ORIGINAL PAPER

NEW BOUNDS AND ESTIMATES FOR THE GAMMA FUNCTION

SHUN-WEI XU[1], CRISTINEL MORTICI[2,3,4,*], CHAO-PING CHEN[1]

Manuscript received: 29.07.2025; Accepted paper: 22.09.2025;
Published online: 30.09.2025.

Abstract. *The aim of this paper is to present new formulas for approximating the factorial function. Our formulas are of simple form and are more accurate than the Stirling's formula and other classical formulas.*

Keywords: *factorial function; gamma function; Stirling's formula; asymptotic series.*
Mathematics Subject Classification: *41A60; 11A55; 40A05.*

1. INTRODUCTION

Undoubtedly, one of the most known and most used formula for approximating the factorial function and its extension – the gamma function Γ – is the Stirling's formula:

$$\Gamma(x+1) \approx \sigma_x := \sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \text{ as } x \to \infty.$$

This elegant expression captures the growth behavior of the factorial function and serves as a bridge between discrete mathematics and continuous analysis. Its utility extends beyond mere approximation; it plays a crucial role in deriving limits, evaluating integrals, and analyzing the behavior of algorithms and distributions.

It seems that its simple form is one of the reasons that makes this formula survived so long. Many researchers provided plenty of extensions and refinements, but a sacrifice of simplicity.

We give in this paper a new formula for approximating the factorial function. Our formula we discuss here remains simple form but is more accurate than the Stirling's formula and other classical formulas. Precisely, we present the new formula:

$$\Gamma(x+1) \approx \mu_x := \sqrt{2\pi}\left(\frac{x^2+x+\frac{1}{6}}{e^2}\right)^{x+\frac{1}{4}}, \text{ as } x \to \infty. \tag{1}$$

Next is a comparison table which shows the superiority of our new formula $\Gamma(x+1) \approx \mu_x$ over Stirling's formula $\Gamma(x+1) \approx \sigma_x$.

[1] Henan Polytechnic University, School of Mathematics and Information Science, 454003 Jiaozuo, China.
E-mail: xswmath@163.com; chenchaoping@sohu.com.
[2] Valahia University of Targoviste, 130004 Targoviste, Romania.
[3] National University of Science and Technology Politehnica Bucuresti, 060042 Bucharest, Romania.
[4] Academy of Romanian Scientists, 050044 Bucharest, Romania.
[*] *Corresponding author: cristinel.mortici@hotmail.com.*

https://doi.org/10.46939/J.Sci.Arts-25.3-a15

Table 1. The superiority of the new formula over Stirling's formula.

x	$\ln(\Gamma(x+1)/\sigma_x)$	$\ln(\Gamma(x+1)/\mu_x)$
10	8.3306×10^{-3}	3.5941×10^{-6}
100	8.3333×10^{-4}	4.1047×10^{-9}
500	1.6667×10^{-4}	3.3234×10^{-11}
1000	8.3333×10^{-5}	4.1604×10^{-12}
2500	3.3333×10^{-5}	2.6651×10^{-13}

We also give a rigorous proof of this fact.

The idea of discovering our formula started from the following equivalent representation of the Stirling's formula:

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n^2}{e^2}\right)^{\frac{n}{2}}, \text{ as } n \to \infty.$$

More exactly, we consider the following family of approximations:

$$\Gamma(n+1) \approx \alpha\left(\frac{n^2+an+b}{e^2}\right)^{\frac{n}{2}+c}, \text{ as } n \to \infty,$$

depending on real parameters a, b, c and $\alpha > 0$. We show that in some sense, the most accurate approximation is obtained for

$$\alpha = \sqrt{2\pi},\ a = 1,\ b = \frac{1}{6},\ c = \frac{1}{4}.$$

2. MOTIVATION

Usually, to an approximation of the form

$$f(n) \approx g(n), \text{ as } n \to \infty \tag{2}$$

(in the sense that $\lim_{n\to\infty} f(n)/g(n) = 1$) we associate the sequence

$$\omega_n = \ln\frac{f(n)}{g(n)}$$

and we say that the approximation formula (2) is better, when the sequence ω_n faster converges to zero.

