ORIGINAL PAPER

NEW CONVERGENT SEQUENCES FOR APPROXIMATING THE EULER-MASCHERONI CONSTANT

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Abstract. *In this paper, we present new sequences that offer improved approximations of the Euler-Mascheroni constant. One of the proposed sequences converges to its limit like* m^{-8} *tends to zero, as* $m \to \infty$*. Furthermore, several new inequalities involving this constant and the Digamma function are established. We also prove that certain functions related to our sequences are completely monotonic. Finally, an open question is posed.*

Keywords: *Euler-Mascheroni constant; Gamma function; rate of convergence; asymptotic expansion; complete monotonicity.*

Mathematics Subject Classification: *33B15; 41A25; 26D15; 41A60.*

1. INTRODUCTION

The Euler-Mascheroni constant γ is one of the most important mathematical constants appearing in number theory, analysis, and special functions. It was first introduced by Leonhard Euler in 1734 and later studied in detail by Lorenzo Mascheroni in 1790 [1]. It is defined by the limiting difference between the partial sum of the harmonic series H_m and the natural logarithm:

$$\gamma = \lim_{m\to\infty}\left(\sum_{r=1}^{m}\frac{1}{r} - \ln m\right) = 0.577216\ldots.$$

Alternative expressions include

$$\gamma = -\Gamma'(1) = -\psi(1) = -\int_0^\infty e^{-s}\ln s\, ds,$$

where $\Gamma(x)$ and $\psi(x)$ denote the Gamma and Digamma functions, respectively, and are defined by:

$$\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}\, ds, x > 0$$

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$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}=\ln x-\int_0^\infty\left[\left(\frac{1}{1-e^{-s}}\right)-\frac{1}{s}\right]e^{-xs}\,ds, x>0. \tag{1}$$

The classical sequence

$$\gamma_m=\sum_{r=1}^{m}\frac{1}{r}-\ln m$$

converges to γ slowly with an error term of order $1/2m$, that is,

$$\gamma_m-\gamma=\frac{1}{2m}-\frac{1}{12m^2}+O(m^{-4}).$$

Many researchers have therefore developed modified or corrected sequences to accelerate this convergence. Typical approaches include logarithmic shifts such as $H_m-\ln(m+\beta)$, rational corrections, and Padé-type approximations. Each improvement aims to cancel successive terms in the asymptotic expansion of H_m, thus obtaining higher convergence rates.

In recent years, several studies have proposed new accelerated sequences that converge to γ significantly faster than the classical harmonic-based sequence. Chen and Mortici [2] introduced in 2012 a family of sequences with logarithmic corrections. Wu and Bercu [3] developed in 2018 a simpler yet highly accurate variant. Cringanu [4] employed continued-fraction corrections in 2025. Moreover, explicit closed-form sequences have been constructed and proven by Batir and Chen [5], Mortici [6], and You and Chen [7]. These results illustrate a clear progression in the development of sequences with increasingly faster convergence to the Euler-Mascheroni constant.

In 2026, Mortici [6] presented the following two new sequences

$$\mu_m=\sum_{r=1}^{m}\frac{1}{r}+\ln\left[\frac{1}{m}+\frac{1}{m+1}\right]=\gamma+\ln 2+\frac{7}{24m^2}+O(m^{-3})$$

and

$$\eta_m=\sum_{r=1}^{m}\frac{1}{r}+\ln\left[\frac{1}{m^2}+\frac{1}{(m+1)^2}\right]=\gamma+\ln\sqrt{2}+\frac{5}{12m^2}+O(m^{-3}).$$

Also, he proved the two inequalities, for all integers $m\geq 1$:

$$\frac{7}{24(m+1)(m+2)}\leq\mu_m-\gamma-\ln 2\leq\frac{7}{24m(m+1)},$$

and

$$\frac{5}{12(m+1)(m+2)}\leq\eta_m-\gamma-\ln\sqrt{2}\leq\frac{5}{12m(m+1)}.$$

The main goal of this paper is to propose new sequences converging to γ with an improved rate of convergence. These sequences are constructed by combining rational and logarithmic corrections that are designed to cancel several leading terms in the asymptotic expansion of H_m. Theoretical error estimates confirm the improvement over recent results presented in [6].

