ORIGINAL PAPER

ON THE BEHAVIOUR OF TOTAL UMBILICAL FIBERS UNDER QUARTER-SYMMETRIC NON-METRIC CONNECTIONS IN RIEMANNIAN SUBMERSIONS

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***Abstract**. In this study, we investigate the geometric properties of Riemannian submersions endowed with quarter-symmetric non-metric connections. We focus specifically on the M-projective curvature tensor and the conformal curvature tensor, providing an analysis of the behaviour of these tensors. We emphasize how these curvature tensors interact with totally umbilical fibers in the setting of Riemannian submersions. We explicitly compute the relevant curvature tensors and examine how the totally umbilical condition influences their relations. This clarifies the connections among the tensors and shows how they affect the geometric structure of the submersions.*

***Keywords:** M-projective curvature tensor; conformal curvature tensor; Riemannian submersion; Quarter-symmetric non-metric connection; totally umbilical fibers.*

***Mathematics Subject Classification:** 53C15; 53C20.*

1. INTRODUCTION

The study of connections is central to differential geometry and has broad relevance across many scientific disciplines. Among the various types of connections, quarter-symmetric non-metric connections are particularly notable due to their distinctive structural properties. The notion of a semi-symmetric linear connection was introduced by Friedmann and Schouten in 1924 [1]. This idea was later extended by Hayden, who introduced the concept of a metric connection on a Riemannian manifold in 1932 [2]. The first systematic study of the semi- symmetric metric connection on Riemannian manifolds was first introduced by Yano [3] in 1970. Since then, many authors have developed and refined this line of research [4–6], contributing to a deeper understanding of both semi- symmetric and non- metric connection structures [7].

The concept of submersion has long been a foundational tool in the development of differential geometry, especially in the study of smooth manifolds and maps between them. Many important contributions to this area can be found in the literature (see [8–11]), highlighting its central role in the theory of manifold structures. The notation of Riemannian submersions, which are specific types of maps between Riemannian manifolds that preserve the length of horizontal vectors, was first formalized in the influential works of O'Neill [12]

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and Gray [13]. Their pioneering research provided a solid foundation for subsequent studies, including those involving manifolds endowed with more general differentiable structures.

Over time, the theory of Riemannian submersions has become an active research area, with applications in various branches of geometry and mathematical physics. Many researchers have continued to explore this topic (see [14–20]), with particular emphasis on submersions equipped with various types of connections. For example, Akyol and Beyendi studied Riemannian submersions in the context of semi-symmetric non-metric connections [21], while Sarı investigated semi-invariant Riemannian submersions involving related connection types [22]. Additionally, Demir and Sarı examined the behaviour of Riemannian submersions under semi-symmetric metric connections [23].

In 1975, Golab introduced the concept of a quarter-symmetric connection on differentiable manifolds [24]. Building on this idea, Demir and Sarı extended the study of Riemannian submersions to quarter-symmetric non-metric connections. In their 2021 work, they investigated the associated tensor fields and computed relevant curvature quantities in detail [25].

Curvature tensors are fundamental tools in both mathematics and theoretical physics, as they describe the geometry of space and, in particular, the gravitational field. The Riemannian curvature tensor is the most familiar example, but related objects such as the Ricci tensor and the scalar curvature are also of significant importance. Moreover, new curvature structures like the concircular curvature tensor, introduced by Mishra [26], and the *M*- projective curvature tensor, studied by Pokhariyal, Mishra [27], and Ojha [28], have enriched the classification of Riemannian and semi-Riemannian manifolds. The theory of M-projective curvature tensors has been further developed by several authors, including Jawarneh and Tashtoush [29].

In recent years, Ayar and Akyol [30] have focused on the study of new curvature tensors in the context of Riemannian submersions. Their contributions are important in understanding how curvature behaves under submersion maps and in exploring the geometrical effects of these mappings under various connections.

Motivated by this line of research, the present study first reviews the basic notions of Riemannian submersions and quarter- symmetric non- metric connections, and their interaction. Within this context, we carry out detailed computations for several types of curvature tensors, with a particular focus on *M*-projective and conformal curvature tensors. We also examine the behavior of these tensors when the fibers of the submersion are totally umbilical, thereby providing a geometric analysis of this special case.

2. PRELIMINARIES

Let M_1 be a differentiable manifold equipped with a linear connection $\tilde{\nabla}$. The connection $\tilde{\nabla}$ is called symmetric if its torsion tensor $\tilde{T}$, defined by

$$\tilde{T}(X_1, X_2) = \tilde{\nabla}_{X_1} X_2 - \tilde{\nabla}_{X_2} X_1 - [X_1, X_2],$$

vanishes for all vector fields X_1 and X_2 on M_1. If the torsion does not vanish identically, the connection is non-symmetric [3].

Let (M_1, g_1) be an n-dimensional Riemannian manifold. A linear connection $\tilde{\nabla}$ on M_1 is called a quarter- symmetric connection if its torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(X_1, X_2) = \eta(X_2)\varphi X_1 - \eta(X_1)\varphi X_2,$$

where η is a differentiable 1-form and φ is a (1,1)- type tensor field. Furthermore, if the connection $\tilde{\nabla}$ also satisfies

$$\left(\tilde{\nabla}_{X_1} g_1\right)(X_2, X_3) = 0,$$

for all vector fields $X_1, X_2, X_3 \in \Gamma(TM_1)$, then the connection is referred to as a quarter-symmetric metric connection [24].

Now, consider the linear connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\varphi X_1, \tag{2.1}$$

where ∇ denotes the Levi-Civita connection associated with g_1, and the 1-form η is defined via the relation $\eta(X_2) = g_1(U_1, X_2)$ for a vector field $U_1 \in \Gamma(TM_1)$. Using equation (2.1), the torsion tensor associated with $\tilde{\nabla}$ is computed as

$$\tilde{T}(X_1, X_2) = \eta(X_2)\varphi X_1 - \eta(X_1)\varphi X_2. \tag{2.2}$$

Furthermore, the non-metricity of $\tilde{\nabla}$ is characterized by the identity

$$\left(\tilde{\nabla}_{X_1} g_1\right)(X_2, X_3) = -\eta(X_2) g_1(\varphi X_1, X_3) - \eta(X_3) g_1(X_2, \varphi X_1), \tag{2.3}$$

which also follows from equation (2.1). Therefore, the connection $\tilde{\nabla}$ defined by (2.1) is referred to as a quarter-symmetric non-metric connection, as the connection $\tilde{\nabla}$ satisfies both (2.2) and (2.3) [24,25].

A differentiable transformation $f: M_1 \to M_2$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2), with dimensions m_1 and m_2, respectively, is termed a Riemannian submersion if the following conditions are satisfied:
i.) f has maximal rank,
ii.) The transformation f_{*p} preserves the lengths of horizontal vectors $X_p \in \Gamma(\mathcal{H}_p)$ at each point $p \in M_1$.

On the other hand, for $q \in M_2$, $f^{-1}(q)$ is $(m_1 - m_2)$ dimensional submanifold of M_1. The submanifolds $f^{-1}(q)$ is called a fiber of the submersion. A vector field on M_1 is termed a vertical vector field if it is always tangent to the fibers; if it is orthogonal to the fibers, it is called a horizontal vector field. If the vector field X_1 is horizontal on M_1 and X_1 is f −related to the vector field $X_1{}'$ on the manifold M_2, then X_1 is called the fundamental vector field [31].

Lemma 2.1.1. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, and let $f: M_1 \to M_2$ be a Riemannian submersion. In this case, the following expressions hold:
i.) $g_1(X_1, X_2) = g_2(X_1{}', X_2{}') \circ f$,
ii.) For the fundamental vector field $h[X_1, X_2]$, $f_* h[X_1, X_2] = [X_1{}', X_2{}'] \circ f$,
iii.) The fundamental vector field $h(\nabla_{X_1} X_2)$ is f −related to $(\nabla'_{X_1{}'} X_2{}')$ where ∇ and ∇' are the Levi-Civita connections on M_1 and M_2, respectively,
iv.) For any vertical vector field U_1, $[X_1, U_1]$ is vertical,
where X_1 and X_2 are basic vector fields that are f −related to $X_1{}'$ and $X_2{}'$, respectively [32].

The distribution $\mathcal{V}$ corresponds to the foliation of M_1 by setting $\mathcal{V}_p = ker f_{*p}$ for any $p \epsilon M_1$. At each point p, $\mathcal{V}_p$ is defined as the vertical space, where $\mathcal{V}$ represents the vertical distribution. The sections of $\mathcal{V}$ are referred to as a Lie subalgebra $\chi^v(M_1)$ of the tangent bundle $\chi(M_1)$.

The complementary distribution of $\mathcal{V}$ determined by the Riemannian metric g_1 is denoted by $\mathcal{H}$. Hence, for any $p \epsilon M_1$, the orthogonal decomposition $T_p(M_1) = \mathcal{V}_p \oplus \mathcal{H}_p$ holds, where $\mathcal{H}_p$ is referred to as the horizontal space at p. Given any vector field $E \in \chi(M_1)$, we denote its vertical and horizontal components by vE and hE, respectively [32].

O'Neill tensor fields are determined by a Riemannian submersion $f: M_1 \to M_2$. The fundamental tensor fields are defined as follows:

$$\mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \tag{2.4}$$

$$\mathcal{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF, \tag{2.5}$$

for any $E, F \in \chi(M_1)$, where v and h represent the vertical and horizontal projections, respectively. Moreover

$$\nabla_{U_1} U_2 = \mathcal{T}_{U_1} U_2 + v\nabla_{U_1} U_2, \tag{2.6}$$

$$\nabla_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + h\nabla_{U_1} X_1, \tag{2.7}$$

$$\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + v\nabla_{X_1} U_1, \tag{2.8}$$

$$\nabla_{X_1} X_2 = \mathcal{A}_{X_1} X_2 + h\nabla_{X_1} X_2, \tag{2.9}$$

where $X_1, X_2 \in \chi^h(M_1)$; $U_1, U_2 \in \chi^v(M_1)$. Furthermore, if X_1 is a basic vector field, then, $h\nabla_{U_1} X_1 = h\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1$. We note that $\mathcal{T}_{U_1} U_2 = \mathcal{T}_{U_2} U_1$ [31].

Definition 2.1.2. Let (M_1, g_1) be a Riemannian manifold with Levi-Civita connection ∇. In this case, the Riemannian curvature tensor R is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are vector fields on M_1.

At a point $p \in M_1$, the sectional curvature K_p of the Riemannian manifold with metric g_1, is defined as follows:

$$K_p = \frac{g_1(R(X_1, X_2)X_2, X_1)}{\|X_1\|^2 \|X_2\|^2 - g_1(X_1, X_2)^2}. \tag{2.10}$$

Furthermore, the Ricci curvature, denoted as Ric, is characterized as follows:

$$Ric: C_2^\infty(TM_1) \to C_0^\infty(TM_1) \text{ by } Ric(X_1, X_2) = \sum_{i=1}^{m} g_1(R(X_1, e_i)e_i, X_2).$$

Thus, the scalar curvature τ is obtained as

$$\tau = \sum_{j=1}^{m} R_i(e_j, e_j) = \sum_{j=1}^{m} \sum_{i=1}^{m} g_1(R(e_i, e_j)e_j, e_i), \tag{2.11}$$

where $\{e_1, e_2, \dots, e_m\}$ denotes any local orthonormal frame for the tangent bundle TM_1. [33].

In this work, $S(X_1, X_2)$ is used to denote $Ric(X_1, X_2)$.

Definition 2.1.3. Let (M_1, g_1) be a Riemannian manifold. A local orthonormal frame $\{X_i, U_j\}_{1\le i\le n, 1\le j\le r}$ on M_1 is called an f −adapted frame if each vector field X_i is horizontal and each U_j is vertical. [32].

Lemma 2.1.4. Given two Riemannian manifolds (M_1, g_1) and (M_2, g_2), let f be a Riemannian submersion between them. Then we have:

$$\sum_{i=1}^{n} g_1(\mathcal{T}_{U_1} X_i, \mathcal{T}_{U_2} X_i) = \sum_{j=1}^{r} g_1(\mathcal{T}_{U_1} U_j, \mathcal{T}_{U_2} U_j), \tag{2.12}$$

$$\sum_{i=1}^{n} g_1(\mathcal{A}_{X_1} X_i, \mathcal{A}_{X_2} X_i) = \sum_{j=1}^{r} g_1(\mathcal{A}_{X_1} U_j, \mathcal{A}_{X_2} U_j), \tag{2.13}$$

$$\sum_{i=1}^{n} g_1(\mathcal{A}_{X_1} X_i, \mathcal{T}_{U_1} X_i) = \sum_{j=1}^{r} g_1(\mathcal{A}_{X_1} U_j, \mathcal{T}_{U_1} U_j), \tag{2.14}$$

where $X_1, X_2 \in \chi^h(M_1)$, $U_1, U_2 \in \chi^v(M_1)$, and $\{X_i, U_j\}$ is an f −adapted frame on (M_1, g_1) [32].

Definition 2.1.5. Let (M_1, g_1) be a Riemannian manifold and let $\{U_j\}$ be a local orthonormal frame of the vertical distribution. In this case, we introduce the horizontal vector field $\mathcal{N}$ on (M_1, g_1), defined as follows [32]:

$$\mathcal{N} = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j. \tag{2.15}$$

Let us write

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\varphi X_1, \tag{2.16}$$

where X_1, X_2 are any vector fields on M_1, η is a 1 −form, and φ is a (1,1)-type tensor field.

Note that throughout this paper, we abbreviate the Riemannian submersions endowed with quarter-symmetric non-metric connections as Q-SNMC.

Theorem 2.1.6. Let $f: M_1 \to M_2$ be a Riemannian submersion from a Riemannian manifold M_1 to a Riemannian manifold M_2 equipped with a Q-SNMC. In this case, expression (2.16) yields

$$\tilde{\mathcal{T}}(E, F) = \tilde{\mathcal{T}}_E F = \mathcal{T}_E F + \eta(vF)h\varphi(vE) + \eta(hF)v\varphi(vE), \tag{2.17}$$

$$\tilde{\mathcal{A}}(E, F) = \tilde{\mathcal{A}}_E F = \mathcal{A}_E F + \eta(hF)v\varphi(hE) + \eta(vF)h\varphi(hE), \tag{2.18}$$

for tensor fields of type (1,2) $\mathcal{T}$ and $\mathcal{A}$ on M_1, with respect to $\tilde{\nabla}$, where $E, F \in \chi(M_1)$ [25].

Corollary 2.1.7. Consider (M_1, g_1) and (M_2, g_2) as Riemannian manifolds, and let $\tilde{\nabla}$ denote a Q-SNMC. Furthermore, suppose $f: M_1 \to M_2$ is a Riemannian submersion mapping from a Riemannian manifold M_1 to another Riemannian manifold M_2 endowed with a Q-SNMC. In this case, the following equations are obtained:

$$\tilde{\mathcal{T}}_{U_1}U_2 = \tilde{\mathcal{T}}_{U_2}U_1 + \eta(vU_2)h\varphi(vU_1) - \eta(vU_1)h\varphi(vU_2), \tag{2.19}$$

$$\tilde{\mathcal{T}}_{U_1}X_1 = \mathcal{T}_{U_1}X_1 + \eta(hX_1)v\varphi(vU_1), \tag{2.20}$$

$$\tilde{\mathcal{A}}_{X_1}X_2 = -\tilde{\mathcal{A}}_{X_2}X_1 + \eta(hX_2)v\varphi(hX_1) + \eta(hX_1)v\varphi(hX_2), \tag{2.21}$$

$$\tilde{\mathcal{A}}_{X_1}U_1 = \mathcal{A}_{X_1}U_1 + \eta(vU_1)h\varphi(hX_1), \tag{2.22}$$

where $U_1, U_2 \in \chi^v(M_1)$, $X_1, X_2 \in \chi^h(M_1)$ [25].

