ORIGINAL PAPER

EVOLUTION AND INEXTENSIBILITY OF TIMELIKE CURVES IN MINKOWSKI 3-SPACE: A POSITIONAL ADAPTED FRAME APPROACH

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Abstract. *This paper investigates the inextensible flow of timelike curves in three-dimensional Minkowski space. We first establish the evolution equation of a timelike curve using a positively oriented orthonormal moving frame, under the condition that the momentum vector is non-vanishing. Next, we derive the condition for inextensibility and use it to obtain the evolution equations of the positional adapted frame and its curvatures. Finally, we present explicit mathematical formulas characterizing the requirements for a timelike curve to admit an inextensible flow in Minkowski space.*
Keywords: *Inextensible flow; positional adapted frame; Minkowski space.*

1. INTRODUCTION

The study of geometric flows has a long history and remains an active area of research. Flows of curves and surfaces arise naturally in physics, chemistry, and biology, where they describe diverse nonlinear phenomena. Moreover, this concept is mainly used in the study of dynamic systems, applied mathematical modelling, vibration analysis. In this context, an inextensible flow refers to a curve evolution in which the velocity vector's magnitude (or equivalently, the arc-length) remains constant. The notion was first introduced by Sasai (1984) in the framework of modified orthogonal systems and has since found applications in curve and surface evolution problems, including computer vision and image processing. There are those in the scientific community who have carried out a variety of in-depth investigations on this subject as a flow should not expand along its curve if its velocity vector is inextensible. For example, Latifi examined inextensible flow of curves in Minkowski 3-space. A partial differential equation including curvature and torsion is used to explain what is needed and enough for an inextensible curve flow [1]. Gökmen has been the focus of research about the formulation of the inextensible flow of curves in Euclidean space, and it provided the necessary criteria for how an inextensible curve can be defined in [2]. Baş provided the required circumstances for inextensible flows of that type of curve in [3]. Bartels studied inextensible flow (IF) in order to approximate the elastic flow of inextensible curves, a numerical scheme is developed, and it is demonstrated that approximations converge to accurate solutions of the nonlinear time-dependent partial differential equation in [4]. Yıldız, in [5], investigated inextensible curve flows (IFC) in three-dimensional Lie groups where the curvatures are involved in a partial differential equation that expressed the necessary and sufficient circumstances for IFC. In Gaber [6], the motion of curves in three-dimensional spherical space S^3 is studied and the evolution equations for curvatures as well as the

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evolution equations of the orthonormal frame are obtained and additionally, determined the curves from their fundamental equations and provided some clear examples of motions of inextensible curves in S^3.In Körpınar [7], has examined and present a new method for IF of timelike curves in a 4-dimensional LP-Sasakian manifold that is conformally symmetric, quasi-conformally flat, and conformally straight, and employed Fermi-Walker parallellism in space to calculate some IF in [8]. Then, Körpınar subsequently discovered a Fermi-Walker derivative in the trajectory of a particle with charge under the influence of a field of magnetization considering inextensible flows in $\mathbb{R}_1^3$, examined normal spherical indicatrices (images) in his work [9], explored the typical spherical indicatrices' geometric characteristics. Moreover, provided some fresh definitions of curvatures using a few partial differential equations in $\mathbb{R}_1^3$. Yüzbaşı categorized and specified lightlike ruled surfaces and described the inextensible evolution of a lightlike curve on a lightlike tangent developable surface in [10]. Li has used Hamilton's principle to construct governing equations for three-dimensional in nature of fluid-conveying pipes with various beginning configurations using the Green-Lagrange strain tensor, extensible theory, and plug-flow [11]. Along a space curve that is the component of an alternative frame in [12], Savić and Eren showed the evolutions of ruled surfaces produced by the fundamental normal, the fundamental normal's derivative, and the Darboux vector fields.