2. MAIN RESULTS

A fundamental tool for analyzing the rate of convergence is the following lemma, first stated in [8]:

Lemma 1. Let $(M_m)_{m\geq 1}$ be a sequence converging to zero. Suppose that for some $r > 1$,

$$\lim_{m\to\infty} m^r(M_m - M_{m+1}) = h \in \mathbb{R} \setminus \{0\}.$$

Then

$$\lim_{m\to\infty} m^{r-1}M_m = \frac{h}{r-1}.$$

This lemma is particularly useful when M_m is defined as a sum, since the difference sequence $M_m - M_{m+1}$ often takes a simpler form. Its primary importance lies in providing a practical method to determine the speed of convergence of a sequence. Moreover, it serves as a foundation for constructing corrections or modified sequences that accelerate convergence. By analyzing how the differences between consecutive terms scale with m, one can design new sequences that converge faster to the desired limit.

Inspired by Mortici [8], our first step is to consider the following sequence:

$$A_m = \sum_{r=1}^{m}\frac{1}{r} - \ln\frac{a_2m^2 + a_1m + a_0}{a_3 + m}, m \in \mathbb{N};\ a_0, a_1, a_2, a_3 \in \mathbb{R}$$

and hence

$$\begin{aligned}
A_m - A_{m+1} &= -\frac{1}{m+1} + \ln\frac{\big((m+1)(a_2(m+1)+a_1)+a_0\big)(a_3+m)}{(m(a_2m+a_1)+a_0)(a_3+m+1)} \\
&= \frac{-\frac{a_1}{a_2} + a_3 + \frac{1}{2}}{m^2} + \frac{\frac{a_1^2 + (a_1 - 2a_0)a_2}{a_2^2} - a_3(a_3+1) - \frac{2}{3}}{m^3} \\
&+ \frac{a_3^3 + \frac{3a_3^2}{2} + a_3 - \frac{2a_1^3 + 3(a_1 - 2a_0)a_2a_1 + 2(a_1 - 3a_0)a_2^2}{2a_2^3} + \frac{3}{4}}{m^4} \\
&+ \frac{1}{5a_2^4m^5}\Big(5a_1^4 + 10a_2a_1^3 + 10a_2(a_2 - 2a_0)a_1^2 + 5a_2^2(a_2 - 6a_0)a_1 \\
&- a_2^2\big(-10a_0^2 + 20a_2a_0 + a_2^2(5a_3(a_3+1)(a_3^2 + a_3 + 1) + 4)\big)\Big) \\
&+ O(m^{-6}), m \to \infty.
\end{aligned}$$

To obtain the best rate of convergence according to Lemma 1, we should choose the constants a_0, a_1, a_2, and a_3 to satisfy the following system of equations:

$$-\frac{a_1}{a^2} + a_3 + \frac{1}{2} = 0$$

$$\frac{a_1^2 + (a_1 - 2a_0)a_2}{a_2^2} - a_3(a_3+1) - \frac{2}{3} = 0$$

$$a_3^3+\frac{3a_3^2}{2}+a_3-\frac{2a_1^3+3(a_1-2a_0)a_2a_1+2(a_1-3a_0)a_2^2}{2a_2^3}+\frac{3}{4}=0,$$

which have the following solution:

$$a_0=\frac{7a_2}{24}, a_1=a_2, a_3=\frac{1}{2} \text{ and } a_2\neq 0 \text{ is arbitrary.}$$

Hence

$$A_m-A_{m+1}=-\frac{37}{1440m^5}+O(m^{-6}), m\to\infty.$$

or

$$\lim_{m\to\infty} m^5(A_m-A_{m+1})=-\frac{37}{1440}.$$

The harmonic number H_m has the asymptotic series

$$H_m\sim \ln m+\gamma+\frac{1}{2m}-\sum_{r=1}^{\infty}\frac{B_{2r}}{2rm^{2r}}, \text{as } m\to\infty,$$

(see, e.g., [1,9,10]), where B_{2r} are the Bernoulli numbers and then

$$\lim_{m\to\infty} A_m=\lim_{m\to\infty}\left(\gamma-\ln a_2-\frac{37}{5760m^4}+O(m^{-5})\right)=\gamma-\ln a_2.$$