Theorem 2.1.8. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, with $f: (M_1, g_1) \to (M_2, g_2)$ a Riemannian submersion. Then, from equation (2.16) the following equations are derived:

$$\tilde{\nabla}_{U_1}U_2 = \tilde{\mathcal{T}}_{U_1}U_2 + \hat{\nabla}_{U_1}U_2 - \eta(U_2)\varphi(U_1), \tag{2.23}$$

$$\tilde{\nabla}_{U_1}X_1 = \mathcal{T}_{U_1}X_1 + h\tilde{\nabla}_{U_1}X_1 + \eta(X_1)v\varphi(U_1), \tag{2.24}$$

$$\tilde{\nabla}_{X_1}U_1 = \mathcal{A}_{X_1}U_1 + v\tilde{\nabla}_{X_1}U_1 + \eta(U_1)h\varphi(X_1), \tag{2.25}$$

$$\tilde{\nabla}_{X_1}X_2 = \mathcal{A}_{X_1}X_2 + h\tilde{\nabla}_{X_1}X_2 + \eta(X_2)v\varphi(X_1), \tag{2.26}$$

where $U_1, U_2 \in \chi^v(M_1)$, $X_1, X_2 \in \chi^h(M_1)$, and $\hat{\nabla}_{U_1}U_2 = v\tilde{\nabla}_{U_1}U_2$ [25] and [34].

Theorem 2.1.9. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, with $f: (M_1, g_1) \to (M_2, g_2)$ a Riemannian submersion. Let $\tilde{R}$, R' and $\hat{R}$ be the Riemannian curvature tensors of M_1, M_2, and the fiber $(f^{-1}(x), \hat{g}_{1x})$ with respect to a Q-SNMC, respectively. Hence, we obtain

$$\begin{aligned} g_1\big(\tilde{R}(U_1,U_2)U_3,U_4\big) &= g_1(\hat{R}(U_1,U_2)U_3,U_4) - g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),U_4) + g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4) \\ &+ g_1\big(\eta(\tilde{\mathcal{T}}_{U_2}U_3)v\varphi(U_1),U_4\big) - g_1\big(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4\big) + g_1(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),U_4) \\ &- g_1\big(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4\big) - g_1\big(\eta(\tilde{\mathcal{T}}_{U_1}U_3)v\varphi(U_2),U_4\big) + g_1\big(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4\big) \\ &+ g_1(\eta(U_3)\varphi([U_1,U_2]),U_4), \end{aligned} \tag{2.27}$$

$$\begin{aligned} g_1\big(\tilde{R}(U_1,U_2)U_3,X_1\big) &= g_1\big(\tilde{\mathcal{T}}_{U_1}\hat{\nabla}_{U_2}U_3,X_1\big) - g_1\big(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),X_1\big) + g_1(h\tilde{\nabla}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,X_1) \\ &- g_1\big(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),X_1\big) - g_1\big(\tilde{\mathcal{T}}_{U_2}\hat{\nabla}_{U_1}U_3,X_1\big) + g_1\big(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),X_1\big) \\ &- g_1(h\tilde{\nabla}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,X_1) + g_1\big(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),X_1\big) \\ &- g_1\big(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1\big) + g_1(\eta(U_3)\varphi([U_1,U_2]),X_1), \end{aligned} \tag{2.28}$$

$$\begin{aligned} g_1\big(\tilde{R}(X_1,X_2)X_3,X_4\big) &= g_1(R'(X_1,X_2)X_3,X_4) + g_1\big(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3\big) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3) \\ &+ g_1(\eta\big(\mathcal{A}_{X_2}X_3\big)h\varphi(X_1),X_4) - g_1(\eta\big(\mathcal{A}_{X_1}X_3\big)h\varphi(X_2),X_4) + g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4\big) \\ &- g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4\big), \end{aligned} \tag{2.29}$$

$$\begin{gathered} g_1\big(\tilde{R}(X_1,X_2)X_3,U_1\big)=g_1\big(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1\big)+g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{X_2}X_3,U_1) \\ +g_1(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(X_1),U_1)+g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1\big)-g_1\big(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1\big) \\ -g_1\big(\mathcal{A}_{X_2}h\tilde{\nabla}_{X_1}X_3,U_1\big)-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_3\big)v\varphi(X_2),U_1\big)-g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1\big) \\ -g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1), \end{gathered} \tag{2.30}$$

$$\begin{gathered} g_1\big(\tilde{R}(X_1,U_1)X_2,U_2\big) \\ =g_1\big(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2\big)+g_1\big(\mathcal{A}_{X_1}h\tilde{\nabla}_{U_1}X_2,U_2\big)+g_1(\eta(h\tilde{\nabla}_{U_1}X_2)v\varphi(X_1),U_2) \\ +g_1\big(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2\big)-g_1\big(\hat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2\big)+g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_1),U_2\big) \\ -g_1(\mathcal{T}_{U_1}h\tilde{\nabla}_{X_1}X_2,U_2)-\eta\big(h\tilde{\nabla}_{X_1}X_2\big)g_1(v\varphi(U_1),U_2) \\ -g_1(\tilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2)-g_1\big(\mathcal{T}_{[X_1,U_1]}X_2,U_2\big)-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2), \end{gathered} \tag{2.31}$$

$$\begin{gathered} g_1\big(\tilde{R}(X_1,X_2)U_1,X_3\big) \\ =g_1(h\big(\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3\big)+g_1\big(\mathcal{A}_{X_1}v\tilde{\nabla}_{X_2}U_1,X_3\big)+g_1(\eta(v\tilde{\nabla}_{X_2}U_1)h\varphi(X_1),X_3) \\ +g_1(\tilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)-g_1\big(\mathcal{A}_{X_2}v\tilde{\nabla}_{X_1}U_1,X_3\big) \\ -g_1\big(\eta\big(v\tilde{\nabla}_{X_1}U_1\big)h\varphi(X_2),X_3\big)-g_1\big(\tilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3\big) \\ -g_1\big(\mathcal{A}_{[X_1,X_2]}U_1,X_3\big)-g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3), \end{gathered} \tag{2.32}$$

$$\begin{gathered} g_1\big(\tilde{R}(X_1,U_1)U_2,U_3\big) \\ =g_1\big(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_1}U_2,U_3\big)-g_1\big(\mathcal{A}_{X_1}U_3,\tilde{\mathcal{T}}_{U_1}U_2\big) \\ +g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_2)v\varphi(X_1),U_3) \\ -g_1\big(\tilde{\nabla}_{X_1}\eta(U_2)\varphi(U_1),U_3\big)+g_1\big(\mathcal{T}_{U_1}U_3,\mathcal{A}_{X_1}U_2\big)-g_1\big(\eta\big(\mathcal{A}_{X_1}U_2\big)v\varphi(U_1),U_3\big) \\ -g_1(\hat{\nabla}_{U_1}v\tilde{\nabla}_{X_1}U_2,U_3)+g_1\big(\eta\big(v\tilde{\nabla}_{X_1}U_2\big)\varphi(U_1),U_3\big)-g_1\big(\tilde{\nabla}_{U_1}\eta(U_2)h\varphi(X_1),U_3\big) \\ -g_1\big(\hat{\nabla}_{[X_1,U_1]}U_2,U_3\big)+g_1(\eta(U_2)\varphi([X_1,U_1]),U_3), \end{gathered} \tag{2.33}$$

where $U_1,U_2,U_3,U_4\in\chi^v(M_1)$, $X_1,X_2,X_3,X_4\in\chi^h(M_1)$ [25] *and* [34].

Proposition 2.1.10. Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with $f:(M_1,g_1)\to(M_2,g_2)$ a Riemannian submersion. Let $\tilde{S}$, S' and $\hat{S}$ be the Ricci tensors of M_1, M_2, and the fiber $(f^{-1}(x),\hat{g}_{1x})$ with respect to a Q-SNMC, respectively. Hence, we get

$$\begin{gathered} \tilde{S}(U_1,U_2)=\hat{S}(U_1,U_2)-g_1\big(\tilde{\mathcal{N}},\mathcal{T}_{U_1}U_2\big)+g_1\big(\eta\big(\tilde{\mathcal{N}}\big)v\varphi(U_1),U_2\big) \\ +\sum\nolimits_i\{-g_1(\eta\big(\hat{\nabla}_{U_i}U_i\big)\varphi(U_1),U_2)-g_1\big(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_2\big) \\ +g_1(\eta\big(\hat{\nabla}_{U_1}U_i\big)(\varphi(U_i),U_2)+g_1\big(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_2\big)-g_1(\eta\big(\tilde{\mathcal{T}}_{U_1}U_i\big)v\varphi(U_i),U_2) \\ +g_1\big(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_2\big)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_2)\}- \\ \sum\nolimits_j\{g_1\left(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_2\right) \\ -g_1\left(\mathcal{A}_{X_j}U_2,h\tilde{\nabla}_{U_1}X_j\right)+g_1\big(\eta\big(h\tilde{\nabla}_{U_1}X_j\big)v\varphi\big(X_j\big),U_2\big)+g_1\left(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(U_1),U_2\right) \\ -g_1\left(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_2\right)+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)g_1(\varphi(U_1),U_2)\right)+g_1\left(\mathcal{T}_{U_1}U_2,h\tilde{\nabla}_{X_j}X_j\right) \\ -g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)g_1(v\varphi(U_1),U_2)\right)-g_1(\tilde{\nabla}_{U_1}\eta\big(X_j\big)v\varphi\big(X_j\big),U_2)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_2\right) \\ -g_1\big(\eta\big(X_j\big)v\varphi\big(\big[X_j,U_1\big]\big),U_2\big)\}, \end{gathered} \tag{2.34}$$

$$
\begin{aligned}
\tilde{S}(X_1,X_2) &= S'(X_1',X_2')\circ f + g_1\big(\mathcal{N}, h\tilde{\nabla}_{X_1}X_2\big) - \sum_i \{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2, U_i) \\
&- g_1(\mathcal{A}_{X_1}U_i, h\tilde{\nabla}_{U_i}X_2) + g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_2\big)v\varphi(X_1), U_i\big) + g_1\big(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i), U_i\big) \\
&- g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2, U_i\big) + g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_i), U_i\big) - g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_2\big)v\varphi(U_i), U_i\big) \\
&- g_1\big(\tilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1), U_i\big) - g_1\big(\mathcal{T}_{[X_1,U_i]}X_2, U_i\big) - g_1(\eta(X_2)v\varphi([X_1,U_i]), U_i)\} \\
&+ \sum_j \{-g_1\big(\mathcal{A}_{X_1}X_j, \mathcal{A}_{X_2}X_j\big) - g_1(\mathcal{A}_{X_1}X_2, \mathcal{A}_{X_j}X_j) + g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1), X_2\right) \\
&- g_1\big(\eta\big(\mathcal{A}_{X_1}X_j\big)h\varphi\big(X_j\big), X_2\big) + g_1\big(\tilde{\nabla}_{X_1}\eta\big(X_j\big)v\varphi\big(X_j\big), X_2\big) \\
&- g_1\left(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1), X_2\right)\},
\end{aligned} \tag{2.35}
$$

$$
\begin{aligned}
\tilde{S}(X_1,U_1) &= \sum_i \{g_1\big(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i, U_1\big) - g_1\big(\mathcal{A}_{X_1}U_1, h\tilde{\nabla}_{X_i}X_i\big) \\
&+ g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1), U_1) \\
&+ g_1\big(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i), U_1\big) - g_1\big(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i, U_1\big) + g_1\big(\mathcal{A}_{X_i}U_1, h\tilde{\nabla}_{X_1}X_i\big) \\
&- g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i), U_1) - g_1\big(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1), U_1\big) \\
&- g_1(\eta(X_i)v\varphi([X_1,X_i], U_1)\} + \sum_j \{g_1\left(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j, U_1\right) \\
&- g_1\big(\tilde{\nabla}_{X_1}\eta\big(U_j\big)\varphi\big(U_j\big), U_1\big) + g_1\left(\mathcal{T}_{U_j}U_1, \mathcal{A}_{X_1}U_j\right) - g_1\big(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi\big(U_j\big), U_1\big) \\
&- g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j, U_1) + g_1\big(\eta\big(v\tilde{\nabla}_{X_1}U_j\big)\varphi\big(U_j\big), U_1\big) - g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_1), U_1) \\
&- g_1(\hat{\nabla}_{[X_1,U_j]}U_j, U_1) + g_1(\eta\big(U_j\big)\varphi\big(\big[X_1,U_j\big], U_1\big) \\
&- g_1\big(\tilde{\mathcal{N}}, \mathcal{A}_{X_1}U_1\big) + g_1\big(\eta\big(\tilde{\mathcal{N}}\big)v\varphi(X_1), U_1\big)\},
\end{aligned} \tag{2.36}
$$

where, $U_1, U_2 \in \chi^v(M_1)$, $X_1, X_2 \in \chi^h(M_1)$, $\tilde{\mathcal{N}} = \sum_{j=1}^{r} \tilde{\mathcal{T}}_{U_j}U_j$, and $\{X_i, U_j\}$ is an f −adapted frame on (M_1, g_1) [34].