Recent works reveal that knowledge on inextensible flows keeps to expand swiftly, with more diversified geometric settings and analytical methodologies contributing to the continued development of the area. Bartels investigated the inextensible flows of normal magnetic particles in space, obtaining several new results and illustrating their main findings with appropriate examples [13]. Furthermore, the inextensible flow of a curve on S^2 was analyzed using a modified orthogonal Saban frame, where the authors first introduced this frame and established its relations with the classical Frenet frame [14]. Yakut subsequently characterized the inextensible curve flow on the unit sphere and derived the corresponding geodesic curvature within this framework. Additionally, inextensible flows of curves in four-dimensional pseudo Galilean space are expressed by Almaz and Öztekin, and the necessary and sufficient conditions of these curve flows are given as partial differential equations [15]. Also, the directional derivatives are defined in accordance with the Serret–Frenet frame in G_1^4, the extended Serret–Frenet relations are expressed by using Frenet formulas. Furthermore, the bending elastic energy functions are characterized for the same particle according to curve $\alpha(s,t)$. In addition, in the present work, Gaber focused on studying the evolution of null Cartan and pseudo null curves using the Bishop frame in Minkowski space $\mathbb{R}_1^2$ [16] and obtained the necessary and sufficient conditions for the null Cartan and pseudo null curves to be inextensible curves (the arc-length is preserved). In addition, Gaber derived the time evolution equations of the Bishop frame (TEEsBF) for these curves. Eren evaluated a combined inextensible evolutions of partner-ruled surfaces produced by vector pairs selected from an evolving space curve's tangent, primary normal, binormal, and Darboux vectorsin manuscript [17]. For similar studies by the author, please see [18–21]. By means of these constructed structures, the articulated mechanisms can be created for complex activities, since a pair of solid links of an arm move out two ruled surfaces simultaneously while twisting or translating according to a specific rule. In the contemporary literature, inextensible flows have been examined across a variety of spaces and geometric configurations. By incorporating the local geometry of the curve, the positional adaptive frame approach used in our work provides a more intrinsic and precise method for expressing inextensible flow, and it offers a very different structure from previous works. This approach makes the functions of the tangent, normal, binormal, and velocity functions easily observable. It also clarifies how derivatives and curvature–torsion terms participate during the curve’s dynamic evolution. Consequently, it provides increased precision and adaptability for both numerical simulations and theoretical research. For instance, Özen demonstrated that there was a close relationship between a moving point particle of constant mass and its trajectory, emphasizing that moving frames adapted to such trajectories serve as powerful tools in kinematics theory. Based on this relation, he introduced a new moving frame, called the Positional Adapted Frame, for trajectories with non-vanishing angular momentum and investigated several fundamental topics by means of this frame [22].

In this paper, we first recall the general properties of spacelike, timelike, and lightlike curves, and present the Frenet equations in Minkowski space. We then establish conditions under which a timelike curve admits an inextensible flow, expressed in terms of the coefficients of the transition matrix. Finally, we provide a formal definition of inextensible flow in this setting.

2. PRELIMINARIES

Assume that the real vector space with its typical vector topology is $\mathbb{R}^3$. Denote by $E = \{e_1, e_2, e_3\}$ usual base of $\mathbb{R}^3$. Using it a vector and its coordinates will be defined: (x, y, z) or $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ for all the article. Operations performed in Minkowski space are generally defined over the inner product, so the definition of the inner product is clearly a cornerstone for the mathematical work to be done on this topic. The Minkowski space is the metric space $\mathbb{R}^3_1$, where the metric tensor $\langle \, , \rangle$ is given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3$$

where $x, y \in \mathbb{R}^3$. The metric tensor $\langle \, , \rangle$ is called Lorentzian inner product depending on certain adequate conditions, vectors in Minkowski space have several types a vector x that is in Minkowski space is called,

$$\begin{gathered} Spacelike\ if\ \langle x, x \rangle > 0\ or\ x = 0 \\ Timelike\ if\ \langle x, x \rangle < 0 \\ Lightlike\ if\ \langle x, x \rangle = 0\ and\ x \neq 0 \end{gathered} \tag{1}$$

with the Lorentzian inner product, the space $\mathbb{R}^3_1$ [23]. The idea of the vector's norm, or length, is necessary in all branches of mathematics that deal with vectors, and it differs according to the space in which one works. The following is the definition of a vector x vector's norm in Minkowski space.

Definition 2.1. For all vector $x \in \mathbb{R}^3_1$, Lorentzian norm of x is defined as in the follow for the nonnegative vector x, [24]

$$\|x\|_L = \sqrt{|\langle x, x \rangle|}. \tag{2}$$

Assuming that α is the arc-lenght parametrized space curve along an interval, the method for determining of the curve α when it is arc-lenght parametrized, is as follows:

$$T = \frac{\alpha'}{\|\alpha'\|}, N = \frac{\alpha''}{\|\alpha''\|} \text{ and } B = T \times N \tag{3}$$

here, T is referred to as the curve's tangent vector field, N as its normal vector field, and B as its binormal vector field. These vectors fluctuate with the curve throughout time [25]. In the event of a timelike curve, all the operations to be carried out will differ since the Frenet matrix form to be obtained will differ. Thus, the following is a definition of the timelike vector set defined as; Consider that Ω is the collection of $\mathbb{R}_1^3$'s timelike vectors. The set of timelike vectors that forms a cone for each $x \in \Omega$ is represented as $\Omega(x) = \{y \in \Omega : \langle x, y\rangle < 0\}$. This set is non-empty since $x \in \Omega(x)$, consequently, if and only if $\langle x, y\rangle < 0$, two timelike vectors, x and y, are in the same timelike cone [23]. On top of that, working with timelike vectors causes differences in frenet matrix shape as in the below: Proceed to say that σ is a timelike curve in $\mathbb{R}_1^3$ with parameter s and Lorentz inner product. The Frenet equations of this curve, as determined by the Frenet frame, is expressed below in terms of the curvatures of the curve,

$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{4}$$

this matrix is the matrix form of the Frenet-Serret equations and reflects the relationship valid for a timelike curve [25].