Now, by choosing $a_2=1$ (for sake of simplicity) and applying Lemma 1, we obtain the following result:

Theorem 1. The sequence

$$\lambda_m=\sum_{r=1}^{m}\frac{1}{r}-\ln\frac{24\,m^2+24\,m+7}{12(2m+1)}, m\in\mathbb{N}^*$$

converges to γ at a rate of convergence m^{-4}. More precisely,

$$\lim_{m\to\infty} m^4(\lambda_m-\gamma)=-\frac{37}{5760}.$$

The Digamma function satisfies the functional relation

$$\psi(x+1)=\psi(x)+\frac{1}{x},$$

so, we can see that

$$\sum_{r=1}^{m}\frac{1}{r}=\gamma+\psi(m+1), m\in\mathbb{N}^*,$$

or

$$\lambda_m-\gamma=\psi(m+1)-\ln\frac{24\,m^2+24\,m+7}{12\,(2\,m+1)}, m\in\mathbb{N}^*.$$

A function $U: (0,\infty) \to \mathbb{R}$ is called completely monotonic (see, e.g., [11]) if it has derivatives of all orders and satisfies

$$(-1)^r U^{(r)}(x) \geq 0, x > 0, r = 0,1,2, \dots .$$

A classical result states that such a function can be represented in the form of a Laplace integral,

$$U(x) = \int_0^\infty e^{-xs}\, d\nu(s),$$

where ν is a nonnegative and bounded measure, defined on $[0,\infty)$. In other words, every completely monotonic function on $(0,\infty)$ is precisely the Laplace transform of a nonnegative measure, and conversely, any Laplace transform of a nonnegative measure is completely monotonic.

Now we can give the following result:

Theorem 2. The function

$$P_1(x) = -\psi(x+1) + \ln\frac{24\,x^2 + 24\,x + 7}{12\,(2\,x+1)}, x > 0$$

is completely monotonic on x > 0 and hence

$$0 < P_1(x) < \gamma - \ln\frac{12}{7}, x > 0.$$

Moreover, the sequence $\lambda_m = \gamma - P_1(m)$ is increasing, and

$$1 - \ln\frac{55}{36} < \lambda_m < \gamma, m \in \mathbb{N}.$$

Proof: Using the integral representation (1) and the relation

$$\ln\frac{24\,x^2 + 24\,x + 7}{12\,(2\,x+1)} - \ln x = \int_0^\infty \left(e^{-s/2} - 2e^{-\frac{s}{2}}\cos\frac{s}{2\sqrt{6}} + 1\right)\frac{e^{-xs}}{s}\,ds$$

we deduce that

$$P_1(x) = \int_0^\infty \left(e^{s/2}s + e^s - 2(e^s - 1)\cos\frac{s}{2\sqrt{6}} - 1\right)\frac{e^{-s/2}e^{-xs}}{(e^s-1)s}\,ds.$$

Using the inequality

$$\cos s \geq 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!}, s \geq 0$$

(see, e.g., [12]), we get

$$\cos\frac{s}{2\sqrt{6}} - \frac{288 - 5s^2}{s^2 + 288} > \frac{432s^6 - s^8}{9953280(s^2 + 288)} > 0, 0 < s < \sqrt{432} \approx 20.7846.$$

Also

$$\cos\frac{s}{2\sqrt{6}}-\frac{288-5s^2}{s^2+288}>-1-\frac{288-5s^2}{s^2+288}=\frac{4(s^2-144)}{s^2+288}>0\,,s>12.$$

Then, by putting together the cases $0<s<20$ and $s>12$, we obtain:

$$\cos\frac{s}{2\sqrt{6}}-\frac{288-5s^2}{s^2+288}>0\,,s>0$$

Now

$$\begin{aligned}e^{s/2}s+e^s-2(e^s-1)\cos\frac{s}{2\sqrt{6}}-1&>e^{\frac{s}{2}}s+e^s-2(e^s-1)\left(\frac{288-5s^2}{s^2+288}\right)-1\\&>\frac{e^{\frac{s}{2}}s^3+11e^s s^2-11s^2+288e^{\frac{s}{2}}s-288e^s+288}{s^2+288}\\&>\frac{1}{s^2+288}\Bigg[\frac{37s^5}{120}\\&+\sum_{r=6}^{\infty}\{8r(r^2-3r+74)+2^r(11r^2-11r-288)\}\frac{2^{-r}s^r}{r!}\Bigg]>0\,,s>0.\end{aligned}$$

