frac{1}{q}} \\
&\leq \frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\mu)\mu^{1/s}d\mu+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\mu)(1-\mu)^{1/s}d\mu\right. \\
&\qquad\left.-k(\varpi-\hbar)^2\int_0^1\mu(1-\mu)^2 d\mu\right)^{\frac{1}{q}} \\
&\quad+\frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\int_0^1\mu.\mu^{1/s}d\mu+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\int_0^1\mu(1-\mu)^{1/s}d\mu\right.
\end{aligned}$$

$$-k(\varpi-\hbar)^2\int_0^1 \mu(1-\mu)^2 d\mu\Bigg)^{\frac{1}{q}}$$

$$=\frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s^2}{(s+1)(2s+1)}+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s}{2s+1}-\frac{k}{12}(\varpi-\hbar)^2\right)^{\frac{1}{q}}$$

$$+\frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s}{2s+1}+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s^2}{(s+1)(2s+1)}-\frac{k}{12}(\varpi-\hbar)^2\right)^{\frac{1}{q}},$$

where

$$\begin{aligned}\int_0^1 (1-\mu)|1-2\mu|^p d\mu &= \int_0^1 \mu|1-2\mu|^p d\mu=\frac{1}{2(p+1)},\\ \int_0^1 (1-\mu)\mu^{1/s} d\mu &= \int_0^1 \mu(1-\mu)^{1/s} d\mu=\frac{s^2}{(s+1)(2s+1)},\\ \int_0^1 (1-\mu)(1-\mu)^{1/s} d\mu &= \int_0^1 \mu.\mu^{1/s} d\mu=\frac{s}{2s+1}.\end{aligned}$$

Thus, the proof is completed.

Corollary 11. If one takes $n=1$ and $k=0$ in the inequality (14), then one gets the following inequality:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|$$

$$\leq \frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\Lambda'(\hbar)|^q+2|\Lambda'(\varpi)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2|\Lambda'(\hbar)|^q+|\Lambda'(\varpi)|^q}{6}\right)^{\frac{1}{q}}\right].$$

This inequality coincides with the inequality of Theorem 3.2 in [26, Theorem 3.2].

Corollary 12. If one takes $\mathrm{k}=0$ in (14), then

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|$$

$$\leq \frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s^2}{(s+1)(2s+1)}+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s}{2s+1}\right)^{\frac{1}{q}}$$

$$+\frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s}{2s+1}+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s^2}{(s+1)(2s+1)}\right)^{\frac{1}{q}}.$$

This inequality coincides with the inequality in [25, Theorem 3.2]

Corollary 13. If one takes $n=1$ in (14), then

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|$$
$$\leq \frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q+2|\Lambda'(\varpi)|^q}{6}-\frac{k}{12}(\varpi-\hbar)^2\right)^{\frac{1}{q}}$$
$$+\frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q+2|\Lambda'(\varpi)|^q}{6}-\frac{k}{12}(\varpi-\hbar)^2\right)^{\frac{1}{q}}.$$

Remark 7. The inequality (14) gives better results than the inequality (12). Indeed, using the inequality $\upsilon^\alpha+\omega^\alpha\leq 2^{1-\alpha}(\upsilon+\omega)^\alpha, \upsilon,\omega\in[0,\infty)$, $0<\alpha\leq 1$, one gets

$$\frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s^2}{(s+1)(2s+1)}+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s}{2s+1}-\frac{k}{12}(\varpi-\hbar)^2\right)^{\frac{1}{q}}$$
$$+\frac{\varpi-\hbar}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s}{2s+1}+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s^2}{(s+1)(2s+1)}-\frac{k}{12}(\varpi-\hbar)^2\right)^{\frac{1}{q}}$$
$$\leq\frac{\varpi-\hbar}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}2\left[\frac{1}{2}\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}+\frac{1}{2}\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\frac{s}{s+1}-\frac{1}{2}\frac{k}{6}(\varpi-\hbar)^2\right]^{\frac{1}{q}}$$
$$=\frac{\varpi-\hbar}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}A(|\Lambda'(\hbar)|^q,|\Lambda'(\varpi)|^q)-\frac{k}{6}(\varpi-\hbar)^2\right)^{\frac{1}{q}},$$

which is the required.