3. EVOLUTION OF TIMELIKE CURVES

A family of curves parametrized by time may be regarded as a curve evolving with respect to time. Each curve in such a family can be represented by a mapping $\sigma(u, t)$, where u denotes the space parameter u and the evolution parameter t. With two components in this definition set, σ can be expressed as follows: An evolution equation, a differential equation that depicts the evolution of $\sigma(u, t)$ over time, is defined by

$$\sigma' = \frac{d\sigma}{dt} = f_1 T + f_2 N + f_3 B \tag{5}$$

represents the flows of curve α where the velocity functions by Frenet frame $\{T, N, B\}$, are f_1, f_2, f_3 the curvatures alter the values of the velocity functions and a flow should not expand along its curve if its velocity vector is inextensible. In other words, the velocity vector's magnitude (or the distance based on the derivative) must stay constant. A curve evolution of $\sigma(u, t)$ and its flow $\frac{\partial \sigma}{\partial t}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ are said to be inextensible if $\frac{\partial}{\partial t}\left|\frac{\partial \sigma}{\partial u}\right| \equiv 0$ [26].

3.1. POSITIONAL ADAPTED FRAME ($\wp\mathcal{A}\mathcal{F}$) OF A TIMELIKE CURVE IN $\mathbb{R}_1^3$

The trajectory of the moving particle can be parameterized with $\sigma = \sigma(s)$ for the unit speed. For a timelike curve $\sigma = \sigma(s)$, the $\{\wp, \mathcal{A}, \mathcal{F}\}$ form an orthonormal moving frame known as a $\wp$ timelike, $\mathcal{A}$ and $\mathcal{F}$ are spacelike vectors. The following describes the relationship between the positional adapted frame and the Frenet frame in Minkowski 3-space;

$$\begin{pmatrix} \wp \\ \mathcal{A} \\ \mathcal{F} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{6}$$

If $\wp$ is timelike, $\mathcal{A}$ and $\mathcal{F}$ are spacelike vectors, then we have insert the reference for the following equation

$$\begin{pmatrix} \wp \\ \mathcal{A} \\ \mathcal{F} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cosh\theta & sinh\theta \\ 0 & sinh\theta & cosh\theta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{7}$$

The derivation formulas of the $\wp\mathcal{A}\mathcal{F}$ for a timelike curve in Minkowski 3-space are given below:

$$\frac{\partial}{\partial t}\begin{pmatrix} \wp \\ \mathcal{A} \\ \mathcal{F} \end{pmatrix} = \begin{pmatrix} 0 & \hbar_1 & \hbar_2 \\ -\hbar_1 & 0 & \hbar_3 \\ -\hbar_2 & \hbar_3 & 0 \end{pmatrix} \begin{pmatrix} \wp \\ \mathcal{A} \\ \mathcal{F} \end{pmatrix}, \tag{8}$$

where $\hbar_1, \hbar_2$ and $\hbar_3$ are $\wp\mathcal{A}\mathcal{F}$ apparatuses in $\mathbb{R}_1^3$ and are obtained by

$$\begin{aligned} \hbar_1 &= \kappa(s)cosh\theta \\ \hbar_2 &= \kappa(s)sinh\theta \\ \hbar_3 &= \tau(s) - \theta'. \end{aligned} \tag{9}$$

Between vectors $\wp$ and B, the angle is denoted by θ. Then we have the equalities as the below satisfies, [27],

$$\wp \times \mathcal{A} = \mathcal{F}, \mathcal{A} \times \mathcal{F} = -\wp, \mathcal{F} \times \wp = \mathcal{A}, \tag{10}$$

and

$$\langle \wp, \wp \rangle = -1 \ \langle \mathcal{A}, \mathcal{A} \rangle = 1 \ \langle \mathcal{F}, \mathcal{F} \rangle = 1 \tag{11}$$

$$\langle \wp, \mathcal{A} \rangle = \langle \mathcal{A}, \mathcal{F} \rangle = \langle \mathcal{F}, \wp \rangle = 0. \tag{12}$$

3.2. INEXTENSIBLE FLOWS OF TIMELIKE CURVES WITH POSITIONAL ADAPTED FRAME

Let $\sigma: I \times (0,1] \to \mathbb{R}_1^3$ be one parameter family of timelike curve respect to the arc length parameter and $\{\wp, \mathcal{A}, \mathcal{F}\}$ be its positional adapted frame. Let u denote the parametrization variable and the evolution parameter t of the curve σ. Then the relation between u and the arc-length parameter of σ can be expressed as follows:

$$s(u) = \int_0^u \left|\frac{\partial \sigma}{\partial u}\right| du, \tag{13}$$

where

$$\omega = \left|\frac{\partial \sigma}{\partial u}\right| = \sqrt{\left|\langle \frac{\partial \sigma}{\partial u}, \frac{\partial \sigma}{\partial u} \rangle\right|}. \tag{14}$$