Hence $P_1(x)$ is completely monotonic on $x>0$ and then it is decreasing for $x>0$. Using the asymptotic series

$$\psi(x)\sim\ln x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}+\cdots\,,x\to\infty, \quad (2)$$

(see, e.g., [9]), we get

$$\lim_{x\to\infty}P_1(x)=\lim_{x\to\infty}\left(\frac{37}{5760x^4}-\frac{37}{2880x^5}+O(x^{-6})\right)=0.$$

Hence

$$0<P_1(x)<P_1(0)=\gamma-\ln\frac{12}{7}\,,x>0.$$

As a consequence, the sequence $\lambda_m=\gamma-P_1(m)$ is increasing and

$$\gamma-P_1(1)<\lambda_m-P_1(m)<\gamma\,,m\in\mathbb{N},$$

which is

$$1-\ln\frac{55}{36}<\lambda_m<\gamma\,,m\in\mathbb{N}.$$

Our second step is to consider the following sequence:

$$D_m=\sum_{r=1}^{m}\frac{1}{r}-\ln\frac{24m^2+24m+7}{12(2m+1)}+\frac{b_0}{m^3(b_1+m)},b_0,b_1\in\mathbb{R}$$

and hence

$$\begin{aligned} D_m - D_{m+1} &= \frac{b_0}{m^3(b_1+m)} - \frac{b_0}{(m+1)^3(b_1+m+1)} - \frac{1}{m+1} \\ &+ \ln\frac{(2m+1)(24m^2+72m+55)}{(2m+3)(24m^2+24m+7)} \\ &= \frac{4b_0 - \frac{37}{1440}}{m^5} + \frac{\frac{37}{288} - 5b_0(b_1+2)}{m^6} + O(m^{-7}), m \to \infty. \end{aligned}$$

To obtain the best rate of convergence according to Lemma 1, we should choose the constants b_0, and b_1 to satisfy the following system of equations:

$$4b_0 - \frac{37}{1440} = 0$$
$$\frac{37}{288} - 5b_0(b_1+2) = 0,$$

which has the following solution:

$$b_0 = \frac{37}{5760} \text{ and } b_1 = 2.$$

By applying Lemma 1, we obtain the following result:

Theorem 3. The sequence

$$\sigma_m = \sum_{r=1}^{m}\frac{1}{r} - \ln\frac{24\,m^2+24\,m+7}{12\,(2\,m+1)} + \frac{\frac{37}{5760}}{m^3(2+m)}, m \in \mathbb{N}^*$$

converges to γ at a rate of convergence m^{-6}. More precisely,

$$\lim_{m\to\infty} m^6(\sigma_m - \gamma) = \frac{19531}{1451520}.$$

The relation between $\psi(m)$ and σ_m is given by

$$\sigma_m - \gamma = \psi(m+1) - \ln\frac{24\,m^2+24\,m+7}{12\,(2\,m+1)} + \frac{\frac{37}{5760}}{m^3(2+m)}, m \in \mathbb{N}^*.$$

Now we will get the following result:

Theorem 4. The following inequality holds

$$\psi(x+1) - \ln\frac{24\,x^2+24\,x+7}{12\,(2\,x+1)} > -\frac{37}{5760\,\left(x^3(x+2)\right)}, x > 0.$$

As a consequence,

$$\lambda_m - \gamma > -\frac{37}{5760\, m^3(m+2)}, m \in \mathbb{N}^*.$$

Proof: Consider the function

$$P_2(x) = \psi(x+1) - \ln\frac{24\,x^2 + 24\,x + 7}{12\,(2\,x+1)} + \frac{37}{5760\,\left(x^3(x+2)\right)}\,, x > 0.$$

