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OSCILLATION CRITERIA FOR THE COMPACTNESS OF RIEMANNIAN MANIFOLDS

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Abstract. *In this paper, we prove some generalizations of the Ambrose (or Myers) theorem for complete Riemannian manifolds. We observe that the problem of finding the Ricci curvature conditions that guarantee the compactness of the manifold is reduced to the problem of finding the proper oscillation conditions of second-order linear differential equations. The proof of the theorems is based on the Riccati comparison theorem and some related oscillation conditions.*

Keywords: *Compactness theorems; oscillation; Riccati comparison theorem.*

1. INTRODUCTION

In 1957, Ambrose [1] established a significant generalization of the classical Myers theorem [2]. According to this theorem, the condition $Ric(\gamma'(t),\gamma'(t)) \geq (n-1)k > 0$ is replaced by a condition involving the integral of the Ricci tensor along geodesics. This condition arises from the oscillation condition of second-order linear differential equations.

Theorem 1. (Ambrose) Suppose there exists a point p in a complete Riemannian manifold M for which every geodesic $\gamma(t)$ emanating from p satisfies

$$\int_0^\infty Ric(\gamma'(t),\gamma'(t))\,dt = \infty. \tag{1}$$

Then M is compact. The above theorem does not require the Ricci tensor to be everywhere nonnegative. Galloway [3] generalized Theorem 1 as follows:

Theorem 2. Suppose there is a point p in a complete Riemannian manifold M such that along each geodesic $\gamma(t):[0,\infty) \to M$ parametrized by arc length with $\gamma(0) = p$ the condition

$$\int_0^\infty t^\lambda\, Ric(\gamma'(t),\gamma'(t))\,dt = \infty \tag{2}$$

holds for some $0 \leq \lambda < 1$. Then M is compact.

In fact, many conditions on the Ricci tensor are closely related to the oscillation theory of second-order linear differential equations. We now consider the second-order linear differential equation

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$$\left(r_1(t)y'(t)\right)' + r_2(t)y(t) = 0, \tag{3}$$

where r_1 and r_2 are real-valued functions, with $r_1(t) > 0$ for all $t > 0$. The equation (3) is said to be *oscillatory* if it has a nontrivial solution that has an infinite number of zeros in the interval $(0, \infty)$. It is well known that if one solution of (3) is oscillatory, then all solutions hold this property [4].

To prove that a complete Riemannian manifold is compact, we need geometric assumptions based on the oscillation conditions of second-order linear differential equations. At this point, we shall present a few of the oscillation conditions that will be utilized in our main results.

Theorem 3. (Leighton, [5]) Let $r_1(t)$ and $r_2(t)$ be continuous functions with $r_1(t) > 0$ for $t \in (0, \infty)$. If both conditions

$$\int_a^\infty \frac{dt}{r_1(t)} dt = \infty \text{ and } \int_a^\infty r_2(t)dt = \infty \tag{4}$$

hold, then the differential equation (3) is oscillatory.

Theorem 4. (Moore, [6]) If the condition

$$\frac{1}{4} < c \le G(t) \le d < +\infty \text{ or } \frac{1}{4} < c \le H(t) \tag{5}$$

holds, then the differential equation (3) is oscillatory. Here, $G(t)$ and $H(t)$ are given by

$$G(t) = \left(1 + \int_a^t \frac{d\eta}{r_1(\eta)}\right) \int_t^\infty r_2(\eta)\, d\eta \text{ and } H(t) = \int_t^\infty \frac{d\eta}{r_1(\eta)} \int_a^t r_2(\eta)\, d\eta. \tag{6}$$

In addition, if $r_2(t) \ge 0$, the function $G(t)$ does not need to be bounded from above.

Theorem 5. (Nehari, [7]) The differential equation $y\,(t)'' + p(t)y(t) = 0$ is a non-oscillatory equation over the interval (a, ∞). If $\beta > 1$ and $0 \le \alpha < 1$, then

$$\begin{aligned} (t-a)^{1-\beta} \int_a^t (\eta - a)^\beta\, p(\eta) d\eta + (t-a)^{1-\alpha} \int_t^\infty (\eta - a)^\alpha\, p(\eta) d\eta \\ \le \frac{\beta - \alpha}{4}\left(1 + \frac{1}{(\beta - 1)(1 - \alpha)}\right) \end{aligned} \tag{7}$$

holds. Here, $p(t) > 0$.