Theorem 2.1.11. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, with $f: (M_1, g_1) \to (M_2, g_2)$ a Riemannian submersion. Let $\tilde{\tau}$, τ', $\hat{\tau}$ denote the scalar curvatures of M_1, M_2, and the fibre $(f^{-1}(x), \hat{g}_{1x})$, respectively, and let $\{X_i, U_j\}$ be an f −adapted frame on (M_1, g_1). Then, the scalar curvature of the Riemannian manifold M_1 with a Q-SNMC $\tilde{\nabla}$ is as follows [34]:

$$
\begin{aligned}
\tilde{\tau} &= \tau'\circ f + \hat{\tau} - g_1\big(\tilde{\mathcal{N}}, \mathcal{N}\big) + \sum_i \{g_1\big(\mathcal{N}, h\tilde{\nabla}_{X_i}X_i\big) - g_1\big(v\tilde{\nabla}_{X_i}\mathcal{T}_{U_i}X_i, U_i\big) \\
&- g_1\big(\mathcal{A}_{X_i}U_i, h\tilde{\nabla}_{U_i}X_i\big) \\
&+ g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_i\big)v\varphi(X_i), U_i\big) + g_1\big(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(U_i), U_i\big) - g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_i}X_i, U_i\big) \\
&+ g_1(\eta\big(\mathcal{A}_{X_i}X_i\big)\varphi(U_i), U_i) - g_1\big(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(U_i), U_i\big) \\
&- g_1\big(\tilde{\nabla}_{U_i}\eta(X_i)v\varphi(X_i), U_i\big) - g_1(\mathcal{T}_{[X_i,U_i]}X_i, U_i) \\
&- g_1(\eta(X_i)v\varphi([X_i,U_i]), U_i)\} + \sum_{i,j} \{-g_1(\mathcal{A}_{X_i}X_j, \mathcal{A}_{X_i}X_j) - g_1(\mathcal{A}_{X_i}X_i, \mathcal{A}_{X_j}X_j) \\
&+ g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_i), X_i\right) - g_1(\eta\big(\mathcal{A}_{X_i}X_j\big)h\varphi\big(X_j\big), X_i) + g_1\big(\tilde{\nabla}_{X_i}\eta\big(X_j\big)v\varphi\big(X_j\big), X_i\big) \\
&- g_1\left(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_i), X_i\right) - g_1(\eta\big(\hat{\nabla}_{U_i}U_i\big)\varphi\big(U_j\big), U_j) - \\
&g_1(\tilde{\nabla}_{U_j}\eta(U_i)\varphi(U_i), U_j) + \\
&g_1\left(\eta\left(\hat{\nabla}_{U_j}U_i\right)\varphi(U_i), U_j\right) + g_1\left(\tilde{\mathcal{T}}_{U_j}U_i, \mathcal{T}_{U_i}U_j\right) - g_1\left(\eta\left(\tilde{\mathcal{T}}_{U_j}U_i\right)v\varphi(U_i), U_j\right)
\end{aligned} \tag{2.37}
$$

$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_j),U_j)\ +g_1(\eta(U_i)\varphi([U_j,U_i]),U_j)\}$$
$$-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_j}X_j,U_j)-g_1(\mathcal{A}_{X_j}U_j,h\tilde{\nabla}_{U_j}X_j)$$
$$+g_1\left(\eta\left(h\tilde{\nabla}_{U_j}X_j\right)v\varphi(X_j),U_j\right)+g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_j),U_j\right)-g_1\left(\hat{\nabla}_{U_j}\mathcal{A}_{X_j}X_j,U_j\right)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_j),U_j\right)+g_1\left(\mathcal{N},h\tilde{\nabla}_{X_j}X_j\right)-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_j),U_j\right)$$
$$-g_1\left(\tilde{\nabla}_{U_j}\eta(X_j)v\varphi(X_j),U_j\right)-g_1\left(\mathcal{T}_{[X_j,U_j]}X_j,U_j\right)-g_1(\eta(X_j)v\varphi([X_j,U_j]),U_j)$$
$$+g_1(\eta(\tilde{\mathcal{N}})v\varphi(U_j),U_j)\}.$$

3. COMPUTATION OF CERTAIN CURVATURE TENSORS IN RIEMANNIAN SUBMERSIONS ENDOWED WITH A QUARTER-SYMMETRIC NON-METRIC CONNECTION

This section is devoted to investigating the relationships of the $M-$projective and conformal curvature tensors among the total space, base manifold, and fibers on a Riemannian submersion. Furthermore, we get a corollary for the specific case in which the fibres are assumed to be totally umbilical.

3.1. $M-$PROJECTIVE CURVATURE TENSOR IN RIEMANNIAN SUBMERSIONS ENDOWED WITH A QUARTER-SYMMETRIC NON-METRIC CONNECTION

Let M_1 and V^n be an n- dimensional manifold and space, respectively. For every $X_1,X_2,X_3\in\chi(M_1)$, the M-Projective curvature tensor field of M_1 is

$$\tilde{W}(X_1,X_2)X_3=\tilde{R}(X_1,X_2)X_3-\frac{1}{2(n-1)}\begin{bmatrix}\tilde{S}(X_2,X_3)X_1-\tilde{S}(X_1,X_3)X_2\\+g_1(X_2,X_3)\tilde{Q}X_1-g_1(X_1,X_3)\tilde{Q}X_2\end{bmatrix},\tag{3.1.1}$$

where, $\tilde{Q},\tilde{R}$ and $\tilde{S}$ denote the Ricci operator, Riemannian curvature tensor, and Ricci curvature tensor, respectively [27].
We proceed by presenting the principal theorem of this section.

Theorem 3.1.1. Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with $f:(M_1,g_1)\to(M_2,g_2)$ representing a Riemannian submersion endowed with a Q-SNMC, and let $\tilde{R},R'$ and $\hat{R}$ be the Riemannian curvature tensors; $\tilde{S},S'$ and $\hat{S}$ be Ricci tensors of M_1, M_2 and the fiber $(f^{-1}(x),\hat{g}_{1x})$ respectively. Then, for every $U_1,U_2,U_3,U_4\in\chi^v(M_1)$ and $X_1,X_2,X_3,X_4\in\chi^h(M_1)$, we have the following relations for M-Projective curvature tensor:

$$\begin{aligned}g_1(\tilde{W}(X_1,X_2)X_3,X_4)&=g_1(R'(X_1,X_2)X_3,X_4)+g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3)\\-g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3)&+g_1(\eta(\mathcal{A}_{X_2}X_3)h\varphi(X_1),X_4)-g_1(\eta(\mathcal{A}_{X_1}X_3)h\varphi(X_2),X_4)\\&+g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4)-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4)-\\&\frac{1}{2(n-1)}\{g_1(X_1,X_4)[S'(X_2',X_3')of+g_1(\mathcal{N},h\tilde{\nabla}_{X_2}X_3)\\-\sum_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_3,U_i)&-g_1(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_3)+g_1(\eta(h\tilde{\nabla}_{U_i}X_3)v\varphi(X_2),U_i)\end{aligned}\tag{3.1.2}$$

$$
\begin{aligned}
&+g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(U_i),U_i) - g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_3,U_i\big) + g_1\big(\eta\big(\mathcal{A}_{X_2}X_3\big)\varphi(U_i),U_i\big) \\
&-g_1\big(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(U_i),U_i\big) - g_1\big(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_2),U_i\big) - g_1\big(\mathcal{T}_{[X_2,U_i]}X_3,U_i\big) \\
&-g_1(\eta(X_3)v\varphi([X_2,U_i]),U_i)\} + \sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_3}X_j) \\
&-g_1\left(\mathcal{A}_{X_2}X_3,\mathcal{A}_{X_j}X_j\right) + g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_3\right) \\
&-g_1\big(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi\big(X_j\big),X_3\big) + g_1(\tilde{\nabla}_{X_2}\eta\big(X_j\big)v\varphi\big(X_j\big),X_3) \\
&- g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_2),X_3)\}] \\
&-g_1(X_2,X_4)\left[S'(X_1',X_3')of + g_1\big(\mathcal{N},h\tilde{\nabla}_{X_1}X_3\big) - \sum_i\{g_1(v\,\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_3,U_i\right) \\
&-g_1\big(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_3\big) + g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_3\big)v\varphi(X_1),U_i\big) \\
&+g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(U_i),U_i\big) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_3,U_i) \\
&+g_1\big(\eta\big(\mathcal{A}_{X_1}X_3\big)\varphi(U_i),U_i\big)-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_3\big)v\varphi(U_i),U_i\big) - g_1(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_1),U_i) \\
&-g_1\big(\mathcal{T}_{[X_1,U_i]}X_3,U_i\big) - g_1(\eta(X_3)v\varphi([X_1,U_i]),U_i)\} + \sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_3}X_j) \\
&-g_1\left(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_j}X_j\right) + g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_3\right) - g_1(\eta\big(\mathcal{A}_{X_1}X_j\big)h\varphi\big(X_j\big),X_3) \\
&+g_1(\tilde{\nabla}_{X_1}\eta\big(X_j\big)v\varphi\big(X_j\big),X_3) - g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1),X_3)\}] + g_1(X_2,X_3)[S'(X_1',X_4')of \\
&+g_1(\mathcal{N},h\tilde{\nabla}_{X_1}X_4) - \sum_i\{g_1\big(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_4,U_i\big) \\
&-g_1\big(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_4\big) + g_1(\eta\big(h\tilde{\nabla}_{U_i}X_4\big)v\varphi(X_1),U_i) \\
&+g_1\big(\tilde{\nabla}_{X_1}\eta(X_4)v\varphi(U_i),U_i\big) - g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_4,U_i\big) + g_1\big(\eta\big(\mathcal{A}_{X_1}X_4\big)\varphi(U_i),U_i\big) \\
&-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_4\big)v\varphi(U_i),U_i\big) - g_1\big(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_1),U_i\big) - g_1(\mathcal{T}_{[X_1,U_i]}X_4,U_i) \\
&-g_1(\eta(X_4)v\varphi([X_1,U_i]),U_i)\} + \textstyle\sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_4}X_j) - g_1\left(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_j}X_j\right) \\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_4\right) - g_1(\eta\big(\mathcal{A}_{X_1}X_j\big)h\varphi\big(X_j\big),X_4) \\
&+ g_1(\tilde{\nabla}_{X_1}\eta\big(X_j\big)v\varphi\big(X_j\big),X_4) \\
&-g_1\left(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1),X_4\right) - g_1(X_1,X_3)[\,S'(X_2',X_4')of + g_1(\mathcal{N},h\tilde{\nabla}_{X_2}X_4) \\
&-\sum_i\{g_1(v\,\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_4,U_i) - g_1\big(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_4\big) + g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_4\big)v\varphi(X_2),U_i\big) \\
&+g_1(\tilde{\nabla}_{X_2}\eta(X_4)\,v\varphi(U_i),U_i) - g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_4,U_i\big) + g_1\big(\eta\big(\mathcal{A}_{X_2}X_4\big)\varphi(U_i),U_i\big) \\
&-g_1(\eta\big(h\tilde{\nabla}_{X_2}X_4\big)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_2),U_i) - g_1(\mathcal{T}_{[X_2,U_i]}X_4,U_i) \\
&-g_1(\eta(X_4)v\varphi([X_2,U_i]),U_i)\} + \sum_j\{-g_1\big(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_4}X_j\big) - g_1\left(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_j}X_j\right) \\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_4\right) - g_1(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi\big(X_j\big),X_4) \\
&+g_1(\tilde{\nabla}_{X_2}\eta\big(X_j\big)v\varphi\big(X_j\big),X_4) - g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_2),X_4)\}]\},
\end{aligned}
$$

$$
\begin{aligned}
&g_1\big(\tilde{W}(X_1,X_2)X_3,U_1\big) = g_1(v\big(\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1\big) + g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{X_2}X_3,U_1) \\
&+g_1\big(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(X_1),U_1\big) + g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1) - g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1) \\
&-g_1(\mathcal{A}_{X_2}h\tilde{\nabla}_{X_1}X_3,U_1) - g_1(\eta(h\tilde{\nabla}_{X_1}X_3)v\varphi(X_2),U_1) - g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1) \\
&-g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1) - \frac{1}{2(n-1)}\{g_1(X_2,X_3)[\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1) \\
&-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i) + g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1),U_1) + g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1) \\
&-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1) + g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i) - g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i),U_1)
\end{aligned}
\qquad (3.1.3)
$$

$$
\begin{aligned}
&-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\}+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)\\
&-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j)-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1)\\
&-g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1\right)+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1)-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1)\\
&-g_1\left(\hat{\nabla}_{[X_1,U_j]}U_j,U_1\right)+g_1(\eta(U_j)\varphi([X_1,U_j],U_1)\}-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_1}U_1)\\
&+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(X_1),U_1)]\}-g_1(X_1,X_3)[\sum_i\{g_1(v\,\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)\\
&-g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_2),U_1)+g_1(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)\\
&-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i)-g_1(\eta(h\tilde{\nabla}_{X_2}X_i)v\varphi(X_i),U_1)\\
&-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1)-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)\}+\sum_j\{g_1(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1)\\
&-g_1(\tilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j)-g_1(\eta(\mathcal{A}_{X_2}U_j)v\varphi(U_j),U_1)\\
&-g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1\right)+g_1(\eta(v\tilde{\nabla}_{X_2}U_j)\varphi(U_j),U_1)-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_2),U_1)\\
&-g_1\left(\hat{\nabla}_{[X_2,U_j]}U_j,U_1\right)+g_1(\eta(U_j)\varphi([X_2,U_j],U_1)]\}\\
&-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_2}U_1)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(X_2),U_1]\},
\end{aligned}
$$

$$
\begin{aligned}
&g_1(\widetilde{W}(X_1,U_1)X_2,U_2)=g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2)+g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{U_1}X_2,U_2)\\
&+g_1(\eta(h\tilde{\nabla}_{U_1}X_2)v\varphi(X_1),U_2)+g_1(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2)-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2)\\
&+g_1(\eta(\mathcal{A}_{X_1}X_2)\varphi(U_1),U_2)-g_1(\mathcal{T}_{U_1}h\tilde{\nabla}_{X_1}X_2,U_2)-g_1(\eta(h\tilde{\nabla}_{X_1}X_2)v\varphi(U_1),U_2)\\
&-g_1(\tilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2)-g_1(\mathcal{T}_{[X_1,U_1]}X_2,U_2)-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2)\\
&+\frac{1}{2(n-1)}\{g_1(U_1,U_2)[S'(X_1',X_2')of+g_1(\mathcal{N},h\tilde{\nabla}_{X_1}X_2)-\sum_i\{g_1(v\,\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2,U_i)\\
&-g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_2)+g_1(\eta(h\tilde{\nabla}_{U_i}X_2)v\varphi(X_1),U_i)\\
&+g_1(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i),U_i)-g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2,U_i)\\
&+g_1(\eta(\mathcal{A}_{X_1}X_2)\varphi(U_i),U_i)-g_1(\eta(h(\tilde{\nabla}_{X_1}X_2)v\varphi(U_i),U_i)-g_1(\tilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1),U_i)\\
&-g_1(\mathcal{T}_{[X_1,U_i]}X_2,U_i)-g_1(\eta(X_2)v\varphi([X_1,U_i]),U_i)\}+\sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_2}X_j)\\
&-g_1(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_j}X_j)+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_2\right)-g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_2)\}\\
&+g_1(\tilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_2)-g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_2)]\}\\
&+g_1(X_1,X_2)[\hat{S}(U_1,U_2)-g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_1}U_2)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_1),U_2)\\
&+\{\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_2)\\
&-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_2)+g_1(\eta(\hat{\nabla}_{U_1}U_i)\varphi(U_i),U_2)\\
&+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_2)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_2)\\
&+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_2)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_2)\}\\
&-\sum_j\{g_1\left(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_2\right)-g_1(\mathcal{A}_{X_j}U_2,h\tilde{\nabla}_{U_1}X_j)+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_2)\\
&+g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_2)-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_2)+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_2\right)\\
&+g_1(\mathcal{T}_{U_1}U_2,h\tilde{\nabla}_{X_j}X_j)-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_2\right)-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_2)\\
&-g_1(\mathcal{T}_{[X_j,U_1]}X_j,U_2)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_2)\}]\},
\end{aligned}
\tag{3.1.4}
$$

$$
\begin{aligned}
&g_1(\widetilde{W}(U_1,U_2)U_3,U_4)=g_1(\hat{R}(U_1,U_2)U_3,U_4)-g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),U_4)\\
&+g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)+g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_3)v\varphi(U_1),U_4)-g_1(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4)
\end{aligned}
\tag{3.1.5}
$$