Then

$$\frac{d}{dx}(P_2(x) - P_2(x+1)) = -\frac{w_1(x)}{w_2(x)} < 0\,, x > 0,$$

where

$$w_1(x) = 3749952x^{10} + 42612480x^9 + 205496488x^8 + 551495056x^7 + 908763898x^6 + 959202176x^5 + 659130763x^4 + 295319718x^3 + 85037211x^2 + 14697288x + 1153845$$

and

$$w_2(x) = 2880x^4(x+1)^4(x+2)^2(x+3)^2(2x+1)(2x+3) \times (24x^2 + 24x + 7)(24x^2 + 72x + 55).$$

Hence the function $P_2(x) - P_2(x+1)$ is decreasing with $\lim_{x\to\infty}(P_2(x) - P_2(x+1)) = 0$, so

$$P_2(x) > P_2(x+1), x > 0.$$

Therefore

$$P_2(x) > P_2(x+1) > P_2(x+2) > \cdots > P_2(x+r)\,, r \in \mathbb{N}\,;\ x > 0.$$

Also using the asymptotic series (2), we have

$$\lim_{x\to\infty} P_2(x) = \lim_{x\to\infty}\left(\frac{19531}{1451520x^6} - \frac{22639}{483840x^7} + O(x^{-8})\right) = 0.$$

Consequently, $P_2(x) > 0,\ x > 0$ and the proof is completed. Our third step is to consider the following sequence:

$$C_m = \sum_{r=1}^{m}\frac{1}{r} - \ln\frac{24m^2 + 24m + 7}{12(2m+1)} + \frac{\frac{37}{5760}}{m^3(2+m)} - \frac{c_0}{m^5(c_1+n)}, c_0, c_1 \in \mathbb{R}$$

and hence

$$C_m - C_{m+1} = -\frac{c_0}{m^5(c_1+m)} + \frac{c_0}{(m+1)^5(c_1+m+1)} + \frac{37}{5760m^3(m+2)} - \frac{1}{m+1} - \frac{37}{5760(m+1)^3(m+3)} + \ln\frac{(2m+1)(24m^2+72m+55)}{(2m+3)(24m^2+24m+7)}$$

$$= \frac{\frac{19531}{241920} - 6c_0}{m^7} + \frac{7c_0(c_1+3) - \frac{4217}{6912}}{m^8} + O(m^{-9}), m \to \infty.$$

To obtain the best rate of convergence according to Lemma 1, we should choose the constants c_0, and c_1 to satisfy the following system of equations:

$$\frac{19531}{241920} - 6c_0 = 0$$
$$7c_0(c_1+3) - \frac{4217}{6912} = 0$$

which has the following solution:

$$c_0 = \frac{19531}{1451520} \text{ and } c_1 = \frac{67917}{19531}.$$

By applying Lemma, we obtain the following result:

Theorem 5. The sequence

$$\rho_m = \sum_{r=1}^{m} \frac{1}{r} - \ln\frac{24\,m^2 + 24\,m + 7}{12\,(2\,m+1)} + \frac{37}{5760\,m^3(2+m)} - \frac{19531}{1451520\,m^5\left(m + \frac{67917}{19531}\right)}, m \in \mathbb{N}^*$$

converges to γ at a rate of convergence m^{-8}, since

$$\lim_{m \to \infty} m^8(\rho_m - \gamma) = -\frac{52671129799}{907188387840}.$$

The relation between $\psi(m)$ and ρ_m is given by

$$\rho_m - \gamma = \psi(m+1) - \ln\frac{24\,m^2 + 24\,m + 7}{12\,(2\,m+1)} + \frac{37}{5760\,m^3(2+m)} - \frac{19531}{1451520\,m^5\left(m + \frac{67917}{19531}\right)}, m \in \mathbb{N}^*.$$

Now we will get the following result:

Theorem 6. The following inequality holds

$$\psi(x+1) - \ln\frac{24x^2 + 24x + 7}{12(2x+1)} + \frac{37}{5760x^3(x+2)} < \frac{19531}{1451520x^5\left(x + \frac{67917}{19531}\right)}, x > 0.$$

As a consequence,

$$\lambda_m - \gamma < -\frac{37}{5760m^3(2+m)} + \frac{19531}{1451520m^5\left(m+\frac{67917}{19531}\right)}, m \in \mathbb{N}^*.$$

Proof: Consider the function, for $x > 0$

$$P_3(x) = \psi(x+1) - \ln\left(\frac{24x^2+24x+7}{12(2x+1)}\right) + \frac{37}{5760x^3(x+2)} - \frac{19531}{1451520x^5\left(x+\frac{67917}{19531}\right)}.$$

