ORIGINAL PAPER

IMPROVING THE CLASSICAL CONVERGENCE TO EULER-MASCHERONI SEQUENCE BY MODIFYING THE MIDDLE TERM

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Abstract. *The aim of this paper is to introduce a new sequence that converges to Euler-Mascheroni constant. Some inequalities and asymptotic formulas are provided. Finally, some numerical comparisons are given.*
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1. INTRODUCTION

The Euler-Masheroni constant γ is defined as the limit of the sequence

$$\gamma_n = h_n - \ln n,$$

where

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is the harmonic sum. Numerically, we have: $\gamma = 0.5772156649 \ldots$.

This is an intriguing constant in mathematics. In contrast to other classical constants such as e, or π, it is not even known whether it is rational or not. It seems that the reason why a proof of this fact has not yet been discovered is that no sequences sufficiently rapidly converging to the constant γ have been found.

Consequently, in the recent past, many authors were preoccupied in finding new, fast convergencies to γ.

Drinceanu and Mortici [1] have introduced the sequence

$$\sum_{k=1}^{n}\left(\frac{1}{k+\frac{1}{2}} - \ln\frac{k}{k+1}\right).$$

Mortici [2] modified the last terms of γ_n to introduce the sequences:

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{(6-2\sqrt{6})n} - \ln\left(n + \frac{1}{\sqrt{6}}\right)$$

and

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$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{(6+2\sqrt{6})n} - \ln\left(n - \frac{1}{\sqrt{6}}\right).$$

Both sequences u_n and v_n were shown to converge to γ as n^{-2}. Other results were published in [3-16].

We use here the idea from [2] to define and discuss the sequence:

$$\rho_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n + \frac{1}{96}} + \frac{1}{n+1} + \cdots + \frac{1}{2n} - \ln\left(2n + \frac{1}{2}\right).$$

The sequence ρ_n is obtained by modifying the middle term $1/n$ in the classical sequence λ_{2n} to $1/(n + 1/96)$ and the logarithm term $\ln 2n$ to $\ln(2n + 1/2)$.

2. A FAMILY OF CONVERGENCIES TO EULER-MASCHERONI CONSTANT

We introduce the family of sequences

$$x_n = x_n(a,b) \coloneqq 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n+a} + \frac{1}{n+1} + \cdots + \frac{1}{2n} - \ln(2n+b),$$

where a, b are real parameters. Remark that $x_n(0,0) = \gamma_{2n}$.

We are interested in finding the values of the parameters a, b such that the sequence x_n goes to γ with the fastest speed of convergence.

An important tool for measuring the speed of convergence of a sequence is the following consequence of the Stolz-Cesaro lemma:

Lemma 1. Let x_n be a sequence convergent to zero such that

$$\lim_{n\to\infty} n^k (x_n - x_{n+1}) = l,$$

for some real parameter $k > 1$. Then

$$\lim_{n\to\infty} n^{k-1} x_n = \frac{l}{k-1}.$$

In particular, if $l \neq 0$, then this Lemma says that if $x_n - x_{n+1}$ converges to zero as n^{-k}, then x_n converges to zero as $n^{-(k-1)}$.

Lemma 1 is of great importance in accelerating some convergencies, or to obtain new approximation formulas. For details and further properties, please see [10].

In our case, we have

$$x_n - x_{n+1} = -\frac{1}{n} - \frac{1}{n+1+a} + \frac{1}{n+1} + \frac{1}{n+a} - \frac{1}{2n+1} - \frac{1}{2n+2} - \ln\left(2n + \frac{1}{2}\right) + \ln\left(2n + 2 + \frac{1}{2}\right).$$

By using the standard expansion series of the logarithm function, or directly, by using the Maple software for symbolic computations, we deduce that:

$$x_n - x_{n+1} = -\frac{2b-1}{4n^2} - \frac{48a - 12b - 6b^2 + 7}{24n^3} + \frac{48a - 8b + 48a^2 - 6b^2 - 2b^3 + 5}{16n^4} + O\left(\frac{1}{n^5}\right).$$

