ORIGINAL PAPER

SPACELIKE OR TIMELIKE (NON-NULL) OSCULATING CURVES OF TYPE (N-3) IN LORENTZIAN N-SPACE

ÖZGÜR BOYACIOĞLU KALKAN[1], SÜLEYMAN ŞENYURT[2]

Manuscript received: 09.07.2025; Accepted paper: 02.12.2025;
Published online: 30.03.2026.

Abstract. *In this paper, we study non-null osculating curves of type (n-3) in Lorentzian n-space E_1^n. We obtain necessary and sufficient conditions for non-null curves to be osculating curves of type (n-3) in E_1^n. We analyze these curves in terms of their curvature functions and derive the necessary relations between the curvatures for a curve to be congruent to an osculating curve of type (n-3) in E_1^n. Moreover, we re-characterize non-null osculating curves of type (n-3) using harmonic curvature functions in terms of B_{n-2}.*

Keywords: *non-null osculating curves; Lorentzian n-space; curvatures; harmonic curvature functions.*

Mathematics Subject Classification: *53A35; 53C50; 53B30.*

1. INTRODUCTION

Rectifying, normal and osculating curves are fundamental concepts in differential geometry. These curves provide a deeper understanding of the local geometric behavior of space curves, which is essential for both theoretical and practical applications. The rectifying curve in E^3 is defined as a curve whose position vector always lies in its rectifying plane [1]. Rectifying curves in Euclidean and Minkowski spaces have been extensively studied in [2-5]. In the same way, a curve whose position vector always lies in its normal plane is called a normal curve [6]. Studies in literature investigating the concept of normal curve can be found in [7-10]. Similar to rectifying and normal curves, the position vector of osculating curves invariably lies within their osculating planes. According to this definition, the position vector of the osculating curve can be written as

$$\zeta(s) = c(s)T(s) + d(s)N(s)$$

where $c(s)$ and $d(s)$ are differentiable functions [11]. İlarslan and Nesovic defined osculating curve in [12] as a curve whose position vector always lies in its orthogonal complement of either $B_1^{\perp}$ or $B_2^{\perp}$ of the curve's first or second binormal vector field in Minkowski space-time. They referred to these curves as first-kind and second-kind osculating curves, respectively. Later on, Bektaş et al. defined osculating curves of type (n-3) as a curve whose position vector always lies in its orthogonal complement of (n-3) th binormal vector

[1] Afyon Kocatepe University, Afyon Vocational School, 03200 Afyonkarahisar, Turkey.
E-mail: bozgur@aku.edu.tr.
[2] Ordu University, Faculty of Arts and Science, Department of Mathematics, 52200 Ordu, Turkey.
E-mail: ssenyurt@odu.edu.tr

https://doi.org/10.46939/J.Sci.Arts-26.1-a02

field $B_{n-3}^{\perp}$ in the n-dimensional Euclidean space [13]. In recent years, researchers have investigated different properties of osculating curves [14-17].

The harmonic curvature functions H_j of a curve in n-dimensional Euclidean space is defined by Özdamar and Hacısalihoğlu in [18]. They examined inclined curves using harmonic curvature functions in E^n. Since then, many mathematicians have studied harmonic curvature functions in relation to various geometric concepts, including specific curves in n-dimensional spaces [19-23]. We can see from these studies that the known characterizations of curves can be obtained in simple forms using harmonic curvature functions of curves in n-dimensional spaces.

Osculating curves have been extensively studied in Euclidean and Minkowski spaces. However, their counterparts in Lorentzian n-space, particularly regarding curvature and harmonic curvature functions, have not been thoroughly examined. In this paper, we establish necessary and sufficient conditions for non-null osculating curves of type (n-3) which are located in the orthogonal complement of the (n-3) th binormal vector in the Lorentzian n-space E_1^n. First, we examine non-null osculating curves of type (n-3) with respect to their curvature functions and obtain necessary and sufficient conditions for any curve to be an osculating curve of type (n-3). Moreover, we characterize non-null osculating curves of type (n-3) using harmonic curvature functions in relation to the binormal B_{n-2} in Lorentzian n-space.