The operator corresponding to differentiation with respect to s can be represented in terms of the variable u as:

$$\frac{\partial}{\partial s} = \frac{1}{\omega}\frac{\partial}{\partial u}. \tag{15}$$

Equivalently, it can also be written as:

$$\partial s = \omega \partial u. \tag{16}$$

With the two components in this definition set, σ can be expressed as follows. An evolution equation, which describes the evolution of $\sigma(u,t)$ over time, is defined by:

$$\sigma' = \frac{\partial \sigma}{\partial t} = f_1 \wp + f_2 \mathcal{A} + f_3 \mathcal{F} \tag{17}$$

represents flow of curve σ where the velocity functions f_1, f_2, f_3, within the positional adapted frame $\{\wp, \mathcal{A}, \mathcal{F}\}$, are defined, and the curvature functions alter their values.

Theorem 3.1. Let σ be timelike curve with the arc-length parameter with its flow defined by Eq. (17). Then, the flow is inextensible iff

$$\frac{\partial \omega}{\partial t} = -\frac{\partial f_1}{\partial u} - f_2 \hbar_1 \omega - f_3 \hbar_2 \omega, \tag{18}$$

where $\omega = \left|\frac{\partial \sigma}{\partial u}\right|$.

Proof: Assume that $\frac{\partial \sigma}{\partial u}$ is a smooth flow, then we have

$$\omega = \left|\frac{\partial \sigma}{\partial u}\right| = \left|\langle \frac{\partial \sigma}{\partial u}, \frac{\partial \sigma}{\partial u} \rangle\right|^{1/2}. \tag{19}$$

Given the squares of all sides, it can be expressed as

$$\omega^2 = \langle \frac{\partial \sigma}{\partial u}, \frac{\partial \sigma}{\partial u} \rangle. \tag{20}$$

If the derivative of both sides of the Eq. (20) is taken with respect to the parameter t, we obtain the following equation:

$$\begin{aligned} 2\omega \frac{\partial \omega}{\partial t} &= \frac{\partial}{\partial t}\left(\langle \frac{\partial \sigma}{\partial u}, \frac{\partial \sigma}{\partial u} \rangle\right) \\ &= \langle \frac{\partial}{\partial t}\left(\frac{\partial \sigma}{\partial u}\right), \frac{\partial \sigma}{\partial u} \rangle + \langle \frac{\partial \sigma}{\partial u}, \frac{\partial}{\partial t}\left(\frac{\partial \sigma}{\partial u}\right) \rangle \\ &= 2\langle \frac{\partial \sigma}{\partial u}, \frac{\partial}{\partial t}\left(\frac{\partial \sigma}{\partial u}\right) \rangle \\ &= 2\langle \frac{\partial \sigma}{\partial u}, \frac{\partial}{\partial u}\left(\frac{\partial \sigma}{\partial t}\right) \rangle. \end{aligned} \tag{21}$$

Considering Eq. (21) above, and the evolution equation of the curve $\sigma(u,t)$ in Eq. (17), the partial derivative $\frac{\partial \omega}{\partial t}$ can be calculated as follows:

$$\begin{aligned} \omega \frac{\partial \omega}{\partial t} &= \langle \frac{\partial \sigma}{\partial u}, \frac{\partial}{\partial u}\left(\frac{\partial \sigma}{\partial t}\right) \rangle \\ &= \langle \frac{\partial \sigma}{\partial u}, \frac{\partial}{\partial u}(f_1 \wp + f_2 \mathcal{A} + f_3 \mathcal{F}) \rangle \\ &= \langle \frac{\partial \sigma}{\partial u}, \left(\frac{\partial f_1}{\partial u}\wp + f_1 \frac{\partial \wp}{\partial u} + \frac{\partial f_2}{\partial u}\mathcal{A} + f_2 \frac{\partial \mathcal{A}}{\partial u} + \frac{\partial f_3}{\partial u}\mathcal{F} + f_3 \frac{\partial \mathcal{F}}{\partial u}\right) \rangle. \end{aligned} \tag{22}$$

By dividing both sides of Eq. (22) by ω and considering the derivation formulas of the $\wp\mathcal{A}\mathcal{F}$ in Eq. (8), it can be expressed as

$$\frac{\partial \omega}{\partial t} = \langle \frac{\partial \sigma}{\partial u}, \ell_1 \wp + \ell_2 \boldsymbol{\mathcal{A}} + \ell_3 \boldsymbol{\mathcal{F}} \rangle,$$

where

$$\ell_1 = \frac{\partial f_1}{\partial u} + f_2 \hbar_1 \omega + f_3 \hbar_2 \omega$$

$$\ell_2 = f_1 \hbar_1 \omega + \frac{\partial f_2}{\partial u} - f_3 \hbar_3 \omega$$

$$\ell_3 = f_1 \hbar_2 + f_2 \hbar_3 + \frac{\partial f_3}{\partial u}$$

that is, it can be written as

$$\frac{\partial \omega}{\partial t} = -\frac{\partial f_1}{\partial u} - f_2 \hbar_1 \omega - f_3 \hbar_2 \omega$$

.