We see that the more of the first terms in this expansion vanish, the better convergence speed of the sequence $x_n - x_{n+1}$ is obtained. The fastest possible sequence is obtained for parameters a, b solution of the system:

$$\begin{cases} 2b - 1 = 0 \\ 48a - 12b - 6b^2 + 7 = 0 \end{cases},$$

that is

$$a = \frac{1}{96}, b = \frac{1}{2}.$$

We can give the following general

Theorem 1. a) If $\mathrm{b} \neq 1/2$, then $\mathrm{x_n}$ converges to zero as n^{-1}, since

$$\lim_{n\to\infty} n^2(x_n(a,b) - x_{n+1}(a,b)) = -\frac{2b-1}{4} \ and \ \lim_{n\to\infty} n x_n(a,b) = -\frac{2b-1}{4} \neq 0.$$

b) If $\mathrm{b} = 1/2$ and $\mathrm{a} \neq 1/96$, then $\mathrm{x_n}(\mathrm{a}, 1/2)$ converges to zero as n^{-2}, since

$$\lim_{n\to\infty} n^3\left(x_n\left(a,\frac{1}{2}\right) - x_{n+1}\left(a,\frac{1}{2}\right)\right) = \frac{1}{48} - 2a \ and \ \lim_{n\to\infty} n^2 x_n\left(a,\frac{1}{2}\right) = \frac{1}{96} - a \neq 0.$$

c) If $\mathrm{b} = 1/2$ and $\mathrm{a} = 1/96$, then $\mathrm{x_n}(1/96, 1/2)$ converges to zero as n^{-3}, since

$$\lim_{n\to\infty} n^4\left(x_n\left(\frac{1}{96},\frac{1}{2}\right) - x_{n+1}\left(\frac{1}{96},\frac{1}{2}\right)\right) = -\frac{47}{3072} \ and \ \lim_{n\to\infty} n^3 x_n\left(\frac{1}{96},\frac{1}{2}\right) = -\frac{47}{9216}.$$

3. THE ASYMPTOTIC EXPANSION OF ρ_n

As it is known, the harmonic sequence

$$h_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is closely related to the digamma function ψ, i.e., the logarithmic derivative of the Euler-gamma function:

$$\psi(x) = (\ln \Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}, x > 0.$$

We have $\psi(1) = -\gamma$ and

$$h_n = \gamma + \psi(n) + \frac{1}{n}, n \geq 2.$$

The digamma function has the following asymptotic representation, as $x \to \infty$:

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{j=1}^{\infty} \frac{B_{2j}}{2jx^{2j}} = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots . \tag{1}$$

$B_k's$ are the Bernoulli numbers. For details, see [3].
For the sequence

$$\rho_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n + \frac{1}{96}} + \frac{1}{n+1} + \cdots + \frac{1}{2n} - \ln\left(2n + \frac{1}{2}\right),$$

we have:

$$\begin{aligned} \rho_n &= h_{2n} + \frac{1}{n + \frac{1}{96}} - \frac{1}{n+1} - \ln\left(2n + \frac{1}{2}\right) \\ &= \gamma + \psi(2n) + \frac{1}{2n} + \frac{1}{n + \frac{1}{96}} - \frac{1}{n+1} - \ln\left(2n + \frac{1}{2}\right). \end{aligned} \tag{2}$$

By taking $x = 2n$ in (1), we get:

$$\begin{aligned} \psi(2n) &\sim \ln 2n - \frac{1}{4n} - \sum_{j=1}^{\infty} \frac{B_{2j}}{2^{2j+1} j n^{2j}} \\ &= \ln 2n - \frac{1}{4n} - \frac{1}{48n^2} + \frac{1}{1920n^4} - \frac{1}{1628n^6} + \cdots . \end{aligned} \tag{3}$$

By using also the series

$$\ln\left(2n + \frac{1}{2}\right) = \ln 2n + \frac{1}{4n} - \frac{1}{32n^2} + \frac{1}{96n^3} - \frac{1}{128n^4} + \cdots$$

and

$$\frac{1}{n+r} = \frac{1}{n} - \frac{r}{n^2} + \frac{r^2}{n^3} - \frac{r^3}{n^4} + \cdots, r \in \mathbb{R},$$

we get:

$$\rho_n \sim \gamma + \frac{1}{n^2} - \frac{9263}{9216n^3} + \frac{4430299}{4423680n^4} + \frac{18211}{18432n^5} - \frac{285436179168}{285380444160n^6} + O\left(\frac{1}{n^7}\right).$$

A complete asymptotic series in terms of Bernoulli's numbers can be also given.