$$
\begin{aligned}
&+g_1(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),U_4) - g_1(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4) - g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_3)v\varphi(U_2),U_4)\\
&+g_1(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4) + g_1(\eta(U_3)\varphi([U_1,U_2]),U_4)\\
&-\frac{1}{2(n-1)}\{g_1(U_1,U_4)[\hat{S}(U_2,U_3) - g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_2}U_3) + g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_2),U_3)\\
&+\sum\nolimits_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_2),U_3) - g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_3)\\
&+g_1(\eta(\hat{\nabla}_{U_2}U_i)\varphi(U_i),U_3)\\
&+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_3) - g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_3) + g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_3)\\
&+g_1(\eta(U_i)\varphi([U_2,U_i]),U_3) - \sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_3) - g_1(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_2}X_j)\\
&+g_1(\eta(h\tilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_3) + g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_3) - g_1(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_3)\\
&+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_3) + g_1(\mathcal{T}_{U_2}U_3,h\tilde{\nabla}_{X_j}X_j) - g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_3\right)\\
&-g_1(\tilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_3) - g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_3) - g_1(\eta(X_j)v\varphi([X_j,U_2]),U_3)\\
&-g_1(U_2,U_4)[\hat{S}(U_1,U_3) - g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_1}U_3) + g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_1),U_3)\\
&+\sum\nolimits_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_3) - g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_3)\\
&+g_1(\eta(\hat{\nabla}_{U_1}U_i)\varphi(U_i),U_3)\\
&+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_3) - g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_3) + g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_3)\\
&+g_1(\eta(U_i)\varphi([U_1,U_i]),U_3) - \sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_3) - g_1(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_1}X_j)\\
&+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_3) + g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_3\right) - g_1\left(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_3\right)\\
&+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_3) + g_1(\mathcal{T}_{U_1}U_3,h\tilde{\nabla}_{X_j}X_j) - g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_3\right)\\
&-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_3) - g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_3\right) - g_1(\eta(X_j)v\varphi([X_j,U_1]),U_3)\\
&+g_1(U_2,U_3)[\hat{S}(U_1,U_4) - g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_1}U_4) + g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_1),U_4)\\
&+\sum\nolimits_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_4) - g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_4)\\
&+g_1(\eta(\hat{\nabla}_{U_1}U_i)\varphi(U_i),U_4)\\
&+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_4) - g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_4) + g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_4)\\
&+g_1(\eta(U_i)\varphi([U_1,U_i]),U_4)\} - \sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_4) - g_1(\mathcal{A}_{X_j}U_4,h\tilde{\nabla}_{U_1}X_j)\\
&+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_4) + g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_4\right) - g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_4)\\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_4\right) + g_1(\mathcal{T}_{U_1}U_4,h\tilde{\nabla}_{X_j}X_j) - g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_4)\\
&-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_4) - g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_4\right) - g_1\left(\eta(X_j)v\varphi\left([X_j,U_1]\right),U_4\right)\}\\
&-g_1(U_1,U_3)\{\hat{S}(U_2,U_4) - g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_2}U_4) + g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_2),U_4)\\
&+\sum\nolimits_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_2),U_4) - g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_4)\\
&+g_1(\eta(\hat{\nabla}_{U_2}U_i)\varphi(U_i),U_4) + g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_4) - g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_4)\\
&+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_4) + g_1(\eta(U_i)\varphi([U_2,U_i]),U_4)\} - \sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_4)\\
&-g_1(\mathcal{A}_{X_j}U_4,h\tilde{\nabla}_{U_2}X_j) + g_1(\eta(h\tilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_4) + g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_4)\\
&-g_1(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_4) + g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_4\right) + g_1(\mathcal{T}_{U_2}U_4,h\tilde{\nabla}_{X_j}X_j)\\
&-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_4\right) - g_1(\tilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_4) - g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_4)
\end{aligned}
$$

$$-g_1(\eta(X_j)v\varphi([X_j,U_2]),U_4)\}]\},$$

$$\begin{aligned}
g_1(\widetilde{W}(U_1,U_2)U_3,X_1) &= g_1(\tilde{\mathcal{T}}_{U_1}\hat{\nabla}_{U_2}U_3,X_1) - g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),X_1) \\
&+g_1(h\tilde{\nabla}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,X_1) - g_1(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),X_1) - g_1(\tilde{\mathcal{T}}_{U_2}\hat{\nabla}_{U_1}U_3,X_1) \\
&+g_1(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),X_1) - g_1(h\tilde{\nabla}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,X_1) + g_1(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),X_1) \\
&-g_1(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1)+ \\
&g_1(\eta(U_3)\varphi([U_1,U_2]),X_1) - \frac{1}{2(n-1)}\{g_1(U_2,U_3)[\textstyle\sum_i\{g_1(v\,\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1) \\
&-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i) + g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1) + g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1) \\
&-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1) + g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i) - g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1) \\
&-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1) - g_1(\eta(X_i)v\varphi([X_1,X_i],U_1) + \sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1) \\
&-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1) + g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j) - g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1) \\
&-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1) + g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1) - g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1) \\
&-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1) + g_1(\eta(U_j)\varphi([X_1,U_j],U_1) - g_1(\tilde{\mathcal{N}},\mathcal{A}_{X_1}U_1) \\
&+g_1(\eta(\tilde{\mathcal{N}})v\varphi(X_1),U_1) \\
&-g_1(U_1,U_3)[\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_2) - g_1(\mathcal{A}_{X_1}U_2,h\tilde{\nabla}_{X_i}X_i) \\
&+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1,U_2) \\
&+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_2) - g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_2) + g_1(\mathcal{A}_{X_i}U_2,h\tilde{\nabla}_{X_1}X_i) \\
&-g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_2) - g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_2) \\
&-g_1(\eta(X_i)v\varphi([X_1,X_i],U_2)\} \\
&+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_2) - g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_2) + g_1(\mathcal{T}_{U_j}U_2,\mathcal{A}_{X_1}U_j) \\
&-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_2) - g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_2) + g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_2) \\
&-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_2) - g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_2) + g_1(\eta(U_j)\varphi([X_1,U_j]),U_2) \\
&-g_1(\tilde{\mathcal{N}},\mathcal{A}_{X_1}U_2) + g_1(\eta(\tilde{\mathcal{N}})v\varphi(X_1),U_2)]\},
\end{aligned} \tag{3.1.6}$$

$$\begin{aligned}
g_1(\widetilde{W}(X_1,X_2)U_1,X_3) &= g_1(h\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3) + g_1(\mathcal{A}_{X_1}v\tilde{\nabla}_{X_2}U_1,X_3) \\
&+g_1(\eta(v\tilde{\nabla}_{X_2}U_1)h\varphi(X_1),X_3) + g_1(\tilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3) - g_1(h\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3) \\
&-g_1(\mathcal{A}_{X_2}v\tilde{\nabla}_{X_1}U_1,X_3) - g_1(\eta(v\tilde{\nabla}_{X_1}U_1)h\varphi(X_2),X_3) - g_1(\tilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3) \\
&-g_1(\mathcal{A}_{[X_1,X_2]}U_1,X_3) - g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3) \\
&-\frac{1}{2(n-1)}\{g_1(X_1,X_3)[\sum_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1) - g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i) \\
&+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_2),U_1) + g_1(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1) - g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1) \\
&+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i) - g_1(\eta(h\tilde{\nabla}_{X_2}X_i)v\varphi(X_i),U_1) - g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1) \\
&-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1) + \sum_j\{g_1(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1) - g_1(\tilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1) \\
&+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j) - g_1(\eta(\mathcal{A}_{X_2}U_j)v\varphi(U_j),U_1) - g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1) \\
&+g_1(\eta(v\tilde{\nabla}_{X_2}U_j)\varphi(U_j),U_1) - g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_2),U_1) - g_1(\hat{\nabla}_{[X_2,U_j]}U_j,U_1) \\
&+g_1(\eta(U_j)\varphi([X_2,U_j],U_1) - g_1(\tilde{\mathcal{N}},\mathcal{A}_{X_2}U_1)\} + g_1(\eta(\tilde{\mathcal{N}})v\varphi(X_2),U_1)] \\
&-g_1(X_2,X_3)[\sum_i\{g_1(v\,\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1) - g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i) \\
&+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)
\end{aligned} \tag{3.1.7}$$

$$+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i)$$
$$-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i),U_1)-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)$$
$$+\sum_j\{g_1\left(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1\right)-g_1(\tilde{\nabla}_{X_1}\eta\big(U_j\big)\varphi\big(U_j\big),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j)$$
$$-g_1\big(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi\big(U_j\big),U_1\big)-g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1\right)+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi\big(U_j\big),U_1)$$
$$-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_1),U_1)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)+g_1(\eta\big(U_j\big)\varphi\big([X_1,U_j],U_1)\}$$
$$-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_1}U_1)+g_1(\eta\big(\widetilde{\mathcal{N}}\big)v\varphi(X_1),U_1)]\}.$$

Proof: For X_1, X_2, X_3, $X_4\in\chi^h(M_1)$, if the equation (3.1.1) is multiplied by X_4, then we have

$$g_1\big(\widetilde{W}(X_1,X_2)X_3,X_4\big)=g_1(\tilde{R}(X_1,X_2)X_3,X_4)-\frac{1}{2(n-1)}[\tilde{S}(X_2,X_3)g_1(X_1,X_4)$$
$$-\tilde{S}(X_1,X_3)g_1(X_2,X_4)+g_1(X_2,X_3)\tilde{S}(X_1,X_4)-g_1(X_1,X_3)\tilde{S}(X_2,X_4)].$$

By explicitly substituting equations (2.29) and (2.35) into the expression above, we obtain the desired result, completing the proof of equation (3.1.2). The verification follows directly from straightforward algebraic manipulation and the relevant definitions. Analogous methods allow equations (3.1.3)–(3.1.7) to be verified straightforwardly. Their derivations rely on the same basic relations, and hence the details are omitted for brevity.

It is worth noting that if the tensor $\mathcal{T}_UV=g_1(U,V)H$, where H denotes the mean curvature vector field of the fiber and $U,V\epsilon\Gamma(kerf_*)$ [35], then the Riemannian submersion has totally umbilical fibers. In this case, it follows that $\mathcal{N}=0$. Moreover, as established in the work of Doğru [36], the condition $\mathcal{N}=0$ further implies that $\mathcal{N}'=0$.
Without providing a detailed proof, we now present the following corollary related to the case of totally umbilical fibers.

Corollary 3.1.2. Let (M_1,g_1) and (M_2,g_2) be Riemannian manifolds, with $f:(M_1,g_1)\rightarrow(M_2,g_2)$ representing a Riemannian submersion endowed with a Q-SNMC. If the Riemannian submersion has totally umbilical fibers, then the M −projective curvature tensor is given by

$$g_1\big(\widetilde{W}(X_1,X_2)X_3,X_4\big)=g_1(R'(X_1,X_2)X_3,X_4)+g_1\big(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3\big)-g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3)$$
$$+g_1\big(\eta\big(\mathcal{A}_{X_2}X_3\big)h\varphi(X_1),X_4\big)-g_1\big(\eta\big(\mathcal{A}_{X_1}X_3\big)h\varphi(X_2),X_4\big)+g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4)$$
$$-g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4\big)-\frac{1}{2(n-1)}\{g_1(X_1,X_4)[\,S'(X_2',X_3')of-\sum_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_3,U_i)$$
$$-g_1\big(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_3\big)+g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_3\big)v\varphi(X_2),U_i\big)+g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(U_i),U_i)$$
$$-g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_3,U_i\big)+g_1\big(\eta\big(\mathcal{A}_{X_2}X_3\big)\varphi(U_i),U_i\big)-g_1\big(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(U_i),U_i\big)$$
$$-g_1\big(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_2),U_i\big)-g_1\big(\mathcal{T}_{[X_2,U_i]}X_3,U_i\big)-g_1(\eta(X_3)v\varphi([X_2,U_i]),U_i)\}$$
$$+\sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_3}X_j)-g_1\left(\mathcal{A}_{X_2}X_3,\mathcal{A}_{X_j}X_j\right)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_3)$$
$$-g_1\big(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi\big(X_j\big),X_3\big)+g_1\big(\tilde{\nabla}_{X_2}\eta\big(X_j\big)v\varphi\big(X_j\big),X_3\big)-g_1\big(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_2),X_3\big)\}]$$
$$-g_1(X_2,X_4)[S'(X_1',X_3')of-\sum_i\{g_1(v\,\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_3,U_i)-g_1\big(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_3\big)$$
$$+g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_3\big)v\varphi(X_1),U_i\big)+g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(U_i),U_i\big)-g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_3,U_i)$$
$$+g_1\big(\eta\big(\mathcal{A}_{X_1}X_3\big)\varphi(U_i),U_i\big)-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_3\big)v\varphi(U_i),U_i\big)-g_1(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_1),U_i)$$

$$-g_1\big(\mathcal{T}_{[X_1,U_i]}X_3,U_i\big)-g_1(\eta(X_3)v\varphi([X_1,U_i]),U_i)\}+\sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_3}X_j)$$
$$-g_1\left(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_j}X_j\right)+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_3\right)-g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_3)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_3)-g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_3)\}]+g_1(X_2,X_3)[S'(X_1',X_4')of$$
$$-\sum_i\{g_1\big(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_4,U_i\big)-g_1\big(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_4\big)+g_1(\eta\big(h\tilde{\nabla}_{U_i}X_4\big)v\varphi(X_1),U_i)$$
$$+g_1\big(\tilde{\nabla}_{X_1}\eta(X_4)v\varphi(U_i),U_i\big)-g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_4,U_i\big)+g_1\big(\eta\big(\mathcal{A}_{X_1}X_4\big)\varphi(U_i),U_i\big)$$
$$-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_4\big)v\varphi(U_i),U_i\big)-g_1\big(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_1),U_i\big)-g_1(\mathcal{T}_{[X_1,U_i]}X_4,U_i)$$
$$-g_1(\eta(X_4)v\varphi([X_1,U_i]),U_i)\}+\sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_4}X_j)-g_1\left(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_j}X_j\right)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_4\right)-g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_4)+g_1(\tilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_4)$$
$$-g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_4\right)-g_1(X_1,X_3)[\,S'(X_2',X_4')of-\sum_i\{g_1(v\,\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_4,U_i)$$
$$-g_1\big(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_4\big)+g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_4\big)v\varphi(X_2),U_i\big)+g_1(\tilde{\nabla}_{X_2}\eta(X_4)\;v\varphi(U_i),U_i)$$
$$-g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_4,U_i\big)+g_1\big(\eta\big(\mathcal{A}_{X_2}X_4\big)\varphi(U_i),U_i\big)-g_1(\eta\big(h\tilde{\nabla}_{X_2}X_4\big)v\varphi(U_i),U_i)$$
$$-g_1(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_2),U_i)\;-g_1(\mathcal{T}_{[X_2,U_i]}X_4,U_i)-g_1(\eta(X_4)v\varphi([X_2,U_i]),U_i)\}$$
$$+\sum_j\{-g_1\big(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_4}X_j\big)-g_1\left(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_j}X_j\right)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_4)$$
$$-g_1\big(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi\big(X_j\big),X_4\big)+g_1(\tilde{\nabla}_{X_2}\eta(X_j)v\varphi(X_j),X_4)-g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_2),X_4)\}]\},$$

$$g_1\big(\widetilde{W}(X_1,X_2)X_3,U_1\big)=g_1\big(v\big(\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1\big)+g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{X_2}X_3,U_1)$$
$$+g_1\big(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(X_1),U_1\big)+g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1)-g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1)$$
$$-g_1(\mathcal{A}_{X_2}h\tilde{\nabla}_{X_1}X_3,U_1)-g_1(\eta(h\tilde{\nabla}_{X_1}X_3)v\varphi(X_2),U_1)-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1)$$
$$-g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1)-\frac{1}{2(n-1)}\{g_1(X_2,X_3)[\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i)-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i),U_1)$$
$$-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\}+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)$$
$$-g_1\big(\tilde{\nabla}_{X_1}\eta\big(U_j\big)\varphi\big(U_j\big),U_1\big)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j)-g_1(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi\big(U_j\big),U_1)$$
$$-g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1\right)+g_1(\eta\big(v\tilde{\nabla}_{X_1}U_j\big)\varphi\big(U_j\big),U_1)-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_1),U_1)$$
$$-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)+g_1(\eta\big(U_j\big)\varphi\big([X_1,U_j],U_1\big)\}-g_1(X_1,X_3)[\sum_i\{g_1(v\,\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i)\;+g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_2),U_1)+g_1(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1\big(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1\big)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i)-g_1(\eta\big(h\tilde{\nabla}_{X_2}X_i\big)v\varphi(X_i),U_1)$$
$$-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1)-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)\}+\sum_j\{g_1(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1)$$
$$-g_1(\tilde{\nabla}_{X_2}\eta\big(U_j\big)\varphi\big(U_j\big),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j)-g_1(\eta\big(\mathcal{A}_{X_2}U_j\big)v\varphi\big(U_j\big),U_1)$$
$$-g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1\right)+g_1\big(\eta\big(v\tilde{\nabla}_{X_2}U_j\big)\varphi\big(U_j\big),U_1\big)-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_2),U_1)$$
$$-g_1\left(\hat{\nabla}_{[X_2,U_j]}U_j,U_1\right)+g_1(\eta\big(U_j\big)\varphi\big([X_2,U_j],U_1\big)]\},$$