Remark 3.2. The flow is inextensible iff

$$\frac{\partial f_1}{\partial s} = -f_2 \hbar_1 - f_3 \hbar_2. \tag{23}$$

Proof: The flow $\partial \sigma / \partial u$ of a curve $\sigma(u,t)$ is said to be inextensible if and only if the arc-length parameter is preserved, that is,

$$\frac{\partial}{\partial t} \left| \frac{\partial \sigma}{\partial u} \right| = 0.$$

Considering Eqs. (13) and (15) together, it can be written as

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial \omega}{\partial t} du = \int_0^u \frac{\partial \omega}{\partial t} du = -\frac{\partial f_1}{\partial u} - f_2 \hbar_1 \omega - f_3 \hbar_2 \omega = 0.$$

Theorem 3.3. The time evolution of the $\wp \mathcal{A} \mathcal{F}$ can be structured in matrix form as shown below:

$$\frac{\partial}{\partial t} \begin{pmatrix} \wp \\ \mathcal{A} \\ \mathcal{F} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3 & \frac{\partial f_3}{\partial s} + f_1 \hbar_2 + f_2 \hbar_3 \\ \left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3 \right) & 0 & \psi \\ \frac{\partial f_3}{\partial s} + f_1 \hbar_2 + f_2 \hbar_3 & -\psi & 0 \end{pmatrix} \begin{pmatrix} \wp \\ \mathcal{A} \\ \mathcal{F} \end{pmatrix}, \tag{24}$$

where

$$\langle \frac{\partial \mathcal{A}}{\partial t}, \mathcal{F} \rangle = \psi. \tag{25}$$

Proof: Assume that the flow of the curve σ is represented by $\frac{\partial \sigma}{\partial t} = (f_1 \wp + f_2 \mathcal{A} + f_3 \mathcal{F})$ then, we have

$$\frac{\partial \wp}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \sigma}{\partial s} \right). \tag{26}$$

Since the curve σ is smooth curve, it can be written as the equality

$$\frac{\partial}{\partial t} \left(\frac{\partial \sigma}{\partial s} \right) = \frac{\partial}{\partial s} \left(\frac{\partial \sigma}{\partial t} \right). \tag{27}$$

By substituting Eq. (17) into Eqs. (26) and (27), we obtain

$$\begin{aligned}\frac{\partial \wp}{\partial t} &= \frac{\partial}{\partial s}(f_1\wp + f_2\mathcal{A} + f_3\mathcal{F}) \\ &= \frac{\partial f_1}{\partial s}\wp + f_1\frac{\partial \wp}{\partial s} + \frac{\partial f_2}{\partial s}\mathcal{A} + f_2\frac{\partial \mathcal{A}}{\partial s} + \frac{\partial f_3}{\partial s}\mathcal{F} + f_3\frac{\partial \mathcal{F}}{\partial s} \\ &= \ell^1\wp + \ell^2\mathcal{A} + \ell^3\mathcal{F},\end{aligned} \tag{28}$$

where

$$\ell^1 = \frac{\partial f_1}{\partial s} + \hbar_1 f_2 + \hbar_2 f_3,$$
$$\ell^2 = \frac{\partial f_2}{\partial s} + f_1\hbar_1 - f_3\hbar_3,$$
$$\ell^3 = \frac{\partial f_3}{\partial s} + f_1\hbar_2 + f_2\hbar_3.$$

In view of Eq. (23), one finds that $\ell^1 = 0$; consequently, Eq. (28) can be expressed as follows:

$$\frac{\partial \wp}{\partial t} = \mathcal{A}\left(\frac{\partial f_2}{\partial s} + f_1\hbar_1 - f_3\hbar_3\right) + \mathcal{F}\left(\frac{\partial f_3}{\partial s} + f_1\hbar_2 + f_2\hbar_3\right). \tag{29}$$

On the other hand, because an orthonormal basis is formed by $\wp\mathcal{A}\mathcal{F}$, we have

$$\langle \wp, \mathcal{A}\rangle = \langle \wp, \mathcal{F}\rangle = \langle \mathcal{A}, \mathcal{F}\rangle = 0. \tag{30}$$