Theorem 11. Let $\Lambda:\mathfrak{I}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\mathfrak{I}^\circ$, $\hbar,\varpi\in\mathfrak{I}^\circ$ with $\hbar<\varpi, q\geq 1$ and assume that $\Lambda'\in L[\hbar,\varpi]$. If $|\Lambda'|^q$ is a strongly n-fractional polynomial convex function with modulus k on $[\hbar,\varpi]$, then the following inequality holds:

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|$$
$$\leq \frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}R_1(s)+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}R_2(s)-\frac{k}{32}(\varpi-\hbar)^2\right)^{\frac{1}{q}} \tag{15}$$
$$+\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}R_2(s)+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}R_1(s)-\frac{k}{32}(\varpi-\hbar)^2\right)^{\frac{1}{q}},$$

where

$$R_1(s)=\frac{s^2\left[\left(\frac{1}{2}\right)^{1+\frac{1}{s}}(5s+1)+1-s\right]}{(s+1)(2s+1)(3s+1)},$$

$$R_2(s)=\frac{s\left[\left(\frac{1}{2}\right)^{1+\frac{1}{s}}s+1+s\right]}{(2s+1)(3s+1)}.$$

Proof: From Lemma 1, the improved power-mean integral inequality and the strongly n-fractional polynomial convexity of $|\Lambda'|^q$, one obtains

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|$$

$$\leq\frac{\varpi-\hbar}{2}\left(\int_0^1(1-\mu)|1-2\mu|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\mu)|1-2\mu||\Lambda'(\mu\hbar+(1-\mu)\varpi)|^q d\mu\right)^{\frac{1}{q}}$$

$$+\frac{\varpi-\hbar}{2}\left(\int_0^1\mu|1-2\mu|d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\mu|1-2\mu||\Lambda'(\mu\hbar+(1-\mu)\varpi)|^q d\mu\right)^{\frac{1}{q}}$$

$$\leq\frac{\varpi-\hbar}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\mu)|1-2\mu|\mu^{1/s}d\mu\right.$$

$$\left.+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\mu)|1-2\mu|(1-\mu)^{1/s}d\mu-k(\varpi-\hbar)^2\int_0^1\mu(1-\mu)^2|1-2\mu|d\mu\right)^{\frac{1}{q}}$$

$$+\frac{\varpi-\hbar}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}\int_0^1\mu|1-2\mu|\mu^{1/s}d\mu\right.$$

$$\left.+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}\int_0^1\mu|1-2\mu|(1-\mu)^{1/s}d\mu-k(\varpi-\hbar)^2\int_0^1\mu^2(1-\mu)|1-2\mu|d\mu\right)^{\frac{1}{q}}$$

$$=\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}R_1(s)+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}R_2(s)-\frac{k}{32}(\varpi-\hbar)^2\right)^{\frac{1}{q}}$$

$$+\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}R_2(s)+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}R_1(s)-\frac{k}{32}(\varpi-\hbar)^2\right)^{\frac{1}{q}},$$

where

$$\int_0^1\mu(1-\mu)^2|1-2\mu|d\mu=\int_0^1\mu^2(1-\mu)|1-2\mu|d\mu=\frac{1}{32},$$

$$\int_0^1(1-\mu)|1-2\mu|d\mu=\int_0^1\mu|1-2\mu|d\mu=\frac{1}{4},$$

$$R_1(s)=\int_0^1(1-\mu)|1-2\mu|\mu^{1/s}d\mu=\int_0^1\mu|1-2\mu|(1-\mu)^{1/s}d\mu$$

$$=\frac{s^2\left[\left(\frac{1}{2}\right)^{1+1/s}(5s+1)+1-s\right]}{(s+1)(2s+1)(3s+1)},$$

$$R_2(s)=\int_0^1\mu|1-2\mu|\mu^{1/s}d\mu=\int_0^1(1-\mu)|1-2\mu|(1-\mu)^{1/s}d\mu$$

$$=\frac{s\left[\left(\frac{1}{2}\right)^{1+1/s}s+1+s\right]}{(2s+1)(3s+1)}.$$

Corollary 14. If one takes $\mathrm{n}=1$ and $\mathrm{k}=0$ in the inequality (15), then one gets the following inequality:

$$\begin{aligned}&\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\\ \leq\quad&\frac{\varpi-\hbar}{8}\left[\left(\frac{|\Lambda'(\hbar)|^q}{4}+\frac{3|\Lambda'(\varpi)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Lambda'(\hbar)|^q}{4}+\frac{|\Lambda'(\varpi)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Also, if one takes $\mathrm{q}=1$ in (15), then one gets

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\leq\frac{\varpi-\hbar}{4}A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|).$$

This inequality coincides with the inequality in [2].