Since the left-hand side of inequality (7) in the theorem above is not negative, this inequality can be considered as two separate inequalities. These are as follows:
For $\alpha = 0$:

$$(t-a)^{1-\beta} \int_a^t (\eta - a)^\beta\, p(\eta) d\eta \le \frac{\beta^2}{4(\beta - 1)} \tag{8}$$

and for $\beta = 2$:

$$(t-a)^{1-\alpha}\int_t^{\infty}(\eta-a)^{\alpha}\,p(\eta)d\eta \le \frac{(2-\alpha)^2}{4(1-\alpha)}. \tag{9}$$

Applying the above oscillation results to a complete Riemannian manifold M, we conclude that the manifold must be compact. In [8], Mastrolia et al. studied the behavior of solutions to a second-order differential equation. Thus, they obtained compactness results with the existence and localization of a zero.

For recent works on this subject, the reader is referred to [9-12] and references therein. The main results of this work are presented in the following sections.

2. COMPACTNESS RESULTS

The Riccati comparison theorem [13] is used in the proofs of the theorems.

Theorem 6. Let (M, g) be a complete and connected Riemannian manifold of dimension n. Suppose there exists a point $p \in M$ such that every geodesic $\gamma(t)$ emanating from p satisfies

$$\int_1^{\infty} e^{-2\sqrt{K}t}\,Ric\big(\gamma'(t),\gamma'(t)\big)\,dt = \infty, \tag{10}$$

where $K > 0$ is a constant. Then M is compact.

Proof: We assume that M is a non-compact Riemannian manifold and let $\gamma(t)$ be a unit speed ray starting from p. For every $t > 0$, $m(t)$ denotes the Laplacian of the distance function from a fixed point $p \in M$, i.e., $m(t) = \Delta\mathrm{r}$. Using the Bochner formula for the Laplacian of the distance function $m(t)$ along a unit speed geodesic $\gamma(t)$, and applying standard comparison techniques, we obtain the following Riccati-type differential inequality:

$$\frac{d}{dt}\left(\frac{m(t)}{n-1}\right)+\left(\frac{m(t)}{n-1}\right)^2+\frac{Ric\big(\gamma'(t),\gamma'(t)\big)}{n-1} \le 0. \tag{11}$$

Let us define a smooth function $F(t)$ by

$$F(t) := f(t)\frac{m(t)}{n-1} \tag{12}$$

for all $t > 0$, where $f \in C^{\infty}(M)$ is a strictly positive function. Then the derivation of $F(t)$ yields

$$F'(t) = f'(t)\frac{m(t)}{n-1} + f(t)\frac{d}{dt}\left(\frac{m(t)}{n-1}\right). \tag{13}$$

Combining (11) with (13), we have

$$F'(t)+f'(t)\left(\frac{m(t)}{n-1}\right)^2+f(t)\frac{Ric\big(\gamma'(t),\gamma'(t)\big)}{n-1}-f'(t)\frac{m(t)}{n-1}\leq 0. \tag{14}$$

The above inequality can be rewritten as

$$\frac{d}{dt}\left(F(t)-\frac{f'(t)}{2}\right)+\frac{1}{f(t)}\left(F(t)-\frac{f'(t)}{2}\right)^2+\psi(t)\leq 0, \tag{15}$$

here

$$\psi(t)=\frac{f''(t)}{2}-\frac{1}{4}\frac{(f'(t))^2}{f(t)}+\ f(t)\frac{Ric\big(\gamma'(t),\gamma'(t)\big)}{n-1}. \tag{16}$$

In inequality (15), if we take $\phi(t):=F(t)-\frac{f'(t)}{2}$, then we obtain the Riccati differential inequality

$$\phi'(t)+\frac{1}{f(t)}\phi^2(t)+\psi(t)\leq 0. \tag{17}$$

We now choose $f(t)=e^{-2\sqrt{K}t}>0$, where $K>0$. Substituting this choice into the equation (16), we obtain

$$\psi(t)=\ \left(K+\frac{Ric\big(\gamma'(t),\gamma'(t)\big)}{n-1}\right)e^{-2\sqrt{K}t}. \tag{18}$$

Then, it follows from (17) that

$$\phi'(t)+\frac{1}{e^{-2\sqrt{K}t}}\phi^2(t)+\left(K+\frac{Ric\big(\gamma'(t),\gamma'(t)\big)}{n-1}\right)e^{-2\sqrt{K}t}\leq 0. \tag{19}$$

On the other hand, we may consider the second-order linear differential equation

$$\left(e^{-2\sqrt{K}t}y'(t)\right)'+\left(K+\frac{Ric\big(\gamma'(t),\gamma'(t)\big)}{n-1}\right)e^{-2\sqrt{K}t}\mathrm{y(t)}=0. \tag{20}$$