$$g_1\big(\widetilde{W}(X_1,U_1)X_2,U_2\big)=g_1\big(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2\big)+g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{U_1}X_2,U_2)$$
$$+g_1(\eta\big(h\tilde{\nabla}_{U_1}X_2\big)v\varphi(X_1),U_2)+g_1(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2)-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2)$$
$$+g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_1),U_2\big)-g_1\big(\mathcal{T}_{U_1}h\tilde{\nabla}_{X_1}X_2,U_2\big)-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_2\big)v\varphi(U_1),U_2)$$

$$
\begin{aligned}
&-g_1(\tilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2)-g_1(\mathcal{T}_{[X_1,U_1]}X_2,U_2)-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2)\\
&+\frac{1}{2(n-1)}\{g_1(U_1,U_2)[S'(X_1',X_2')of-\sum_i\{g_1(v\,\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2,U_i)-g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_2)\\
&+g_1(\eta\left(h\tilde{\nabla}_{U_i}X_2\right)v\varphi(X_1),U_i)+g_1(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i),U_i)-g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2,U_i)\\
&+g_1\left(\eta\left(\mathcal{A}_{X_1}X_2\right)\varphi(U_i),U_i\right)-g_1(\eta(h\left(\tilde{\nabla}_{X_1}X_2\right)v\varphi(U_i),U_i)-g_1(\tilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1),U_i)\\
&-g_1(\mathcal{T}_{[X_1,U_i]}X_2,U_i)-g_1(\eta(X_2)v\varphi([X_1,U_i]),U_i)\}+\sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_2}X_j)\\
&-g_1(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_j}X_j)+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_2\right)-g_1\left(\eta\left(\mathcal{A}_{X_1}X_j\right)h\varphi\left(X_j\right),X_2\right)\}\\
&+g_1\left(\tilde{\nabla}_{X_1}\eta\left(X_j\right)v\varphi\left(X_j\right),X_2\right)-g_1(\tilde{\nabla}_{X_j}\eta\left(X_j\right)v\varphi(X_1),X_2)]\}\\
&+g_1(X_1,X_2)[\hat{S}(U_1,U_2)+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_2)-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_2)\\
&+g_1\left(\eta\left(\hat{\nabla}_{U_1}U_i\right)\varphi(U_i),U_2\right)+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_2)-g_1(\eta\left(\tilde{\mathcal{T}}_{U_1}U_i\right)v\varphi(U_i),U_2)\\
&+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_2)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_2)\}\\
&-\sum_j\{g_1\left(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_2\right)-g_1(\mathcal{A}_{X_j}U_2,h\tilde{\nabla}_{U_1}X_j)+g_1(\eta\left(h\tilde{\nabla}_{U_1}X_j\right)v\varphi\left(X_j\right),U_2)\\
&+g_1(\tilde{\nabla}_{X_j}\eta\left(X_j\right)v\varphi(U_1),U_2)-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_2)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_2)\\
&+g_1(\mathcal{T}_{U_1}U_2,h\tilde{\nabla}_{X_j}X_j)-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_2\right)-g_1(\tilde{\nabla}_{U_1}\eta\left(X_j\right)v\varphi\left(X_j\right),U_2)\\
&-g_1(\mathcal{T}_{[X_j,U_1]}X_j,U_2)-g_1(\eta\left(X_j\right)v\varphi\left(\left[X_j,U_1\right]\right),U_2)\}]\},
\end{aligned}
$$

$$
\begin{aligned}
&g_1\left(\tilde{W}(U_1,U_2)U_3,U_4\right)=g_1\left(\hat{R}(U_1,U_2)U_3,U_4\right)-g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),U_4)\\
&+g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)+g_1(\eta\left(\tilde{\mathcal{T}}_{U_2}U_3\right)v\varphi(U_1),U_4)-g_1(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4)\\
&+g_1(\eta\left(\hat{\nabla}_{U_1}U_3\right)\varphi(U_2),U_4)-g_1(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4)-g_1(\eta\left(\tilde{\mathcal{T}}_{U_1}U_3\right)v\varphi(U_2),U_4)\\
&+g_1\left(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4\right)+g_1(\eta(U_3)\varphi([U_1,U_2]),U_4)\\
&-\frac{1}{2(n-1)}\{g_1(U_1,U_4)[\hat{S}(U_2,U_3)+\sum_i\{-g_1(\eta\left(\hat{\nabla}_{U_i}U_i\right)\varphi(U_2),U_3)\\
&-g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_3)+g_1(\eta\left(\hat{\nabla}_{U_2}U_i\right)\varphi(U_i),U_3)\\
&+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_3)-g_1\left(\eta\left(\tilde{\mathcal{T}}_{U_2}U_i\right)v\varphi(U_i),U_3\right)+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_3)\\
&+g_1(\eta(U_i)\varphi([U_2,U_i]),U_3)-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_3)-g_1(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_2}X_j)\\
&+g_1\left(\eta\left(h\tilde{\nabla}_{U_2}X_j\right)v\varphi\left(X_j\right),U_3\right)+g_1(\tilde{\nabla}_{X_j}\eta\left(X_j\right)v\varphi(U_2),U_3)-g_1(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_3)\\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_3\right)+g_1(\mathcal{T}_{U_2}U_3,h\tilde{\nabla}_{X_j}X_j)-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_3\right)\\
&-g_1(\tilde{\nabla}_{U_2}\eta\left(X_j\right)v\varphi\left(X_j\right),U_3)-g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_3)-g_1\left(\eta\left(X_j\right)v\varphi\left(\left[X_j,U_2\right]\right),U_3\right)\\
&-g_1(U_2,U_4)[\hat{S}(U_1,U_3)+\sum_i\{-g_1\left(\eta\left(\hat{\nabla}_{U_i}U_i\right)\varphi(U_1),U_3\right)\\
&-g_1\left(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_3\right)+g_1(\eta\left(\hat{\nabla}_{U_1}U_i\right)\varphi(U_i),U_3)\\
&+g_1\left(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_3\right)-g_1(\eta\left(\tilde{\mathcal{T}}_{U_1}U_i\right)v\varphi(U_i),U_3)+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_3)\\
&+g_1(\eta(U_i)\varphi([U_1,U_i]),U_3)-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_3)-g_1(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_1}X_j)\\
&+g_1\left(\eta\left(h\tilde{\nabla}_{U_1}X_j\right)v\varphi\left(X_j\right),U_3\right)+g_1\left(\tilde{\nabla}_{X_j}\eta\left(X_j\right)v\varphi(U_1),U_3\right)-g_1\left(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_3\right)\\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_3\right)+g_1(\mathcal{T}_{U_1}U_3,h\tilde{\nabla}_{X_j}X_j)-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_3\right)\\
&-g_1(\tilde{\nabla}_{U_1}\eta\left(X_j\right)v\varphi\left(X_j\right),U_3)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_3\right)-g_1\left(\eta\left(X_j\right)v\varphi\left(\left[X_j,U_1\right]\right),U_3\right)\}]
\end{aligned}
$$

$$+g_1(U_2,U_3)[\hat{S}(U_1,U_4)+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_4)$$
$$-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_4)+g_1(\eta(\hat{\nabla}_{U_1}U_i)\varphi(U_i),U_4)$$
$$+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_4)+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_4)$$
$$+g_1(\eta(U_i)\varphi([U_1,U_i]),U_4)\}-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_4)-g_1(\mathcal{A}_{X_j}U_4,h\tilde{\nabla}_{U_1}X_j)$$
$$+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_4)+g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_4\right)-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_4)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_4\right)+g_1(\mathcal{T}_{U_1}U_4,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_4)$$
$$-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_4)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_4\right)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_4)$$
$$-g_1(U_1,U_3)\{\hat{S}(U_2,U_4)+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_2),U_4)-g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_4)$$
$$+g_1(\eta(\hat{\nabla}_{U_2}U_i)\varphi(U_i),U_4)+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_4)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_4)+g_1(\eta(U_i)\varphi([U_2,U_i]),U_4)\}-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_4)$$
$$-g_1(\mathcal{A}_{X_j}U_4,h\tilde{\nabla}_{U_2}X_j)+g_1(\eta(h\tilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_4)+g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_4)$$
$$-g_1(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_4)+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_4\right)+g_1(\mathcal{T}_{U_2}U_4,h\tilde{\nabla}_{X_j}X_j)$$
$$-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_4\right)-g_1(\tilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_4)-g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_4)$$
$$-g_1(\eta(X_j)v\varphi([X_j,U_2]),U_4)\}]\},$$

$$g_1(\widetilde{W}(U_1,U_2)U_3,X_1)=g_1(\tilde{\mathcal{T}}_{U_1}\hat{\nabla}_{U_2}U_3,X_1)-g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),X_1)$$
$$+g_1(h\tilde{\nabla}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,X_1)-g_1(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),X_1)-g_1(\tilde{\mathcal{T}}_{U_2}\hat{\nabla}_{U_1}U_3,X_1)$$
$$+g_1(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),X_1)-g_1(h\tilde{\nabla}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,X_1)+g_1(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),X_1)$$
$$-g_1(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1)+g_1(\eta(U_3)\varphi([U_1,U_2]),X_1)-\frac{1}{2(n-1)}\{g_1(U_2,U_3)[\textstyle\sum_i\{g_1(v\,\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i)-g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)$$
$$-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)$$
$$-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j)-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1)$$
$$-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1)-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1)$$
$$-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)+g_1(\eta(U_j)\varphi([X_1,U_j],U_1)\}$$
$$-g_1(U_1,U_3)[\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_2)-g_1(\mathcal{A}_{X_1}U_2,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1,U_2)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_2)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_2)+g_1(\mathcal{A}_{X_i}U_2,h\tilde{\nabla}_{X_1}X_i)$$
$$-g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_2)-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_2)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_2)\}$$
$$+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_2)-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_2)+g_1(\mathcal{T}_{U_j}U_2,\mathcal{A}_{X_1}U_j)$$
$$-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_2)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_2)+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_2)$$
$$-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_2)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_2)+g_1(\eta(U_j)\varphi([X_1,U_j]),U_2)]\},$$

$$g_1(\widetilde{W}(X_1,X_2)U_1,X_3)=g_1(h\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3)+g_1(\mathcal{A}_{X_1}v\tilde{\nabla}_{X_2}U_1,X_3)$$
$$+g_1(\eta(v\tilde{\nabla}_{X_2}U_1)h\varphi(X_1),X_3)+g_1(\tilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)$$
$$-g_1(\mathcal{A}_{X_2}v\tilde{\nabla}_{X_1}U_1,X_3)-g_1(\eta(v\tilde{\nabla}_{X_1}U_1)h\varphi(X_2),X_3)-g_1(\tilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3)$$

$$-g_1\big(\mathcal{A}_{[X_1,X_2]}U_1, X_3\big) - g_1(\eta(U_1)h\varphi([X_1,X_2]), X_3)$$
$$-\frac{1}{2(n-1)}\{g_1(X_1,X_3)[\sum_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i, U_1) - g_1(\mathcal{A}_{X_2}U_1, h\tilde{\nabla}_{X_i}X_i)$$
$$+g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_2), U_1) + g_1(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i), U_1) - g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i, U_1)$$
$$+g_1(\mathcal{A}_{X_i}U_1, h\tilde{\nabla}_{X_2}X_i) - g_1(\eta\big(h\tilde{\nabla}_{X_2}X_i\big)v\varphi(X_i), U_1) - g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2), U_1)$$
$$-g_1(\eta(X_i)v\varphi([X_2,X_i], U_1) + \sum_j\{g_1(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j, U_1) - g_1(\tilde{\nabla}_{X_2}\eta\big(U_j\big)\varphi\big(U_j\big), U_1)$$
$$+g_1\left(\mathcal{T}_{U_j}U_1, \mathcal{A}_{X_2}U_j\right) - g_1\big(\eta\big(\mathcal{A}_{X_2}U_j\big)v\varphi\big(U_j\big), U_1\big) - g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j, U_1)$$
$$+g_1\big(\eta\big(v\tilde{\nabla}_{X_2}U_j\big)\varphi\big(U_j\big), U_1\big) - g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_2), U_1) - g_1(\hat{\nabla}_{[X_2,U_j]}U_j, U_1)$$
$$+g_1\big(\eta\big(U_j\big)\varphi\big([X_2,U_j\,]\big), U_1\big) - g_1(X_2,X_3)[\sum_i\{g_1(v\,\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i, U_1)$$
$$-g_1\big(\mathcal{A}_{X_1}U_1, h\tilde{\nabla}_{X_i}X_i\big) + g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1), U_1)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i), U_1) - g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i, U_1) + g_1(\mathcal{A}_{X_i}U_1, h\tilde{\nabla}_{X_1}X_i)$$
$$-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i), U_1) - g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1), U_1) - g_1(\eta(X_i)v\varphi([X_1,X_i], U_1)$$
$$+\sum_j\{g_1\left(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j, U_1\right) - g_1(\tilde{\nabla}_{X_1}\eta\big(U_j\big)\varphi\big(U_j\big), U_1) + g_1(\mathcal{T}_{U_j}U_1, \mathcal{A}_{X_1}U_j)$$
$$-g_1\big(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi\big(U_j\big), U_1\big) - g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j, U_1\right) + g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi\big(U_j\big), U_1)$$
$$-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_1), U_1) - g_1(\hat{\nabla}_{[X_1,U_j]}U_j, U_1) + g_1\left(\eta\big(U_j\big)\varphi\big(\big[X_1,U_j\big], U_1\big)\right)\}]\}$$

for every $U_1, U_2, U_3, U_4 \in \chi^v(M_1)$ and $X_1, X_2, X_3, X_4 \in \chi^h(M_1)$.

3.2. CONFORMAL CURVATURE TENSOR

Let M_1 be an n- dimensional manifold. In this context, within the n- dimensional space V, for every $X, Y, Z \in \chi(M_1)$, the conformal curvature tensor field of M_1 is defined as follows:

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-2)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y - g_1(X,Z)\tilde{Q}Y + g_1(Y,Z)\tilde{Q}X] + \frac{\tilde{\tau}}{(n-1)(n-2)}\{g_1(Y,Z)X - g_1(X,Z)Y\}, \tag{3.2.1}$$

where $\tilde{Q}$, $\tilde{R}$ and $\tilde{S}$ denote the Ricci operator, Riemannian curvature tensor, and Ricci curvature tensor, respectively [27].