4. SOME ESTIMATES FOR ρ_n

We give the following:

Theorem 2. The following inequalities hold true, for every integer n ≥ 1281:

$$\gamma + \frac{1}{n^2} - \frac{9263}{9216n^3} + \frac{\alpha}{n^4} < \rho_n < \gamma + \frac{1}{n^2} - \frac{9263}{9216n^3} + \frac{\beta}{n^4},$$

where

$$\alpha = 1 \text{ and } \beta = \frac{4430299}{4423680} = 1.0015 \ldots$$

(the right-hand side inequality holds true for every integer n ≥ 1).

Proof: It is well-known that by truncation the asymptotic series (3), under- and upper-approximations are obtained. We use the following:

$$\ln 2n - \frac{1}{4n} - \frac{1}{48n^2} < \psi(2n) < \ln 2n - \frac{1}{4n} - \frac{1}{48n^2} + \frac{1}{1920n^4}.$$

By replacing these estimates in (2), we deduce that

$$\left(\ln 2n - \frac{1}{4n} - \frac{1}{48n^2}\right) + \frac{1}{2n} + \frac{1}{n + \frac{1}{96}} - \frac{1}{n+1} - \ln\left(2n + \frac{1}{2}\right) < \rho_n - \gamma$$
$$< \left(\ln 2n - \frac{1}{4n} - \frac{1}{48n^2} + \frac{1}{1920n^4}\right) + \frac{1}{2n} + \frac{1}{n + \frac{1}{96}} - \frac{1}{n+1} - \ln\left(2n + \frac{1}{2}\right).$$

Now it suffices to prove that $u(n) < 0$ and $v(n) > 0$, where:

$$u(n) = \left(\frac{1}{n^2} - \frac{9263}{9216n^3} + \frac{1}{n^4}\right)$$
$$- \left(\left(\ln 2n - \frac{1}{4n} - \frac{1}{48n^2}\right) + \frac{1}{2n} + \frac{1}{n + \frac{1}{96}} - \frac{1}{n+1} - \ln\left(2n + \frac{1}{2}\right)\right)$$

and

$$v(n) = \left(\frac{1}{n^2} - \frac{9263}{9216n^3} + \frac{4430299}{4423680n^4}\right)$$
$$- \left(\left(\ln 2n - \frac{1}{4n} - \frac{1}{48n^2} + \frac{1}{1920n^4}\right) + \frac{1}{2n} + \frac{1}{n + \frac{1}{96}} - \frac{1}{n+1} - \ln\left(2n + \frac{1}{2}\right)\right)$$

We have:

$$u'(n) = \frac{P(n - 1281)}{3072n^5(4n+1)(n+1)^2(96n+1)^2} > 0$$

and

$$v'(n) = \frac{Q(n)}{221184n^5(4n+1)(n+1)^2(96n+1)^2} < 0,$$

where

$$P(n) = 441856n^5 + 2264749436n^4 + 4353\,285594903\,n^3 + 3719610\,438960439n^2$$
$$+ 1192537757\,849212232n + 555358623\,998399148$$

and

$$Q(n) = 174681666n + 9058259079n^2 + 43\,672503236n^3$$
$$+40\,776597024n^4 + 885599.$$

It follows that u is strictly increasing and v is strictly decreasing. Moreover, using

$$\lim_{n\to\infty} u(n) = \lim_{n\to\infty} v(n) = 0,$$

we deduce that $u < 0$ and $v > 0$ and the proof is completed.

5. NUMERICAL COMPARISON

We give in this section a numerical table that shows the superiority of our new sequence ρ_n over the sequences u_n and v_n:

n	$\gamma - \rho_n$	$\gamma - u_n$	$v_n - \gamma$
11	3.7306×10^{-6}	1.6013×10^{-5}	1.8096×10^{-5}
100	5.0849×10^{-9}	2.2528×10^{-8}	2.2833×10^{-8}
250	3.2601×10^{-10}	1.4476×10^{-9}	1.4555×10^{-9}
500	4.0775×10^{-11}	1.8120×10^{-10}	1.8169×10^{-10}
10000	5.0997×10^{-15}	2.2679×10^{-14}	2.2682×10^{-14}

We can see that the sequence ρ_n is of one order higher than u_n and v_n.

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