Because of the assumption (10) of Theorem 6 and the fact that $\int_a^\infty \frac{dt}{e^{-2\sqrt{K}t}}dt=\infty$ holds, the oscillation conditions of Leighton given by Theorem 3 are satisfied. Thus, the differential equation (20) is oscillatory. Then, its solutions have an infinite number of zeros for $t>0$. Let be a nontrivial solution $y\colon[0,\infty)\to\mathbb{R}$ to (20) such that $y(t_1)=y(t_2)=0$ with $0<t_1<t_2$. On the other hand, for all $t\in(t_1,t_2)$ the function

$$\mu(t)=e^{-2\sqrt{K}t}\ \frac{y'(t)}{y(t)} \tag{21}$$

is the solution of the Riccati differential equation

$$\mu'(t)+\frac{1}{e^{-2\sqrt{K}t}}\mu^2(t)+\left(K+\frac{Ric\big(\gamma'(t),\gamma'(t)\big)}{n-1}\right)e^{-2\sqrt{K}t}=0. \tag{22}$$

To use the Riccati comparison theorem for the equations (19) and (22), we must note the following fact: Since every point on a geodesic $\gamma(t)$ is outside the cut locus, the distance function is differentiable along the geodesic $\gamma(t)$. Therefore, for $t_1\in(0,\infty)$

$$\begin{aligned}\phi(t_1)\coloneqq F(t_1)-\frac{f'(t_1)}{2}&=f(t_1)\frac{m(t_1)}{n-1}-\frac{f'(t_1)}{2}\\ &=e^{-2\sqrt{K}t_1}\,\frac{m(t_1)}{n-1}+\sqrt{K}e^{-2\sqrt{K}t_1}\\ &=constant\end{aligned} \tag{23}$$

and so

$$\lim_{t\to t_1^+}\phi(t)=\phi(t_1)=constant. \tag{24}$$

With the equality $y(t_1)=y(t_2)=0$ and using the definition of $\mu(t)$ given in (21) we get

$$\lim_{t\to t_1^+}\mu(t)=\infty. \tag{25}$$

From the above two inequalities, we obtain

$$\lim_{t\to t_1^+}\phi(t)\le\lim_{t\to t_1^+}\mu(t). \tag{26}$$

Thus, for a sufficiently small positive constant ε the inequality

$$\phi(t_1+\varepsilon)\le\mu(t_1+\varepsilon) \tag{27}$$

is ensured on $[t_1+\varepsilon,t_2)$. In that case, the Riccati comparison theorem gives an inequality

$$\phi(t)\le\mu(t) \tag{28}$$

on $[t_1+\varepsilon,t_2)$. As stated previously, every point on a geodesic $\gamma(t)$ is outside the cut locus. In this case, the function $\phi(t)$ is defined at the point t_2. Therefore, we have

$$\lim_{t\to t_2^-}\phi(t)=\phi(t_2)=constant \tag{29}$$

and also

$$\lim_{t\to t_2^-}\mu(t)=-\infty \tag{30}$$

by the definition of $\mu(t)$. However, this result contradicts the inequality (28) obtained from the Riccati comparison theorem. It seems that the Laplacian of the distance function blows up in finite time in the negative direction, so the Hessian also blows up in the same direction.

Hence, every geodesic ray emanating from $p = \gamma(0)$ has a point conjugate to p along ray. Then M is bounded and, hence, compact (see Lemma 1 in [1]). □

The following theorem is the second result of this paper.

Theorem 7. Let (M, g) be a complete and connected Riemannian manifold of dimension n. Suppose there exists a point $p \in M$ such that every geodesic $\gamma(t)$ emanating from p satisfies

$$\frac{1}{t}\int_1^t s^2 \frac{Ric\big(\gamma'(s), \gamma'(s)\big)}{n-1} ds \geq c > \frac{1}{4}. \tag{31}$$

Then M is compact.

Proof. Doing a similar computation, such as Theorem 6, and choosing $f(t) = t^2 > 0$ in expression (17) we achieve the following Riccati differential inequality

$$\phi'(t) + \frac{1}{t^2}\phi^2(t) + \frac{Ric\big(\gamma'(t), \gamma'(t)\big)}{n-1}\, t^2 \leq 0. \tag{32}$$

On the other hand, we may consider the second-order linear differential equation

$$\big(t^2 z'(t)\big)' + \frac{Ric\big(\gamma'(t), \gamma'(t)\big)}{n-1} t^2 z(t) = 0. \tag{33}$$