The speed of convergence of the sequence ω_n can be expressed by using the following result first stated in this form by Mortici [9]:

Lemma 1. If the sequence ω_n converges to zero and

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l,$$

for some $k > 1$, then

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}. \tag{3}$$

This Lemma was proved to be an useful tool for accelerating some convergences, or to discover new approximation formulas. See, e.g., [2-15]. According to relation (3), the sequence ω_n converges to zero as $n^{-(k-1)}$ and in this case, we say that the approximation formula (1) is of order $n^{-(k-1)}$. As an example, the associated sequence to the Stirling's formula is:

$$\omega_n = \ln\frac{\Gamma(n+1)}{\sqrt{2\pi}\left(\frac{n}{e}\right)^n} = \ln\Gamma(n+1) - \frac{1}{2}\ln 2\pi - n\ln n + n.$$

By using the asymptotic formula:

$$\ln\Gamma(n+1) \sim \frac{1}{2}\ln 2\pi + \left(n+\frac{1}{2}\right)\ln n - n + \frac{1}{12n} - \frac{1}{360n^3} + \cdots, \text{ as } n\to\infty,$$

(see, e.g., [1, Rel. 6.1.41], it results that ω_n converges to zero as n^{-1}.

In consequence, the Stirling's formula is an approximation of order n^{-1}. We show that our new formula is much more accurate, being of order n^{-3}.

3. THE RESULTS

Now let us denote by

$$w_n = \ln\Gamma(n+1) - \ln\alpha - \left(\frac{n}{2}+c\right)\ln\frac{n^2+an+b}{e^2}$$

the associated sequence of the approximation (1). Thus

$$w_n - w_{n+1} = -\ln(n+1) - \left(\frac{n}{2}+c\right)\ln\frac{n^2+an+b}{e^2} + \left(\frac{n+1}{2}+c\right)\ln\frac{(n+1)^2+a(n+1)+b}{e^2}.$$

After some direct computations, using the standard series of logarithm function, or simpler by using the Maple software, we deduce that:

$$w_n - w_{n+1} = \left(2c-\frac{1}{2}\right)\frac{1}{n} - \left(-\frac{1}{4}a^2 + ca + \frac{1}{2}b + c - \frac{1}{3}\right)\frac{1}{n^2} - \left(2bc - \frac{2}{3}c - a^2c - ab - ac - \frac{1}{2}b + \frac{1}{4}a^2 + \frac{1}{3}a^3 + \frac{1}{4}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

According to Lemma, the fastest sequence w_n is obtained together to the fastest possible sequence $w_n - w_{n+1}$. This case is obtained when the first three coefficients in the asymptotic expansion of $w_n - w_{n+1}$ vanish. Namely, the variables a, b, c should be any solution of the system:

$$\begin{cases} 2c - \frac{1}{2} = 0 \\ -\frac{1}{4}a^2 + ca + \frac{1}{2}b + c - \frac{1}{3} = 0 \\ 2bc - \frac{2}{3}c - a^2c - ab - ac - \frac{1}{2}b + \frac{1}{4}a^2 + \frac{1}{3}a^3 + \frac{1}{4} = 0 \end{cases}.$$

This system has three solutions:

$$a = 1, \quad b = \frac{1}{6}, \quad c = \frac{1}{4};$$

$$a = 1 + \frac{\sqrt{2}}{2}, \quad b = \frac{5}{12} + \frac{\sqrt{2}}{4}, \quad c = \frac{1}{4}$$

and

$$a = 1 - \frac{\sqrt{2}}{2}, \quad b = \frac{5}{12} - \frac{\sqrt{2}}{4}, \quad c = \frac{1}{4};$$

As we are interested in finding an approximation formula of simple form, we will discuss next only the rational solution $a = 1$, $b = 1/6$, $c = 1/4$. From the natural condition

$$\lim_{n\to\infty} \frac{\ln \Gamma(n+1)}{\alpha \left(\frac{n^2 + n + \frac{1}{6}}{e^2} \right)^{\frac{n}{2} + \frac{1}{4}}} = 1,$$

we deduce that $\alpha = \sqrt{2\pi}$. Now the formula (1) is completely justified. The associated sequence

$$w_n = \ln \frac{\Gamma(n+1)}{\sqrt{2\pi} \left(\frac{n^2 + n + \frac{1}{6}}{e^2} \right)^{\frac{n}{2} + \frac{1}{4}}}$$

has the difference sequence

$$w_n - w_{n+1} = \frac{1}{80n^4} - \frac{1}{20n^5} + \frac{599}{4536n^6} + O\left(\frac{1}{n^7}\right).$$

We have

$$\lim_{n\to\infty} n^4 (w_n - w_{n+1}) = \frac{1}{80}$$

and by Lemma 1,

$$\lim_{n\to\infty} n^3 w_n = \frac{1}{240}.$$

In consequence, our new approximation (1) is of order n^{-3}.