We have

$$\frac{d}{dx}(P_3(x) - P_3(x+1)) = \frac{w_3(x)}{w_4(x)} > 0\,, x > 0,$$

where

$$w_3(x) = 6.78094 \times 10^{26}x^{16} + 1.70415 \times 10^{28}x^{15} + 1.93003 \times 10^{29}x^{14} + 1.30664 \times 10^{30}x^{13} + 5.91672 \times 10^{30}x^{12} + 1.90041 \times 10^{31}x^{11} + 4.48176 \times 10^{31}x^{10} + 7.92591 \times 10^{31}x^9 + 1.06425 \times 10^{32}x^8 + 1.09137 \times 10^{32}x^7 + 8.54675 \times 10^{31}x^6 + 5.07761 \times 10^{31}x^5 + 2.25345 \times 10^{31}x^4 + 7.25384 \times 10^{30}x^3 + 1.60299 \times 10^{30}x^2 + 2.17791 \times 10^{29}x + 1.37297 \times 10^{28}.$$

and

$$w_4(x) = 483840x^6(x+1)^6(x+2)^2(x+3)^2(2x+1)(2x+3) \times (19531x+67917)^2(19531x+87448)^2(24x^2+24x+7)(24x^2+72x+55).$$

Hence the function $P_3(x) - P_3(x+1)$ is increasing with $\lim_{x\to\infty}(P_3(x) - P_3(x+1)) = 0$, so

$$P_2(x) < P_2(x+1), x > 0.$$

Therefore

$$P_3(x) < P_3(x+1) < P_3(x+2) < \cdots < P_3(x+r)\,, r \in \mathbb{N}\,;\; x > 0.$$

Also using the asymptotic series (2), we have

$$\lim_{x\to\infty} P_3(x) = \lim_{x\to\infty}\left(-\frac{52671129799}{907188387840x^8} + O(x^{-9})\right) = 0.$$

Consequently, $P_3(x) < 0$, $x > 0$ and the proof is completed. As a consequence of the above results, we get the following

Corollary 1. The following inequalities hold, for all m $\in \mathbb{N}^*$:

$$-\frac{37}{5760m^3(2+m)} < \lambda_m - \gamma < -\frac{37}{5760m^3(2+m)} + \frac{19531}{1451520m^5\left(m+\frac{67917}{19531}\right)}.$$

3. OPEN QUESTION

In this work, several inequalities concerning the functions $P_2(x)$ and $P_3(x)$ have been derived. It remains an open problem to determine whether these functions possess complete monotonicity for all $x > 0$. Readers are encouraged to further investigate this property.

4. CONCLUSIONS

By adjusting the logarithmic term in the classical sequence for γ_m and adding rational terms, we define the sequence

$$\rho_m = \sum_{r=1}^{m} \frac{1}{r} - \ln \frac{24\,m^2 + 24\,m + 7}{12\,(2\,m+1)} + \frac{37}{5760\,m^3(2+m)} - \frac{19531}{1451520\,m^5\left(m + \frac{67917}{19531}\right)}\,, m \in \mathbb{N}^*,$$

which provides an accelerated convergence to the Euler-Mascheroni constant γ with a rate of order m^{-8}. Also, we deduced that the sequence

$$\lambda_m = \sum_{r=1}^{m} \frac{1}{r} - \ln \frac{24\,m^2 + 24\,m + 7}{12\,(2\,m+1)}$$

has the following bounds, for all $m \in \mathbb{N}^*$:

$$-\frac{37}{5760\,m^3(2+m)} < \lambda_m - \gamma < -\frac{37}{5760\,m^3(2+m)} + \frac{19531}{1451520\,m^5\left(m + \frac{67917}{19531}\right)}.$$

We study the monotonicity of certain functions involving the Digamma function and deduce several inequalities for the Digamma function, as stated in Theorem 2, 4 and 6. The approach developed in this study can be extended to further modify our sequence, thereby enhancing its convergence rate. Additionally, we propose an open question for readers interested in the monotonicity properties of specific functions related to the Digamma function.

Remark. Some calculations in this paper were performed using the Maple software for symbolic computation.

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