Theorem 3.2.1. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds with $f: (M_1, g_1) \to (M_2, g_2)$ representing a Riemannian submersion endowed with a Q-SNMC, and let $\tilde{R}$, R' and $\hat{R}$ be Riemannian curvature tensors, $\tilde{S}$, S' and $\hat{S}$ be Ricci tensors of M_1, M_2 and the fiber $(f^{-1}(x), \hat{g}_{1x})$, respectively. Moreover, let $\tilde{\tau}$ be scalar curvature tensor of M_1. Then for any U_1, U_2, U_3, $U_4 \in \chi^v(M_1)$ and X_1, X_2, X_3, $X_4 \in \chi^h(M_1)$, we have the following relations for the conformal curvature tensor:

$$
\begin{aligned}
&g_1(\tilde{C}(X_1,X_2)X_3,X_4) \\
&= g_1(R'(X_1,X_2)X_3,X_4) + g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3) \\
&+g_1(\eta(\mathcal{A}_{X_2}X_3)h\varphi(X_1),X_4) - g_1(\eta(\mathcal{A}_{X_1}X_3)h\varphi(X_2),X_4) \\
&+ g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4) \\
&-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4) - \frac{1}{(n-2)}\{g_1(X_1,X_4)[S'(X_2',X_3')\circ f + g_1(\mathcal{N},h\tilde{\nabla}_{X_2}X_3) \\
&-\sum_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_3,U_i) - g_1(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_3) + g_1(\eta(h\tilde{\nabla}_{U_i}X_3)v\varphi(X_2),U_i) \\
&+g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(U_i),U_i) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_3,U_i) + g_1(\eta(\mathcal{A}_{X_2}X_3)\varphi(U_i),U_i) \\
&-g_1(\eta(h\tilde{\nabla}_{X_2}X_3)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_2),U_i) - g_1(\mathcal{T}_{[X_2,U_i]}X_3,U_i) \\
&-g_1(\eta(X_3)v\varphi([X_2,U_i]),U_i) + \sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_3}X_j) - g_1(\mathcal{A}_{X_2}X_3,\mathcal{A}_{X_j}X_j) \\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_3\right) - g_1(\eta(\mathcal{A}_{X_2}X_j)h\varphi(X_j),X_3) \\
&+ g_1(\tilde{\nabla}_{X_2}\eta(X_j)v\varphi(X_j),X_3) \\
&-g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_2),X_3)] - g_1(X_2,X_4)[S'(X_1{}',X_3{}')\circ f + g_1(\mathcal{N},h\tilde{\nabla}_{X_1}X_3) \\
&-\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_3,U_i) - g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_3) + g_1(\eta(h\tilde{\nabla}_{U_i}X_3)v\varphi(X_1),U_i) \\
&+g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(U_i),U_i) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_3,U_i) + g_1(\eta(\mathcal{A}_{X_1}X_3)\varphi(U_i),U_i) \\
&-g_1(\eta(h\tilde{\nabla}_{X_1}X_3)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_1),U_i) - g_1(\mathcal{T}_{[X_1,U_i]}X_3,U_i) \\
&-g_1(\eta(X_3)v\varphi([X_1,U_i]),U_i) + \sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_3}X_j) - g_1(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_j}X_j) \\
&+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_3) - g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_3) \qquad (3.2.1) \\
&+ g_1(\tilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_3) \\
&-g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_3) - g_1(X_1,X_3)[S'(X_2{}',X_4{}')\circ f + g_1(\mathcal{N},h\tilde{\nabla}_{X_2}X_4) \\
&-\sum_i\{g_1(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_4,U_i) - g_1(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_4) + g_1(\eta(h\tilde{\nabla}_{U_i}X_4)v\varphi(X_2),U_i) \\
&+g_1(\tilde{\nabla}_{X_2}\eta(X_4)v\varphi(U_i),U_i) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_4,U_i) + g_1(\eta(\mathcal{A}_{X_2}X_4)\varphi(U_i),U_i) \\
&-g_1(\eta(h\tilde{\nabla}_{X_2}X_4)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_2),U_i) - g_1(\mathcal{T}_{[X_2,U_i]}X_4,U_i) \\
&-g_1(\eta(X_4)v\varphi([X_2,U_i]),U_i) + \sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_4}X_j) - g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_j}X_j) \\
&+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_4) - g_1(\eta(\mathcal{A}_{X_2}X_j)h\varphi(X_j),X_4) \\
&+ g_1(\tilde{\nabla}_{X_2}\eta(X_j)v\varphi(X_j),X_4) \\
&-g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_2),X_4)] + g_1(X_2,X_3)[S'(X_1{}',X_4{}')\circ f + g_1(\mathcal{N},h\tilde{\nabla}_{X_1}X_4) \\
&-\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_4,U_i) - g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_4) + g_1(\eta(h\tilde{\nabla}_{U_i}X_4)v\varphi(X_1),U_i) \\
&+g_1(\tilde{\nabla}_{X_1}\eta(X_4)v\varphi(U_i),U_i) - g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_4,U_i) + g_1(\eta(\mathcal{A}_{X_1}X_4)\varphi(U_i),U_i) \\
&-g_1(\eta(h\tilde{\nabla}_{X_1}X_4)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_1),U_i) - g_1(\mathcal{T}_{[X_1,U_i]}X_4,U_i) \\
&-g_1(\eta(X_4)v\varphi([X_1,U_i]),U_i) + \sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_4}X_j) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_j}X_j) \\
&+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_4) - g_1(\eta(\mathcal{A}_{X_1}X_j)h\varphi(X_j),X_4) \\
&+ g_1(\tilde{\nabla}_{X_1}\eta(X_j)v\varphi(X_j),X_4) \\
&-g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(X_1),X_4) + \frac{\tilde{\tau}}{(n-1)(n-2)}\{g_1(X_2,X_3)g_1(X_1,X_4) - \\
&g_1(X_1,X_3)g_1(X_2,X_4),
\end{aligned}
$$

$$
\begin{gathered}
g_1\big(\tilde{C}(X_1,X_2)X_3,U_1\big)=g_1\big(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1\big)+g_1\big(\mathcal{A}_{X_1}h\tilde{\nabla}_{X_2}X_3,U_1\big)\\
+g_1(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1)-g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1)\\
-g_1(\mathcal{A}_{X_2}h\tilde{\nabla}_{X_1}X_3,U_1)-g_1(\eta(h\tilde{\nabla}_{X_1}X_3)v\varphi(X_2),U_1)-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1)\\
-g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1)\ -\frac{1}{(n-2)}\{g_1(X_2,X_3)[\sum\nolimits_i g_1(v\,\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)\\
-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)\\
-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1\big(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i\big)-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i),U_1)\\
-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)\\
+\sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)-g_1(\tilde{\nabla}_{X_1}\eta\big(U_j\big)\varphi\big(U_j\big),U_1)\\
+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j)-g_1(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi\big(U_j\big),U_1)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)\\
+g_1(\eta\big(v\tilde{\nabla}_{X_1}U_j\big)\varphi\big(U_j\big),U_1)-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_1),U_1)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)\\
+g_1(\eta\big(U_j\big)\varphi\big(\big[X_1,U_j\big],U_1\big)\}-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_1}U_1)+g_1(\eta\big(\widetilde{\mathcal{N}}\big)v\varphi(X_1),U_1)\\
-g_1(X_1,X_3)[\sum\nolimits_i\{g_1(v\,\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i)\\
+\ g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_2),U_1)\\
+g_1\big(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1\big)-g_1\big(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1\big)+g_1\big(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i\big)\\
-g_1(\eta\big(h\tilde{\nabla}_{X_2}X_i\big)v\varphi(X_i),U_1)\ -g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1)\\
-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)+\sum\nolimits_j\{g_1(v\,\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1)\\
-g_1\big(\tilde{\nabla}_{X_2}\eta\big(U_j\big)\varphi\big(U_j\big),U_1\big)+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j\right)-g_1\big(\eta\big(\mathcal{A}_{X_2}U_j\big)v\varphi\big(U_j\big),U_1\big)\\
-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1)+g_1(\eta\big(v\tilde{\nabla}_{X_2}U_j\big)\varphi\big(U_j\big),U_1)\\
-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_2),U_1)-g_1(\hat{\nabla}_{[X_2,U_j]}U_j,U_1)\\
+g_1\big(\eta\big(U_j\big)\varphi\big(\big[X_2,U_j\big]\big),U_1\big)\}-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_2}U_1)+g_1(\eta\big(\widetilde{\mathcal{N}}\big)v\varphi(X_2),U_1)]\},
\end{gathered}
\tag{3.2.3}
$$

$$
\begin{gathered}
g_1\big(\tilde{C}(U_1,U_2)U_3,U_4\ \big)\\
=g_1\big(\hat{R}(U_1,U_2)U_3,U_4\big)-g_1(\eta\big(\hat{\nabla}_{U_2}U_3\big)\varphi(U_1),U_4)\\
+\ g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)\\
+g_1\big(\eta\big(\tilde{\mathcal{T}}_{U_2}U_3\big)v\varphi(U_1),U_4\big)-g_1\big(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4\big)+g_1(\eta\big(\hat{\nabla}_{U_1}U_3\big)\varphi(U_2),U_4)\\
-g_1(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4)-g_1(\eta\big(\tilde{\mathcal{T}}_{U_1}U_3\big)v\varphi(U_2),U_4)+g_1(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4)\\
+g_1(\eta(U_3)\varphi([U_1,U_2]),U_4)-\frac{1}{(n-2)}\{g_1(U_1,U_4)[\hat{S}(U_2,U_3)-g_1\big(\widetilde{\mathcal{N}},\mathcal{T}_{U_2}U_3\big)\\
+g_1\big(\eta\big(\widetilde{\mathcal{N}}\big)v\varphi(U_2),U_3\big)+\sum\nolimits_i\{-g_1(\eta\big(\hat{\nabla}_{U_i}U_i\big)\varphi(U_2),U_3)\\
-\ g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_3)\\
+g_1\big(\eta\big(\hat{\nabla}_{U_2}U_i\big)\varphi(U_i),U_3\big)+g_1\big(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_3\big)-g_1\big(\eta\big(\tilde{\mathcal{T}}_{U_2}U_i\big)v\varphi(U_i),U_3\big)\\
+\ g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_3)\ +g_1(\eta(U_i)\varphi([U_2,U_i]),U_3)\\
-\sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j\,,U_3)-g_1(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_2}X_j)+g_1(\eta\big(h\tilde{\nabla}_{U_2}X_j\big)v\varphi\big(X_j\big),U_3)\\
+g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(U_2),U_3)-g_1(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_3)\\
+g_1(\eta(\mathcal{A}_{X_j}X_j)\varphi(U_2),U_3)+g_1(\mathcal{T}_{U_2}U_3,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_3)\\
-g_1(\tilde{\nabla}_{U_2}\eta\big(X_j\big)v\varphi\big(X_j\big),U_3)\ -g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_3)-g_1(\eta\big(X_j\big)v\varphi\big(\big[X_j,U_2\big]\big),U_3)\\
-g_1(U_2,U_4)[\hat{S}(U_1,U_3)-g_1\big(\widetilde{\mathcal{N}},\mathcal{T}_{U_1}U_3\big)+g_1(\eta\big(\widetilde{\mathcal{N}}\big)v\varphi(U_1),U_3)
\end{gathered}
\tag{3.2.5}
$$

$$
\begin{aligned}
&+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_3)-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_3)\\
&+g_1(\eta(\hat{\nabla}_{U_1}U_i)\varphi(U_i),U_3)\\
&+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_3)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_3)+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_3)\\
&+g_1(\eta(U_i)\varphi([U_1,U_i]),U_3)-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_3)-g_1(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_1}X_j)\\
&+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_3)+g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_3)-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_3)\\
&+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_3\right)+g_1(\mathcal{T}_{U_1}U_3,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta(h\tilde{\nabla}_{X_j}X_j)v\varphi(U_1),U_3)\\
&-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_3)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_3\right)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_3)\}]\\
&-g_1(U_1,U_3)[\hat{S}(U_2,U_4)-g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_2}U_4)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_2),U_4)\\
&+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_2),U_4)-g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_4)\\
&+g_1(\eta(\hat{\nabla}_{U_2}U_i)\varphi(U_i),U_4)+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_4)\\
&+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_4)+g_1(\eta(U_i)\varphi([U_2,U_i]),U_4)-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_4)\\
&-g_1\left(\mathcal{A}_{X_j}U_4,h\tilde{\nabla}_{U_2}X_j\right)+g_1(\eta(h\tilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_4)+g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_4\right)\\
&-g_1(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_4)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_4)\\
&+g_1(\mathcal{T}_{U_2}U_4,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_4)\\
&-g_1(\tilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_4)-g_1\left(\mathcal{T}_{[X_j,U_2]}X_j,U_4\right)-g_1(\eta(X_j)v\varphi([X_j,U_2]),U_4)\}]\\
&+g_1(U_2,U_3)[\tilde{S}(U_1,U_4)-g_1(\widetilde{\mathcal{N}},\mathcal{T}_{U_1}U_4)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(U_1),U_4)\\
&+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_4)-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_4)\\
&+g_1(\eta(\hat{\nabla}_{U_1}U_i)\varphi(U_i),U_4)+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_4)\\
&+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_4)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_4)\}-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_4)\\
&-g_1\left(\mathcal{A}_{X_j}U_4,h\tilde{\nabla}_{U_1}X_j\right)+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_4)+g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_4\right)\\
&-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_4)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_4)+g_1(\mathcal{T}_{U_1}U_4,h\tilde{\nabla}_{X_j}X_j)\\
&-g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_4)-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_4)\\
&-g_1(\mathcal{T}_{[X_j,U_1]}X_j,U_4)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_4)\}]\\
&+\frac{\tilde{\tau}}{(n-1)(n-2)}\{g_1(U_2,U_3)g_1(U_1,U_4)-g_1(U_1,U_3)g_1(U_2,U_4)\},
\end{aligned}
$$

$$
\begin{aligned}
&g_1(\tilde{C}(U_1,U_2)U_3,X_1)\\
&=g_1(\tilde{\mathcal{T}}_{U_1}\hat{\nabla}_{U_2}U_3,X_1)-g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),X_1)\\
&+g_1(h\tilde{\nabla}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,X_1)\\
&-g_1(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),X_1)-g_1(\tilde{\mathcal{T}}_{U_2}\hat{\nabla}_{U_1}U_3,X_1)+g_1(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),X_1)\\
&-g_1(h\tilde{\nabla}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,X_1)+g_1(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),X_1)-g_1(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1)\\
&-g_1(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1)+g_1(\eta(U_3)\varphi([U_1,U_2]),X_1)\\
&-\frac{1}{(n-2)}\{g_1(U_2,U_3)[\sum_i g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)\\
&+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)\\
&+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i)-g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)
\end{aligned}
\tag{3.2.6}
$$

$$-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)$$
$$-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1)$$
$$+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right)-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)$$
$$+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1)-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)$$
$$+g_1(\eta(U_j)\varphi([X_1,U_j]),U_1)\}-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_1}U_1)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(X_1),U_1)]$$
$$-g_1(U_1,U_3)[\textstyle\sum_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_2)-g_1(\mathcal{A}_{X_1}U_2,h\tilde{\nabla}_{X_i}X_i)+$$
$$g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_2)$$
$$+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_2)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_2)+g_1(\mathcal{A}_{X_i}U_2,h\tilde{\nabla}_{X_1}X_i)$$
$$-g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_2)-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_2)$$
$$-g_1(\eta(X_i)v\varphi([X_1,X_i]),U_2)$$
$$+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_2)-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_2)+g_1\left(\mathcal{T}_{U_j}U_2,\mathcal{A}_{X_1}U_j\right)$$
$$-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_2)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_2)+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_2)$$
$$-g_1\left(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_2\right)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_2)+g_1(\eta(U_j)\varphi([X_1,U_j]),U_2)\}$$
$$-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_1}U_2)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(X_1),U_2)]\},$$