4. A MONOTONE FUNCTION AND SOME INEQUALITIES

In this section we need the following under- and upper approximations of the trigamma function:

$$a(x) < \psi'(x) < b(x) \tag{4}$$

where

$$a(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5},$$
$$b(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}.$$

Recall that the digamma function is the logarithmic derivative of the Euler's gamma function:

$$\psi(x) = (\ln \Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}, \ \ x > 0.$$

The well-known inequalities (4) are obtained by truncation the classical asymptotic series of the trigamma function:

$$\psi'(x) \sim \sum_{k=0}^{\infty} \frac{B_{2k}}{x^{2k+1}}, \ \ x \to \infty$$

($B_j' - s$ are the Bernoulli's numbers). For details see, e.g., [1].

We are in a position to give the following:

Theorem 1.
a) The function

$$s(x) = \ln \frac{\Gamma(x+1)}{\sqrt{2\pi}\left(\frac{x^2 + x + \frac{1}{6}}{e^2}\right)^{\frac{x}{2}+\frac{1}{4}}}$$

is strictly decreasing on $[0, \infty)$.

b) The following inequality holds true, for all x ≥ 0:

$$\lambda \cdot \sqrt{2\pi}\left(\frac{x^2 + x + \frac{1}{6}}{e^2}\right)^{\frac{x}{2}+\frac{1}{4}} < \ln \Gamma(x+1) \leq \ \mu \cdot \sqrt{2\pi}\left(\frac{x^2 + x + \frac{1}{6}}{e^2}\right)^{\frac{x}{2}+\frac{1}{4}},$$

where the constants $\lambda = 1$ and $\mu = 6^{\frac{1}{4}} \cdot \sqrt{\frac{e}{2\pi}} = 1.0294$ ... are sharp.

Proof: a) We have

$$s(x) = \ln\Gamma(x+1) - \frac{1}{2}\ln 2\pi - \left(\frac{x}{2}+\frac{1}{4}\right)\ln\frac{x^2+x+\frac{1}{6}}{e^2},$$

so

$$s''(x) = \psi'(x+1) - \frac{18x(2x^2+3x+1)}{(6x^2+6x+1)^2}.$$

Using (4), we obtain:

$$s''(x) > a(x+1) - \frac{18x(2x^2+3x+1)}{(6x^2+6x+1)^2} = \frac{223x+362x^2+243x^3+54x^4+49}{30(6x+6x^2+1)^2(x+1)^5} > 0.$$

It follows that $s'(x)$ is strictly increasing, with $\lim_{x\to\infty} s'(x) = 0$, so $s'(x) < 0$ on $[0,\infty)$. Thus $s(x)$ is strictly decreasing.

b) The required inequality follows by exponentiating the inequality $s(\infty) < s(x) \le s(0)$, where $s(\infty) = 0$ and

$$s(0) = \frac{1}{4}\ln 6 + \frac{1}{2}\ln\frac{e}{2\pi}.$$

The proof is completed. By using the Maple software, we found that

$$w_n = \frac{1}{240n^3} - \frac{1}{160n^4} + \frac{253}{45360n^5} + O\left(\frac{1}{n^6}\right).$$

We are entitled to give the following:

Theorem 2. The following inequalities hold true, for all $x \ge 1$:

$$\sqrt{2\pi}\left(\frac{x^2+x+\frac{1}{6}}{e^2}\right)^{\frac{x}{2}+\frac{1}{4}} \cdot \exp\left(\frac{1}{240x^3} - \frac{1}{160x^4}\right) \le \Gamma(x+1)$$

$$\le \sqrt{2\pi}\left(\frac{x^2+x+\frac{1}{6}}{e^2}\right)^{\frac{x}{2}+\frac{1}{4}} \cdot \exp\frac{1}{240x^3}.$$

Proof: To prove Theorem 2, it suffices to show that $u(x) < 0$ and $v(x) > 0$, for $x \ge 1$, where

$$u(x) = \ln\Gamma(x+1) - \frac{1}{2}\ln 2\pi - \left(\frac{x}{2}+\frac{1}{4}\right)\ln\frac{x^2+x+\frac{1}{6}}{e^2} - \frac{1}{240x^3}$$

and

$$v(x) = \ln\Gamma(x+1) - \frac{1}{2}\ln 2\pi - \left(\frac{x}{2}+\frac{1}{4}\right)\ln\frac{x^2+x+\frac{1}{6}}{e^2} - \left(\frac{1}{240x^3} - \frac{1}{160x^4}\right).$$