Hence, if the equations in (30) are differentiated sequentially with respect to t. If we first take the derivative of the equation

$$0 = \frac{\partial}{\partial t}\langle \wp, \mathcal{A}\rangle \tag{31}$$

with respect to t then, we have

$$0 = \langle \frac{\partial \wp}{\partial t}, \mathcal{A}\rangle + \langle \wp, \frac{\partial \mathcal{A}}{\partial t}\rangle. \tag{32}$$

If the expression $\frac{\partial \wp}{\partial t}$ is replaced with $\mathcal{A}\left(\frac{\partial f_2}{\partial s} + f_1\hbar_1 - f_3\hbar_3\right) + \mathcal{F}\left(\frac{\partial f_3}{\partial s} + f_1\hbar_2 + f_2\hbar_3\right)$, then

$$0 = \langle\left(\frac{\partial f_2}{\partial s} + f_1\hbar_1 - f_3\hbar_3\right)\mathcal{A} + \left(\frac{\partial f_3}{\partial s} + f_1\hbar_2 + f_2\hbar_3\right)\mathcal{F}, \mathcal{A}\rangle + \langle \wp, \frac{\partial \mathcal{A}}{\partial t}\rangle. \tag{33}$$

Similarly, we have

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\langle \wp, \mathcal{F}\rangle \\ &= \langle \frac{\partial \wp}{\partial t}, \mathcal{F}\rangle + \langle \wp, \frac{\partial \mathcal{F}}{\partial t}\rangle \\ &= \langle\left(\frac{\partial f_2}{\partial s} + f_1\hbar_1 - f_3\hbar_3\right)\mathcal{A} + \left(\frac{\partial f_3}{\partial s} + f_1\hbar_2 + f_2\hbar_3\right)\mathcal{F}, \mathcal{F}\rangle + \langle \wp, \frac{\partial \mathcal{F}}{\partial t}\rangle\end{aligned} \tag{34}$$

and

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\langle \mathcal{A}, \mathcal{F}\rangle \\ &= \langle \frac{\partial \mathcal{A}}{\partial t}, \mathcal{F}\rangle + \langle \mathcal{A}, \frac{\partial \mathcal{F}}{\partial t}\rangle.\end{aligned} \tag{35}$$

Assume that $\langle \frac{\partial \mathcal{A}}{\partial t}, \mathcal{F} \rangle$ is equal to ψ , then from (34) it can be obtained that

$$\langle \mathcal{A}, \frac{\partial \mathcal{F}}{\partial t} \rangle = -\psi.$$

Moreover, given that the equation below holds, it can be written using Eqs. (34) and (35) as follows:

$$\frac{\partial \mathcal{A}}{\partial t} = \left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right)\wp + \psi \mathcal{F} \tag{36}$$

$$\frac{\partial \mathcal{F}}{\partial t} = \left(\frac{\partial f_3}{\partial s} + f_1 \hbar_2 + f_2 \hbar_3\right)\wp - \psi \mathcal{A} \tag{37}$$

Theorem 3.4. The time evolution of the $\wp\mathcal{A}\mathcal{F}$ can be written in differential equation form as follows:

$$\begin{gathered}\frac{\partial \hbar_1}{\partial t} = \hbar_1\left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right) \\ \frac{\partial \hbar_2}{\partial t} = \frac{\partial^2 f_3}{\partial s^2} + \frac{\partial}{\partial s}(f_1 \hbar_2 + f_2 \hbar_3) + \hbar_1 \psi + \hbar_3\left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right) \\ \frac{\partial \hbar_3}{\partial t} = \left(\hbar_1\left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right) - \hbar_1 \psi + \frac{d\psi}{ds}\right)\end{gathered} \tag{36}$$

and

$$\psi = \frac{\partial f_3}{\partial s} + f_1 \hbar_2 + f_2 \hbar_3. \tag{39}$$

Proof: Assume that curve $\frac{\partial \sigma}{\partial t} = (f_1 \wp + f_2 \mathcal{A} + f_3 \mathcal{F})$'s flow is inextensible, then the set of partial differantial equations that follows is valid:

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial \mathcal{A}}{\partial t}\right) &= \frac{\partial}{\partial s}\left(\left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right)\wp + \psi \mathcal{F}\right) \\ &= \left(\frac{\partial^2 f_2}{\partial s^2} + \frac{\partial}{\partial s}(f_1 \hbar_1 - f_3 \hbar_3)\right)\wp + \left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right)\frac{\partial \wp}{\partial s} + \frac{\partial \psi}{\partial s}\mathcal{F} + \psi \frac{\partial \mathcal{F}}{\partial s} \\ &= \left(\frac{\partial^2 f_2}{\partial s^2} + \frac{\partial}{\partial s}(f_1 \hbar_1 - f_3 \hbar_3) + \hbar_2 \psi\right)\wp \\ &\quad + \left(\hbar_1\left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right) + \hbar_3 \psi\right)\mathcal{A} \\ &\quad + \left(\hbar_2\left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right) + \frac{d\psi}{ds}\right)\mathcal{F}\end{aligned} \tag{40}$$