$$g_1(\tilde{C}(X_1,X_2)U_1,X_3)=g_1(h\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3)+g_1(\mathcal{A}_{X_1}v\tilde{\nabla}_{X_2}U_1,X_3)$$
$$+g_1(\eta(v\tilde{\nabla}_{X_2}U_1)h\varphi(X_1),X_3)+g_1(\tilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)$$
$$-g_1(\mathcal{A}_{X_2}v\tilde{\nabla}_{X_1}U_1,X_3)-g_1(\eta(v\tilde{\nabla}_{X_1}U_1)h\varphi(X_2),X_3)-g_1(\tilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3)$$
$$-g_1(\mathcal{A}_{[X_1,X_2]}U_1,X_3)-g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3)$$
$$-\frac{1}{(n-2)}\{g_1(X_1,X_3)[\sum_i g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_2),U_1)$$
$$+g_1(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i)$$
$$-g_1(\eta(h\tilde{\nabla}_{X_2}X_i)v\varphi(X_i),U_1)-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1)$$
$$-g_1(\eta(X_i)v\varphi([X_2,X_i]),U_1)$$
$$+\sum_j\{g_1(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1)-g_1(\tilde{\nabla}_{X_2}\eta(U_j)\varphi(U_j),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j)$$
$$-g_1(\eta(\mathcal{A}_{X_2}U_j)v\varphi(U_j),U_1)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1)+g_1(\eta(v\tilde{\nabla}_{X_2}U_j)\varphi(U_j),U_1), \quad (3.2.7)$$
$$-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_2),U_1)-g_1(\hat{\nabla}_{[X_2,U_j]}U_j,U_1)+g_1(\eta(U_j)\varphi([X_2,U_j]),U_1)\}$$
$$-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_2}U_1)+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(X_2),U_1)]-g_1(X_2,X_3)[\sum_i g_1(v(\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta(h\tilde{\nabla}_{X_i}X_i)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i)$$
$$-g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)$$
$$-g_1(\eta(X_i)v\varphi([X_1,X_i]),U_1)+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)-g_1(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1)$$
$$+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right)-g_1(\eta(\mathcal{A}_{X_1}U_j)v\varphi(U_j),U_1)-g_1\left(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1\right)$$
$$+g_1(\eta(v\tilde{\nabla}_{X_1}U_j)\varphi(U_j),U_1)-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1)$$
$$-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)+g_1(\eta(U_j)\varphi([X_1,U_j]),U_1)\}-g_1(\widetilde{\mathcal{N}},\mathcal{A}_{X_1}U_1)$$
$$+g_1(\eta(\widetilde{\mathcal{N}})v\varphi(X_1),U_1)]\}.$$

Corollary 3.2.2. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and $f:(M_1, g_1) \to (M_2, g_2)$ be a Riemannian submersion endowed with a Q-SNMC. If the Riemannian submersion has totally umbilical fibers, then the conformal curvature tensors according to certain vector fields are given as follows:

$$
\begin{aligned}
g_1\big(\tilde{C}(X_1,X_2)X_3,X_4\big) &= g_1(R'(X_1,X_2)X_3,X_4) + g_1\big(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_1}X_3\big) - g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_2}X_3) \\
&+g_1(\eta\big(\mathcal{A}_{X_2}X_3\big)h\varphi(X_1),X_4) - g_1(\eta\big(\mathcal{A}_{X_1}X_3\big)h\varphi(X_2),X_4) + g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),X_4) \\
&-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),X_4) - \frac{1}{(n-2)}\{g_1(X_1,X_4)[S'(X_2',X_3')\circ f - \sum_i\{ g_1\big(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_3,U_i\big) \\
&-g_1\big(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_3\big) + g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_3\big)v\varphi(X_2),U_i\big) + g_1\big(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(U_i),U_i\big) \\
&-g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_3,U_i) + g_1(\eta\big(\mathcal{A}_{X_2}X_3\big)\varphi(U_i),U_i) \\
&-g_1(\eta(h\tilde{\nabla}_{X_2}X_3)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_2),U_i) \\
&-g_1(\mathcal{T}_{[X_2,U_i]}X_3,U_i) - g_1(\eta(X_3)v\varphi([X_2,U_i]),U_i) + \sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_3}X_j) \\
&-g_1(\mathcal{A}_{X_2}X_3,\mathcal{A}_{X_j}X_j) + g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_3\right) \\
&-g_1(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi\big(X_j\big),X_3) + g_1(\tilde{\nabla}_{X_2}\eta\big(X_j\big)v\varphi\big(X_j\big),X_3) \\
&-g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_2),X_3)] - g_1(X_2,X_4)[S'(X_1{}',X_3{}')\circ f - \textstyle\sum_i\{ g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_3,U_i) \\
&-g_1\big(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_3\big) + g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_3\big)v\varphi(X_1),U_i\big) + g_1\big(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(U_i),U_i\big) \\
&-g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_3,U_i\big) + g_1(\eta\big(\mathcal{A}_{X_1}X_3\big)\varphi(U_i),U_i) \\
&-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_3\big)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_3)v\varphi(X_1),U_i) \\
&-g_1(\mathcal{T}_{[X_1,U_i]}X_3,U_i) - g_1(\eta(X_3)v\varphi([X_1,U_i]),U_i) + \sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_3}X_j) \\
&-g_1(\mathcal{A}_{X_1}X_3,\mathcal{A}_{X_j}X_j) + g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_3) \\
&-g_1(\eta\big(\mathcal{A}_{X_1}X_j\big)h\varphi\big(X_j\big),X_3) + g_1(\tilde{\nabla}_{X_1}\eta\big(X_j\big)v\varphi\big(X_j\big),X_3) \\
&-g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1),X_3) - g_1(X_1,X_3)[S'(X_2{}',X_4{}')\circ f - \textstyle\sum_i\{ g_1(v\tilde{\nabla}_{X_2}\mathcal{T}_{U_i}X_4,U_i) \\
&-g_1(\mathcal{A}_{X_2}U_i,h\tilde{\nabla}_{U_i}X_4) + g_1(\eta(h\tilde{\nabla}_{U_i}X_4)v\varphi(X_2),U_i) + g_1\big(\tilde{\nabla}_{X_2}\eta(X_4)v\varphi(U_i),U_i\big) \\
&-g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_2}X_4,U_i) + g_1(\eta\big(\mathcal{A}_{X_2}X_4\big)\varphi(U_i),U_i) \\
&-g_1(\eta\big(h\tilde{\nabla}_{X_2}X_4\big)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_2),U_i) \\
&-g_1(\mathcal{T}_{[X_2,U_i]}X_4,U_i) - g_1(\eta(X_4)v\varphi([X_2,U_i]),U_i) + \sum_j\{-g_1(\mathcal{A}_{X_2}X_j,\mathcal{A}_{X_4}X_j) \\
&-g_1(\mathcal{A}_{X_2}X_4,\mathcal{A}_{X_j}X_j) + g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_2),X_4) \\
&-g_1(\eta\big(\mathcal{A}_{X_2}X_j\big)h\varphi\big(X_j\big),X_4) + g_1(\tilde{\nabla}_{X_2}\eta\big(X_j\big)v\varphi\big(X_j\big),X_4) \\
&-g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_2),X_4)] + g_1(X_2,X_3)[S'(X_1{}',X_4{}')\circ f - \textstyle\sum_i\{ g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_4,U_i) \\
&-g_1(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_4) + g_1(\eta\big(h\tilde{\nabla}_{U_i}X_4\big)v\varphi(X_1),U_i) + g_1\big(\tilde{\nabla}_{X_1}\eta(X_4)v\varphi(U_i),U_i\big) \\
&-g_1(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_4,U_i) + g_1(\eta\big(\mathcal{A}_{X_1}X_4\big)\varphi(U_i),U_i) \\
&-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_4\big)v\varphi(U_i),U_i) - g_1(\tilde{\nabla}_{U_i}\eta(X_4)v\varphi(X_1),U_i) \\
&-g_1(\mathcal{T}_{[X_1,U_i]}X_4,U_i) - g_1(\eta(X_4)v\varphi([X_1,U_i]),U_i) + \sum_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_4}X_j) \\
&-g_1(\mathcal{A}_{X_1}X_4,\mathcal{A}_{X_j}X_j) + g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_4) \\
&-g_1(\eta\big(\mathcal{A}_{X_1}X_j\big)h\varphi\big(X_j\big),X_4) + g_1(\tilde{\nabla}_{X_1}\eta\big(X_j\big)v\varphi\big(X_j\big),X_4) \\
&-g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1),X_4) + \tfrac{\tilde{\tau}}{(n-1)(n-2)}\{g_1(X_2,X_3)g_1(X_1,X_4) - g_1(X_1,X_3)g_1(X_2,X_4),
\end{aligned}
$$

$$g_1\big(\tilde{C}(X_1,X_2)X_3,U_1\big)=g_1\big(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}X_3,U_1\big)+g_1\big(\mathcal{A}_{X_1}h\tilde{\nabla}_{X_2}X_3,U_1\big)$$
$$+g_1(\eta\big(h\tilde{\nabla}_{X_2}X_3\big)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_3)v\varphi(X_2),U_1)-g_1(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}X_3,U_1)$$
$$-g_1(\mathcal{A}_{X_2}h\tilde{\nabla}_{X_1}X_3,U_1)-g_1(\eta(h\tilde{\nabla}_{X_1}X_3)v\varphi(X_2),U_1)-g_1(\tilde{\nabla}_{X_2}\eta(X_3)v\varphi(X_1),U_1)$$
$$-g_1(\eta(X_3)v\varphi([X_1,X_2]),U_1)-\frac{1}{(n-2)}\{g_1(X_2,X_3)[\sum\nolimits_i g_1(v\,\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1),U_1)+g_1(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1)$$
$$-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1\big(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i\big)-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i),U_1)$$
$$-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)+\sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)$$
$$-g_1\big(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1\big)+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right)-g_1\big(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi(U_j),U_1\big)$$
$$-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)+g_1\big(\eta\big(v\tilde{\nabla}_{X_1}U_j\big)\varphi(U_j),U_1\big)$$
$$-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)$$
$$+g_1(\eta(U_j)\varphi([X_1,U_j],U_1)\}-g_1(X_1,X_3)[\sum\nolimits_i\{g_1(v\,\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i)$$
$$+g_1\big(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_2),U_1\big)+g_1\big(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1\big)-g_1\big(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1\big)$$
$$+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i)-g_1(\eta\big(h\tilde{\nabla}_{X_2}X_i\big)v\varphi(X_i),U_1)$$
$$-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1)-g_1(\eta(X_i)v\varphi([X_2,X_i],U_1)$$
$$+\sum\nolimits_j\{g_1(v\,\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1)-g_1(\tilde{\nabla}_{X_2}\eta\big(U_j\big)\varphi\big(U_j\big),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j)$$
$$-g_1(\eta\big(\mathcal{A}_{X_2}U_j\big)v\varphi\big(U_j\big),U_1)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1)+g_1(\eta\big(v\tilde{\nabla}_{X_2}U_j\big)\varphi\big(U_j\big),U_1)$$
$$-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_2),U_1)-g_1(\hat{\nabla}_{[X_2,U_j]}U_j,U_1)+g_1\big(\eta\big(U_j\big)\varphi\big(\big[X_2,U_j\big]\big),U_1\big)\},$$

$$g_1\big(\tilde{C}(X_1,U_1)X_2,U_2\big)=g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_1}X_2,U_2)+g_1(\mathcal{A}_{X_1}h\tilde{\nabla}_{U_1}X_2,U_2)$$
$$+g_1(\eta\big(h\tilde{\nabla}_{U_1}X_2\big)v\varphi(X_1),U_2)+g_1\big(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_1),U_2\big)-g_1\big(\hat{\nabla}_{U_1}\mathcal{A}_{X_1}X_2,U_2\big)$$
$$+g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_1),U_2\big)-g_1(\mathcal{T}_{U_1}h\tilde{\nabla}_{X_1}X_2,U_2)-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_2\big)v\varphi(U_1),U_2)$$
$$-g_1(\tilde{\nabla}_{U_1}\eta(X_2)v\varphi(X_1),U_2)-g_1(\mathcal{T}_{[X_1,U_1]}X_2,U_2)-g_1(\eta(X_2)v\varphi([X_1,U_1]),U_2)$$
$$+\frac{1}{(n-2)}\{g_1(U_1,U_2)[S'(X_1',X_2')\circ f-\sum\nolimits_i\{g_1(v\tilde{\nabla}_{X_1}\mathcal{T}_{U_i}X_2,U_i)$$
$$-g_1\big(\mathcal{A}_{X_1}U_i,h\tilde{\nabla}_{U_i}X_2\big)+g_1\big(\eta\big(h\tilde{\nabla}_{U_i}X_2\big)v\varphi(X_1),U_i\big)+g_1\big(\tilde{\nabla}_{X_1}\eta(X_2)v\varphi(U_i),U_i\big)$$
$$-g_1\big(\hat{\nabla}_{U_i}\mathcal{A}_{X_1}X_2,U_i\big)+g_1\big(\eta\big(\mathcal{A}_{X_1}X_2\big)\varphi(U_i),U_i\big)$$
$$-g_1\big(\eta\big(h\tilde{\nabla}_{X_1}X_2\big)v\varphi(U_i),U_i\big)-g_1\big(\tilde{\nabla}_{U_i}\eta(X_2)v\varphi(X_1),U_i\big)$$
$$-g_1\big(\mathcal{T}_{[X_1,U_i]}X_2,U_i\big)-g_1(\eta(X_2)v\varphi([X_1,U_i]),U_i)+\sum\nolimits_j\{-g_1(\mathcal{A}_{X_1}X_j,\mathcal{A}_{X_2}X_j)$$
$$-g_1(\mathcal{A}_{X_1}X_2,\mathcal{A}_{X_j}X_j)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)h\varphi(X_1),X_2)$$
$$-g_1(\eta\big(\mathcal{A}_{X_1}X_j\big)h\varphi\big(X_j\big),X_2)+g_1(\tilde{\nabla}_{X_1}\eta\big(X_j\big)v\varphi\big(X_j\big),X_2)$$
$$-g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(X_1),X_2)+g_1(X_1,X_2)[\hat{S}(U_1,U_2)+\sum\nolimits_i\{-g_1\big(\eta\big(\hat{\nabla}_{U_i}U_i\big)\varphi(U_1),U_2\big)$$
$$-g_1\big(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_2\big)+g_1\big(\eta\big(\hat{\nabla}_{U_1}U_i\big)\varphi(U_i),U_2\big)+g_1\big(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_2\big)$$
$$-g_1(\eta\big(\tilde{\mathcal{T}}_{U_1}U_i\big)v\varphi(U_i),U_2)+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_2)$$
$$+g_1(\eta(U_i)\varphi([U_1,U_i]),U_2)-\sum\nolimits_j\{g_1\left(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_2\right)$$
$$-g_1\left(\mathcal{A}_{X_j}U_2,h\tilde{\nabla}_{U_1}X_j\right)+g_1(\eta\big(h\tilde{\nabla}_{U_1}X_j\big)v\varphi\big(X_j\big),U_2)+g_1(\tilde{\nabla}_{X_j}\eta\big(X_j\big)v\varphi(U_1),U_2)$$
$$-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_2)+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_2)+g_1(\mathcal{T}_{U_1}U_2,h\tilde{\nabla}_{X_j}X_j)$$