Because of the assumption (31) of Theorem 7, the oscillation conditions of Moore given by Theorem 4 are satisfied. Thus, the differential equation (33) is oscillatory. Then its solutions have an infinite number of zeros for $t > 0$. Let be a nontrivial solution $z: [0, \infty) \to \mathbb{R}$ to (33) such that $z(t_1) = z(t_2) = 0$ with $0 < t_1 < t_2$. On the other hand, for all $t \in (t_1, t_2)$ the function

$$\varphi(t) = t^2 \frac{z'(t)}{z(t)} \tag{34}$$

is the solution of the Riccati differential equation

$$\varphi'(t) + \frac{1}{t^2}\varphi^2(t) + \frac{Ric\big(\gamma'(t), \gamma'(t)\big)}{n-1} t^2 = 0. \tag{35}$$

The same arguments used in the proof of Theorem 6 show that for any $[t_1 + \varepsilon, t_2)$ the Riccati comparison theorem causes the inequality

$$\phi(t) \leq \varphi(t). \tag{36}$$

Since every point on a geodesic $\gamma(t)$ is outside the cut-locus, the function $\phi(t)$ is defined at the point t_2. Therefore, we have

$$\lim_{t \to t_2^-} \phi(t) = \phi(t_2) = constant \tag{37}$$

and also

$$\lim_{t\to t_2^-}\varphi(t) = -\infty \tag{38}$$

by the definition of $\varphi(t)$. However, this result contradicts the inequality (36) obtained from the Riccati comparison theorem. Due to the same reasons as in Theorem 6, the manifold is compact. □

Finally, using Theorem 5, we prove the following compactness theorem.

Theorem 8. Let (M, g) be a complete and connected Riemannian manifold of dimension n and $Ric \geq -(n-1)(t^2-1)g$. Suppose there exists a point $p \in M$ such that every geodesic $\gamma(t)$ emanating from p satisfies

$$\int_{t_0}^{\infty} t^{\lambda} e^{-t^2} \frac{Ric(\gamma'(s),\gamma'(s))}{n-1} dt > \frac{(2-\lambda)^2}{4(1-\lambda){t_0}^{1-\lambda}} - C > 0, \tag{39}$$

where $t_0 > 0, C > 0$ and $0 \leq \lambda < 1$. Then M is compact.

Proof. Choosing $f(t) = e^{-t^2}$ in expression (17), we obtain the following Riccati differential inequality

$$\phi'(t) + \frac{1}{e^{-t^2}}\phi^2(t) + \left(t^2 - 1 + \frac{Ric(\gamma'(t),\gamma'(t))}{n-1}\right)e^{-t^2} \leq 0, \tag{40}$$

where $t \in (1,\infty)$. Since $\frac{1}{e^{-t^2}}\phi^2(t) \geq \phi^2(t)$, the inequality (40) can be rewritten as

$$\phi'(t) + \phi^2(t) + \left(t^2 - 1 + \frac{Ric(\gamma'(t),\gamma'(t))}{n-1}\right)e^{-t^2} \leq 0. \tag{41}$$

On the other hand, we can consider the second-order linear differential equation

$$h''(t) + \left(t^2 - 1 + \frac{Ric(\gamma'(t),\gamma'(t))}{n-1}\right)e^{-t^2}h(t) = 0. \tag{42}$$

Because of the assumption (39) of Theorem 8 and $Ric \geq -(n-1)(t^2-1)g$, the oscillation conditions of Nehari given by Theorem 5 are satisfied. Thus, the differential equation (42) is oscillatory. Then its solutions have an infinite number of zeros for $t > 0$. Let be a nontrivial solution $h: [0,\infty) \to \mathbb{R}$ to (28) such that $h(t_1) = h(t_2) = 0$ with $0 < t_1 < t_2$. On the other hand, for all $t \in (t_1, t_2)$ the function

$$\sigma(t) = \frac{h'(t)}{h(t)} \tag{43}$$

is the solution of the Riccati differential equation

$$\sigma'(t) + \sigma^2(t) + \left(t^2 - 1 + \frac{Ric(\gamma'(t),\gamma'(t))}{n-1}\right)e^{-t^2} = 0. \tag{43}$$

At this stage, applying the same arguments used in the proofs of the above theorems, we obtain that the manifold is compact. □

3. CONCLUSIONS

In this paper, we extend several compactness theorems on complete Riemannian manifolds by applying oscillation criteria and Riccati comparison techniques. Our results demonstrate that the compactness of a manifold can be determined based on specific conditions of the Ricci curvature and oscillatory behavior of second-order differential equations.

A natural direction for future research is to explore the implications of these theorems in the setting of modified Ricci tensors. Investigating how the oscillation conditions adapt to different variations of the Ricci curvature, such as Bakry-Émery Ricci tensor and weighted manifolds, may lead to further generalizations of compactness criteria. In addition, considering these results in the context of non-smooth spaces or warped product structures can provide new insights into the geometric and analytical properties of manifolds.

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