Corollary 15. If one takes $\mathrm{k}=0$ in (15), then

$$\begin{aligned}&\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\\ \leq\quad&\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}R_1(s)+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}R_2(s)\right)^{\frac{1}{q}}\\ &+\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}R_2(s)+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}R_1(s)\right)^{\frac{1}{q}}.\end{aligned}$$

This inequality coincides with the inequality in [25]. Also, if one takes $\mathrm{q}=1$ in the above inequality, then one gets

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\leq\frac{\varpi-\hbar}{n}\sum_{s=1}^{n}\frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|).$$

Corollary 16. If one takes $\mathrm{n}=1$ in (15), then

$$\begin{aligned}&\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\\ \leq\quad&\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q+3|\Lambda'(\varpi)|^q}{16}-\frac{k}{32}(\varpi-\hbar)^2\right)^{\frac{1}{q}}\\ &+\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{3|\Lambda'(\hbar)|^q+|\Lambda'(\varpi)|^q}{16}-\frac{k}{32}(\varpi-\hbar)^2\right)^{\frac{1}{q}}.\end{aligned}$$

Also, if one takes $\mathrm{q}=1$ in the above inequality, then one gets

$$\left|\frac{\Lambda(\hbar)+\Lambda(\varpi)}{2}-\frac{1}{\varpi-\hbar}\int_{\hbar}^{\varpi}\Lambda(\sigma)d\sigma\right|\leq\frac{\varpi-\hbar}{4}\left(A(|\Lambda'(\hbar)|,|\Lambda'(\varpi)|)-\frac{k}{8}(\varpi-\hbar)^2\right).$$

Remark 8. The inequality (15) gives better result than the inequality (13). Indeed, from the inequality $\upsilon^{\alpha}+\omega^{\alpha}\leq 2^{1-\alpha}(\upsilon+\omega)^{\alpha}, \upsilon,\omega\in[0,\infty)$, $0<\alpha\leq 1$, one gets

$$\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}R_1(s)+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}R_2(s)-\frac{k}{32}(\varpi-\hbar)^2\right)^{\frac{1}{q}}$$

$$+\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Lambda'(\hbar)|^q}{n}\sum_{s=1}^{n}R_2(s)+\frac{|\Lambda'(\varpi)|^q}{n}\sum_{s=1}^{n}R_1(s)-\frac{k}{32}(\varpi-\hbar)^2\right)^{\frac{1}{q}}$$

$$\leq\frac{\varpi-\hbar}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}[R_1(s)+R_2(s)]A(|\Lambda'(\hbar)|^q,|\Lambda'(\varpi)|^q)-\frac{k}{16}(\varpi-\hbar)^2\right)^{\frac{1}{q}},$$

where

$$R_1(s)+R_2(s)=\frac{s(s+2^{1/s})}{2^{1/s}(s+1)(2s+1)}$$

which completes the proof of remark.

5. AN APPLICATION FOR SPECIAL MEANS

Now, we will use the following notations for special means of two nonnegative numbers $\hbar,\varpi$ with $\varpi>\hbar$:

1. The arithmetic mean

$$A:=A(\hbar,\varpi)=\frac{\hbar+\varpi}{2},$$

2. The logarithmic mean

$$L:=L(\hbar,\varpi)=\begin{cases}\dfrac{\varpi-\hbar}{\ln\varpi-\ln\hbar}, & \hbar\neq\varpi\\ \hbar, & \hbar=\varpi\end{cases};\ \hbar,\varpi>0,$$

3. The p-logarithmic mean
4.

$$L_p:=L_p(\hbar,\varpi)=\begin{cases}\left(\dfrac{\varpi^{p+1}-\hbar^{p+1}}{(p+1)(\varpi-\hbar)}\right)^{\frac{1}{p}}, & \hbar\neq\varpi, p\in\mathbb{R}\setminus\{-1,0\}\\ \hbar, & \hbar=\varpi\end{cases};\ \hbar,\varpi>0.$$

Proposition 1. Let $\hbar,\varpi\in[-1,1]$ with $\hbar<\varpi$. Then, the following inequalities are obtained*:*

$$\frac{n}{2\sum_{s=1}^{n}\left(\frac{1}{2}\right)^{1/s}}\left[A^2(\hbar,\varpi)+\frac{k}{12}(\varpi-\hbar)^2\right]\leq L_2^2(\hbar,\varpi)\leq A(\hbar^2,\varpi^2)\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}-\frac{k}{6}(\varpi-\hbar)^2.$$

Proof: The assertion follows from the inequalities (10) for the function

$$\Lambda(\sigma) = \sigma^2, \sigma \in [-1,1].$$

6. CONCLUSION

In this article, the class of strongly n-fractional polynomial convex functions is introduced and related properties are given. Hermite-Hadamard inequalities for the newly defined class of functions are establihed. New refinements of the Hermite−Hadamard inequality, for functions whose first derivatives in absolute value at certain power, are strongly n-fractional polynomial convex. It is demonstrated that the newly obtained upper bounds give better results than the previous ones in the literature.

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