$$-g_1\left(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_2\right)-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_2)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_2\right)$$
$$-g_1(\eta(X_j)(v\varphi([X_j,U_1]),U_2)\}]-\frac{\tilde{\tau}}{(n-1)(n-2)}g_1(X_1,X_2)g_1(U_1,U_2),$$

$$g_1(\tilde{C}(U_1,U_2)U_3,U_4)=g_1(\hat{R}(U_1,U_2)U_3,U_4)-g_1(\eta(\hat{\nabla}_{U_2}U_3)\varphi(U_1),U_4)+g_1(\mathcal{T}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,U_4)$$
$$+g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_3)v\varphi(U_1),U_4)-g_1(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),U_4)+g_1(\eta(\hat{\nabla}_{U_1}U_3)\varphi(U_2),U_4)$$
$$-g_1(\mathcal{T}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_3)v\varphi(U_2),U_4)+g_1(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),U_4)$$
$$+g_1(\eta(U_3)\varphi([U_1,U_2]),U_4)-\frac{1}{(n-2)}\{g_1(U_1,U_4)[\hat{S}(U_2,U_3)+$$
$$\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_2),U_3)-g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_3)$$
$$+g_1(\eta(\hat{\nabla}_{U_2}U_i)\varphi(U_i),U_3)+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_3)-g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_3)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_3)+g_1(\eta(U_i)\varphi([U_2,U_i]),U_3)-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_3)$$
$$-g_1\left(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_2}X_j\right)+g_1(\eta(h\tilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_3)+g_1\left(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_3\right)$$
$$-g_1(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_3)+g_1(\eta(\mathcal{A}_{X_j}X_j)\varphi(U_2),U_3)$$
$$+g_1(\mathcal{T}_{U_2}U_3,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_3)$$
$$-g_1(\tilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_3)-g_1(\mathcal{T}_{[X_j,U_2]}X_j,U_3)-g_1(\eta(X_j)v\varphi([X_j,U_2]),U_3)\}]$$
$$-g_1(U_2,U_4)[\hat{S}(U_1,U_3)+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_3)-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_3)$$
$$+g_1(\eta(\hat{\nabla}_{U_1}U_i)\varphi(U_i),U_3)+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_3)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_3)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_3)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_3)$$
$$-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_3)-g_1(\mathcal{A}_{X_j}U_3,h\tilde{\nabla}_{U_1}X_j)$$
$$+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_3)+g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_3)-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_3)$$
$$+g_1\left(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_3\right)+g_1(\mathcal{T}_{U_1}U_3,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta(h\tilde{\nabla}_{X_j}X_j)v\varphi(U_1),U_3)$$
$$-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_3)-g_1\left(\mathcal{T}_{[X_j,U_1]}X_j,U_3\right)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_3)\}]$$
$$-g_1(U_1,U_3)[\hat{S}(U_2,U_4)+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_2),U_4)-g_1(\tilde{\nabla}_{U_2}\eta(U_i)\varphi(U_i),U_4)$$
$$+g_1(\eta(\hat{\nabla}_{U_2}U_i)\varphi(U_i),U_4)+g_1(\tilde{\mathcal{T}}_{U_2}U_i,\mathcal{T}_{U_i}U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_2}U_i)v\varphi(U_i),U_4)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_2),U_4)+g_1(\eta(U_i)\varphi([U_2,U_i]),U_4)$$
$$-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_2}X_j,U_4)-g_1(\mathcal{A}_{X_j}U_4,h\tilde{\nabla}_{U_2}X_j)$$
$$+g_1(\eta(h\tilde{\nabla}_{U_2}X_j)v\varphi(X_j),U_4)+g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_2),U_4)-g_1(\hat{\nabla}_{U_2}\mathcal{A}_{X_j}X_j,U_4)$$
$$+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_2),U_4)+g_1(\mathcal{T}_{U_2}U_4,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_2),U_4)$$
$$-g_1(\tilde{\nabla}_{U_2}\eta(X_j)v\varphi(X_j),U_4)-g_1\left(\mathcal{T}_{[X_j,U_2]}X_j,U_4\right)-g_1(\eta(X_j)v\varphi([X_j,U_2]),U_4)\}]$$
$$+g_1(U_2,U_3)[\tilde{S}(U_1,U_4)+\sum_i\{-g_1(\eta(\hat{\nabla}_{U_i}U_i)\varphi(U_1),U_4)-g_1(\tilde{\nabla}_{U_1}\eta(U_i)\varphi(U_i),U_4)$$
$$+g_1(\eta(\hat{\nabla}_{U_1}U_i)\varphi(U_i),U_4)+g_1(\tilde{\mathcal{T}}_{U_1}U_i,\mathcal{T}_{U_i}U_4)-g_1(\eta(\tilde{\mathcal{T}}_{U_1}U_i)v\varphi(U_i),U_4)$$
$$+g_1(\tilde{\nabla}_{U_i}\eta(U_i)\varphi(U_1),U_4)+g_1(\eta(U_i)\varphi([U_1,U_i]),U_4)\}$$
$$-\sum_j\{g_1(v\tilde{\nabla}_{X_j}\mathcal{T}_{U_1}X_j,U_4)-g_1(\mathcal{A}_{X_j}U_4,h\tilde{\nabla}_{U_1}X_j)$$

$$+g_1(\eta(h\tilde{\nabla}_{U_1}X_j)v\varphi(X_j),U_4)+g_1(\tilde{\nabla}_{X_j}\eta(X_j)v\varphi(U_1),U_4)-g_1(\hat{\nabla}_{U_1}\mathcal{A}_{X_j}X_j,U_4)$$
$$+g_1(\eta\left(\mathcal{A}_{X_j}X_j\right)\varphi(U_1),U_4)+g_1(\mathcal{T}_{U_1}U_4,h\tilde{\nabla}_{X_j}X_j)-g_1(\eta\left(h\tilde{\nabla}_{X_j}X_j\right)v\varphi(U_1),U_4)$$
$$-g_1(\tilde{\nabla}_{U_1}\eta(X_j)v\varphi(X_j),U_4)-g_1(\mathcal{T}_{[X_j,U_1]}X_j,U_4)-g_1(\eta(X_j)v\varphi([X_j,U_1]),U_4)\}]$$
$$+\frac{\tilde{\tau}}{(n-1)(n-2)}\{g_1(U_2,U_3)g_1(U_1,U_4)-g_1(U_1,U_3)g_1(U_2,U_4)\},$$

$$g_1\big(\tilde{C}(U_1,U_2)U_3,X_1\big)=g_1(\tilde{\mathcal{T}}_{U_1}\hat{\nabla}_{U_2}U_3,X_1)-g_1(\eta\big(\hat{\nabla}_{U_2}U_3\big)\varphi(U_1),X_1)+g_1(h\tilde{\nabla}_{U_1}\tilde{\mathcal{T}}_{U_2}U_3,X_1)$$
$$-g_1\big(\tilde{\nabla}_{U_1}\eta(U_3)\varphi(U_2),X_1\big)-g_1\big(\tilde{\mathcal{T}}_{U_2}\hat{\nabla}_{U_1}U_3,X_1\big)+g_1\big(\eta\big(\hat{\nabla}_{U_1}U_3\big)\varphi(U_2),X_1\big)$$
$$-g_1(h\tilde{\nabla}_{U_2}\tilde{\mathcal{T}}_{U_1}U_3,X_1)+g_1\big(\tilde{\nabla}_{U_2}\eta(U_3)\varphi(U_1),X_1\big)$$
$$-g_1\big(\tilde{\mathcal{T}}_{[U_1,U_2]}U_3,X_1\big)+g_1(\eta(U_3)\varphi([U_1,U_2]),X_1)$$
$$-\frac{1}{(n-2)}\{g_1(U_2,U_3)[\sum\nolimits_i g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)$$
$$+g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1),U_1)+g_1\big(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1\big)$$
$$-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i)-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i),U_1)$$
$$-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)-g_1(\eta(X_i)v\varphi([X_1,X_i],U_1)+\sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)$$
$$-g_1\big(\tilde{\nabla}_{X_1}\eta(U_j)\varphi(U_j),U_1\big)+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right)-g_1\big(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi(U_j),U_1\big)$$
$$-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)+g_1(\eta\big(v\tilde{\nabla}_{X_1}U_j\big)\varphi(U_j),U_1)$$
$$-g_1(\tilde{\nabla}_{U_j}\eta(U_j)h\varphi(X_1),U_1)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)$$
$$+g_1(\eta(U_j)\varphi([X_1,U_j]),U_1)\}]-g_1(U_1,U_3)[\sum\nolimits_i g_1(v\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_2)-g_1\big(\mathcal{A}_{X_1}U_2,h\tilde{\nabla}_{X_i}X_i\big)$$
$$+g_1\big(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1),U_2\big)+g_1\big(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_2\big)-g_1\big(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_2\big)$$
$$+g_1(\mathcal{A}_{X_i}U_2,h\tilde{\nabla}_{X_1}X_i)-g_1(\eta\big(h\tilde{\nabla}_{X_1}X_i\big)v\varphi(X_i),U_2)$$
$$-g_1\big(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_2\big)-g_1(\eta(X_i)v\varphi([X_1,X_i]),U_2)$$
$$+\sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_2)-g_1(\tilde{\nabla}_{X_1}\eta\big(U_j\big)\varphi\big(U_j\big),U_2)+g_1(\mathcal{T}_{U_j}U_2,\mathcal{A}_{X_1}U_j)$$
$$-g_1(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi\big(U_j\big),U_2)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_2)+g_1\big(\eta\big(v\tilde{\nabla}_{X_1}U_j\big)\varphi\big(U_j\big),U_2\big)$$
$$-g_1\left(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_1),U_2\right)-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_2)+g_1(\eta\big(U_j\big)\varphi\big([X_1,U_j]\big),U_2)\}]\},$$

$$g_1\big(\tilde{C}(X_1,X_2)U_1,X_3\big)=g_1\big(h\tilde{\nabla}_{X_1}\mathcal{A}_{X_2}U_1,X_3\big)+g_1(\mathcal{A}_{X_1}v\tilde{\nabla}_{X_2}U_1,X_3)$$
$$+g_1(\eta\big(v\tilde{\nabla}_{X_2}U_1\big)h\varphi(X_1),X_3)+g_1(\tilde{\nabla}_{X_1}\eta(U_1)h\varphi(X_2),X_3)-g_1(h\tilde{\nabla}_{X_2}\mathcal{A}_{X_1}U_1,X_3)$$
$$-g_1(\mathcal{A}_{X_2}v\tilde{\nabla}_{X_1}U_1,X_3)-g_1\big(\eta\big(v\tilde{\nabla}_{X_1}U_1\big)h\varphi(X_2),X_3\big)-g_1\big(\tilde{\nabla}_{X_2}\eta(U_1)h\varphi(X_1),X_3\big)$$
$$-g_1\big(\mathcal{A}_{[X_1,X_2]}U_1,X_3\big)-g_1(\eta(U_1)h\varphi([X_1,X_2]),X_3)$$
$$-\frac{1}{(n-2)}\{g_1(X_1,X_3)[\sum\nolimits_i g_1\,(v\tilde{\nabla}_{X_2}\mathcal{A}_{X_i}X_i,U_1)$$
$$-g_1(\mathcal{A}_{X_2}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_2),U_1)$$
$$+g_1(\tilde{\nabla}_{X_2}\eta(X_i)v\varphi(X_i),U_1)-g_1(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_2}X_i,U_1)+g_1(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_2}X_i)$$
$$-g_1\big(\eta\big(h\tilde{\nabla}_{X_2}X_i\big)v\varphi(X_i),U_1\big)-g_1\big(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_2),U_1\big)-g_1(\eta(X_i)v\varphi([X_2,X_i]),U_1)$$
$$+\sum\nolimits_j\{g_1(v\tilde{\nabla}_{X_2}\hat{\nabla}_{U_j}U_j,U_1)-g_1(\tilde{\nabla}_{X_2}\eta\big(U_j\big)\varphi\big(U_j\big),U_1)+g_1(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_2}U_j)$$
$$-g_1(\eta\big(\mathcal{A}_{X_2}U_j\big)v\varphi\big(U_j\big),U_1)-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_2}U_j,U_1)+g_1(\eta(v\tilde{\nabla}_{X_2}U_j)\varphi\big(U_j\big),U_1),$$
$$-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_2),U_1)-g_1(\hat{\nabla}_{[X_2,U_j]}U_j,U_1)+g_1(\eta\big(U_j\big)\varphi\big([X_2,U_j]\big),U_1)\}]$$

$$-g_1(X_2,X_3)[\sum_i g_1(v(\tilde{\nabla}_{X_1}\mathcal{A}_{X_i}X_i,U_1)-g_1(\mathcal{A}_{X_1}U_1,h\tilde{\nabla}_{X_i}X_i)+g_1(\eta\big(h\tilde{\nabla}_{X_i}X_i\big)v\varphi(X_1),U_1)$$
$$+g_1\big(\tilde{\nabla}_{X_1}\eta(X_i)v\varphi(X_i),U_1\big)-g_1\big(v\tilde{\nabla}_{X_i}\mathcal{A}_{X_1}X_i,U_1\big)+g_1\big(\mathcal{A}_{X_i}U_1,h\tilde{\nabla}_{X_1}X_i\big)$$
$$-g_1(\eta(h\tilde{\nabla}_{X_1}X_i)v\varphi(X_i),U_1)-g_1(\tilde{\nabla}_{X_i}\eta(X_i)v\varphi(X_1),U_1)$$
$$-g_1(\eta(X_i)v\varphi([X_1,X_i]),U_1)+\sum_j\{g_1(v\tilde{\nabla}_{X_1}\hat{\nabla}_{U_j}U_j,U_1)$$
$$-g_1\big(\tilde{\nabla}_{X_1}\eta\big(U_j\big)\varphi\big(U_j\big),U_1\big)+g_1\left(\mathcal{T}_{U_j}U_1,\mathcal{A}_{X_1}U_j\right)-g_1\big(\eta\big(\mathcal{A}_{X_1}U_j\big)v\varphi\big(U_j\big),U_1\big)$$
$$-g_1(\hat{\nabla}_{U_j}v\tilde{\nabla}_{X_1}U_j,U_1)+g_1(\eta\big(v\tilde{\nabla}_{X_1}U_j\big)\varphi\big(U_j\big),U_1)-g_1(\tilde{\nabla}_{U_j}\eta\big(U_j\big)h\varphi(X_1),U_1)$$
$$-g_1(\hat{\nabla}_{[X_1,U_j]}U_j,U_1)+g_1(\eta\big(U_j\big)\varphi\big(\big[X_1,U_j\big]\big),U_1)\}]\},$$

where U_1, U_2, U_3, $U_4 \in \chi^v(M_1)$ and X_1, X_2, X_3, $X_4 \in \chi^h(M_1)$.

4. CONCLUSIONS

In this paper, we have investigated Riemannian submersions endowed with quarter-symmetric non-metric connections, with particular emphasis on the *M*- projective and conformal curvature tensors. We derived the fundamental relations between the base manifold and the fibers for the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature. We analyzed the special case where the fibers of the Riemannian submersion are totally umbilical. By applying the condition that the horizontal vector field $\mathcal{N}$ vanishes ($\mathcal{N}=0$) in the presence of totally umbilical fibers, we derived simplified corollaries for both the *M*-projective and conformal curvature tensors. These results provide a deeper insight into how the geometric structure of the fibers and the choice of connection influence the curvature characteristics of the total space. This work opens new directions for future studies on Riemannian submersions with non-metric connections, particularly in the context of generalized curvature tensors and special classes of fibers.

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