with the other hand by positional adapted frame, it can be written as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{A}}{\partial s}\right) &= \left(\hbar_1\left(\frac{\partial f_2}{\partial s} + f_1 \hbar_1 - f_3 \hbar_3\right) + \hbar_3\left(\frac{\partial f_3}{\partial s} + f_1 \hbar_1 + f_2 \hbar_3\right)\right)\wp \\ &\quad + \left(\frac{\partial \hbar_1}{\partial t} - \hbar_3 \psi\right)\mathcal{A} + \left(\hbar_1 \psi + \frac{\partial \hbar_3}{\partial t}\right)\mathcal{F}.\end{aligned} \tag{41}$$

Since

$$\frac{\partial}{\partial s}\left(\frac{\partial \mathcal{A}}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{A}}{\partial s}\right) \tag{42}$$

and moreover we can show that the equation below satisfies,

$$\frac{\partial \hbar_1}{\partial t}=\hbar_1\left(\frac{\partial f_2}{\partial s}+f_1\hbar_1-f_3\hbar_3\right) \tag{43}$$

$$\frac{\partial \hbar_3}{\partial t}=\hbar_1\left(\frac{\partial f_2}{\partial s}+f_1\hbar_1-f_3\hbar_3\right)-\hbar_1\psi+\frac{d\psi}{ds}. \tag{44}$$

In addition, the following relation holds:

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial \mathcal{F}}{\partial t}\right)&=\frac{\partial}{\partial s}\left(\left(\frac{\partial f_3}{\partial s}+f_1\hbar_1+f_2\hbar_3\right)\wp-\psi\,\mathcal{A}\right)\\ &=\left(\frac{\partial^2 f_3}{\partial s^2}+\frac{\partial}{\partial s}(f_1\hbar_2+f_2\hbar_3)\right)\wp+\frac{\partial\wp}{\partial s}+\frac{\partial\psi}{\partial s}\mathcal{A}+\psi\frac{\partial\mathcal{A}}{\partial s}\\ &=\left(\frac{\partial^2 f_3}{\partial s^2}+\frac{\partial}{\partial s}(f_1\hbar_2+f_2\hbar_3)+\hbar_1\psi\right)\wp+\left(\hbar_1\left(\frac{\partial f_3}{\partial s}+f_1\hbar_2+f_2\hbar_3\right)-\frac{\partial\psi}{\partial s}\right)\mathcal{A}\\ &\quad+\left(\hbar_1\left(\frac{\partial f_3}{\partial s}+f_1\hbar_2+f_2\hbar_3\right)+\hbar_3\psi\right)\mathcal{F}\end{aligned} \tag{45}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{F}}{\partial s}\right)&=\frac{\partial}{\partial t}(\hbar_2\wp-\hbar_3\mathcal{A})\\ &=\left(\frac{\partial\hbar_2}{\partial t}-\hbar_3(\frac{\partial f_2}{\partial s}+f_1\hbar_1-f_3\hbar_3)\right)\wp+\left(\hbar_2\left(\frac{\partial f_2}{\partial s}+f_1\hbar_1-f_3\hbar_3\right)-\frac{\partial\hbar_3}{\partial t}\right)\mathcal{A}\\ &\quad+\left(\hbar_2\left(\frac{\partial f_2}{\partial s}+\frac{\partial f_3}{\partial s}+f_1\hbar_2+f_2\hbar_3\right)-\hbar_3\psi\right)\mathcal{F}.\end{aligned} \tag{46}$$

As a result,

$$\frac{\partial}{\partial s}\left(\frac{\partial \mathcal{F}}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{F}}{\partial s}\right)$$

and furthermore, it can be demonstrated that the following equation holds,

$$\frac{\partial\hbar_2}{\partial t}=\frac{\partial^2 f_3}{\partial s^2}+\frac{\partial}{\partial s}(f_1\hbar_2+f_2\hbar_3)+\hbar_1\psi+\hbar_3\left(\frac{\partial f_2}{\partial s}+f_1\hbar_1-f_3\hbar_3\right)$$

$$\frac{\partial\hbar_3}{\partial t}=\hbar_3\left(\frac{\partial f_2}{\partial s}+f_1\hbar_1-f_3\hbar_3\right)-\hbar_1\left(\frac{\partial f_3}{\partial s}+f_1\hbar_2+f_2\hbar_3\right)+\frac{\partial\psi}{\partial s}$$

and

$$\psi=\frac{\partial f_3}{\partial s}+f_1\hbar_2+f_2\hbar_3.$$

Example 3.5. Let a timelike curve in 3D Minkowski space be given as follows:

$$\sigma(s)=\left(\cos\left(\frac{s}{\sqrt{3}}\right),\sin\left(\frac{s}{\sqrt{3}}\right),\frac{2}{\sqrt{3}}s\right). \tag{47}$$