Note that

$$\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = 0,$$

it suffices to show that the function $u(x)$ strictly increasing and $v(x)$ is strictly decreasing for $x \geq 1$. Direct computation yields

$$u'(x) = \psi(x+1) - \frac{1}{2}\ln\frac{x^2+x+\frac{1}{6}}{e^2} - \frac{480x^6+480x^5+120x^4-6x^2-6x-1}{89(6x^2+6x+1)x^4}$$

and

$$v'(x) = \psi(x+1) - \frac{1}{2}\ln\frac{x^2+x+\frac{1}{6}}{e^2} - \frac{480x^7+480x^6+120x^5-6x^3+6x^2+11x+2}{89(6x^2+6x+1)x^4}.$$

Using the following inequalities:

$$\ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} < \psi(x+1) < \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8}, \quad x > 0,$$

we obtain

$$u'(x) > \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} - \frac{1}{2}\ln\frac{x^2+x+\frac{1}{6}}{e^2} - \frac{480x^6+480x^5+120x^4-6x^2-6x-1}{89(6x^2+6x+1)x^4} := U(x)$$

and

$$u'(x) < \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{2}\ln\frac{x^2+x+\frac{1}{6}}{e^2} - \frac{480x^7+480x^6+120x^5-6x^3+6x^2+11x+2}{89(6x^2+6x+1)x^4} := V(x).$$

Direct computation yields

$$U'(x) = -\frac{a(x)}{87x^7(6x^2+6x+1)^2} < 0, \quad V'(x) = \frac{b(x)}{840x^9(6x^2+6x+1)^2} > 0, \quad x \geq 1,$$

where

$$a(x) = 453 + 2508(x-1) + 5011(x-1)^2 + 4720(x-1)^3 + 2140(x-1)^4 + 378(x-1)^5$$

and

$$b(x) = 8483 + 55131(x-1) + 132815(x-1)^2 + 160497(x-1)^3 + 105242(x-1)^4 + 36000(x-1)^5 + 5060(x-1)^6.$$

Hence, the function $U(x)$ is strictly decreasing and $V(x)$ is strictly increasing for $x \geq 1$, and we have, for $x \geq 1$,

$$U(x) \geq \lim_{x \to 1} U(x) = 0.0084752 \ldots > 0 \quad \text{and} \quad V(x) \leq \lim_{x \to 1} V(x) = -0.1235806 \ldots < 0.$$

Thus, we obtain that

$$u'(x) > 0 \quad \text{and} \quad v'(x) < 0 \quad \text{for} \quad x \geq 1.$$

The proof is completed.

REFERENCES

[1] Abramowitz, M., I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series, 55, Washington, 1964.
[2] Chen, C. P., Qi, F., *Tamkang Journal of Mathematics*, **36**(4), 303, 2005.
[3] Chen, C. P., Qi, F., *Proceedings of the American Mathematical Society*, **133**(2), 397, 2005.
[4] Chen, C. P., Elezovic, N., Vuksic, L., *Journal of Classical Analysis*, **2,** 151, 2013.
[5] Chen, C. P., Mortici, C., *Research in the Mathematical Sciences*, **80,** 35, 2025.
[6] Chen, C. P., Mortici, C., *Journal of Science and Arts*, **24**(4), 775, 2024.
[7] Mortici, C., *American Mathematical Monthly*, **117**(5), 434, 2010.
[8] Mortici, C., Jang, X. J., *Filomat*, **7**, 1535, 2015.
[9] Mortici, C., *American Mathematical Monthly*, **117**(5), 434, 2010.
[10] Mortici, C., Cristea, V.C., Lu. D, *Applied Mathematics and Computation*, **240**, 168, 2014.
[11] Mortici, C., *Ramanujan Journal*, **38**(3), 549, 2015.
[12] Mortici, C., *Mathematical and Computer Modelling*, **51**(9-10), 1154, 2010.
[13] Mortici, C., *Computational and Applied Mathematics*, **29**(3), 479, 2010.
[14] Mortici, C., *Ramanujan Journal*, **26**(2), 185, 2011.
[15] Mortici, C., *Ramanujan Journal*, **38**(3), 549, 2015.