The parameter s is already the arc–length parameter. We therefore set,

$$\wp(s) = \frac{\partial \sigma(s)}{\partial s} \tag{48}$$

and regard $\wp$ as the unit timelike tangent vector field along the curve. The second derivative of the timelike curve σ is

$$\wp'(s) = \left(-\frac{1}{3}\cos\left(\frac{s}{\sqrt{3}}\right), -\frac{1}{3}\sin\left(\frac{s}{\sqrt{3}}\right), 0\right)$$

and its Minkowski norm is $\|\wp'(s)\|$=1/9 . Hence the curvature of the curve σ is the constant and it is obtained as, $(s) = 1/3$. More over one easliy checks that, the evolution $\wp$ is

$$\frac{\partial \wp}{\partial t} = \left(\frac{\partial f_1}{\partial s} + \frac{1}{3} f_2\right)\wp + \left(\frac{1}{3} f_1 + \frac{\partial f_2}{\partial s} + \frac{2}{3} f_3\right)\mathcal{A} + \left(-\frac{2}{3} f_2 + \frac{\partial f_3}{\partial s}\right)\mathcal{F},$$

where $f_1 = \sin(s)$, $f_2 = -3\cos(s)$, $f_3 = 0$.

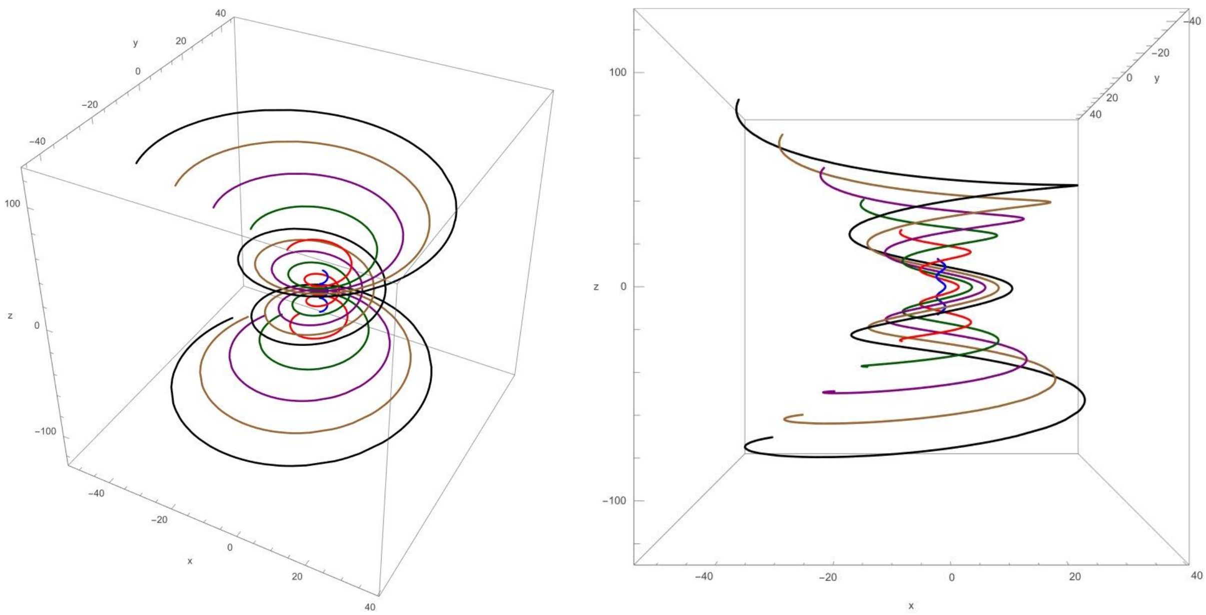

Figure 1. The evolution of $\sigma(u,t)$ is visualized using distinct colors blue $(t = 0)$, red $(t = 1)$, dark green $(t = 2)$, purple $(t = 3)$, and black $(t = 4)$ making each stage of the inextensible flow easily distinguishable.

4. CONCLUSIONS

This study contributes to the analysis of curve and surface evolution in Minkowski space by applying the positional adapted frame to the inextensible flow of timelike curves. We derived the evolution equations for the adapted frame and obtained conditions under which timelike curves admit inextensible flows. In the process, studies that integrate timelike and Frenet frames, together with the Bishop frame, also embrace the concept of IFC. In this research, we used the $\wp\mathcal{AF}$ mathematical framework to work on the IF of a timelike curve, using the $\wp\mathcal{AF}$ framework, this study mainly allowed us to examine the surfaces formed by various timelike curves that were created at each distinct time t of a curve. Additionally, we show some partial differentials with respect to parameter t and coordinates utilizing the $\wp\mathcal{AF}$ of a particular timelike curve in Minkowski space.

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