2. PRELIMINARIES

Let E_1^n denote the Lorentzian n-space. For vectors $X=(x_1,x_2,...,x_n)$ and $Y=(y_1,y_2,...,y_n)$ in E_1^n

$$\langle X,Y\rangle = -x_1y_1 + \sum_{i=2}^{n} x_iy_i$$

It is called the Lorentzian inner product. The vector $v\in E_1^n$ has one of three causal characters; it can be spacelike if $\langle v,v\rangle \rangle 0$ or $v=0$, timelike if $\langle v,v\rangle \langle 0$ and null (lightlike) if $\langle v,v\rangle = 0$ and $v\neq 0$. In Lorentzian geometry, curves are classified according to the causal character of their tangent vectors, which is determined by the Lorentzian metric. A non-null curve is named as a spacelike or timelike curve if its tangent vector is spacelike or timelike, respectively [24].

Let $\zeta=\zeta(s)$ be an arc-length parametrized non-null curve in E_1^n and $\{T(s),N(s),B_1(s),...,B_{n-2}(s)\}$ be the moving frame along ζ, where the vectors $T(s),N(s),B_1(s)$, $B_2(s),...,B_{n-2}(s)$ are mutually orthogonal vectors satisfying

$$\langle T,T\rangle = \varepsilon_1 = \pm 1,\quad \langle N,N\rangle = \varepsilon_2 = \pm 1 \;\; and \;\; \langle B_i,B_i\rangle = \varepsilon_{i+2} = \pm 1,\quad i=\{1,2,...,n-2\}.$$

The Frenet formulas for the curve can be expressed as follows [21]:

$$\begin{aligned}
&T' = \varepsilon_2 k_1 N, \\
&N' = -\varepsilon_1 k_1 T + \varepsilon_3 k_2 B_1, \\
&B_1' = -\varepsilon_2 k_2 N + \varepsilon_4 k_3 B_2, \\
&B_i' = -\varepsilon_{i+1} k_{i+1} B_{i-1} + \varepsilon_{i+3} k_{i+2} B_{i+1}, \\
&B_{n-2}' = -\varepsilon_{n-1} k_{n-1} B_{n-3}.
\end{aligned} \tag{1}$$

The functions $k_i(s)$ are called the i-th curvatures of ζ for $i = \{1,2,...,n-1\}$. Similar to the definition in 4-dimensional Euclidean space, osculating curves in n-dimensional Euclidean space can be defined as curves whose position vectors always lie in the orthogonal complement of one of their binormal vector fields $B_1(s), B_2(s),..., B_{n-3}(s), B_{n-2}(s)$. In [13], the authors introduce the concept of the osculating curve of type (n-3) in n-dimensional Euclidean space as follows:

Definition 2.1. Let $\zeta : I \subset R \to E^n$ be an arc-length parametrized curve in E^n. The curve ζ is called the osculating curve of type (n-3)**,** if its position vector with respect to some chosen origin consistently lies in the orthogonal complement of its (n-3) th binormal vector field $B_{n-3}(s)$. Then, the orthogonal complement of $B_{n-3}(s)$ can be written as $B_{n-3}^{\perp}(s) = \{ v \in E^n \mid \langle v, B_{n-3}(s) \rangle = 0 \}$ and the equation for the position vector of the osculating curve of type (n-3) can be expressed as follows:

$$\zeta(s) = \mu_1(s)T(s) + \mu_2(s)N(s) + \sum_{j=1}^{n-4} \mu_{j+2}(s)B_j(s) + \mu_n(s)B_{n-2}(s)$$

with differentiable real functions $\mu_1, \mu_2, \mu_3, ..., \mu_{n-2}, \mu_n$ [13].

3. NON-NULL OSCULATING CURVES OF TYPE (n-3) IN E_1^n

3.1. NON-NULL OSCULATING CURVES OF TYPE (n-3) WITH CURVATURE FUNCTIONS

In this first subsection, we characterize non-null osculating curves in terms of their curvature functions.

Definition 3.1. The curve $\zeta : I \subset R \to E_1^n$ is called a non-null osculating curve of type (n-3)**,** if its position vector with respect to some chosen origin consistently lies in the orthogonal complement of its (n-3) th binormal vector field $B_{n-3}(s)$. Then, the orthogonal complement of $B_{n-3}(s)$ can be written as $B_{n-3}^{\perp}(s) = \{ v \in E_1^n \mid \langle v, B_{n-3}(s) \rangle = 0 \}$ and the equation for the position vector of the non-null osculating curve of type (n-3) can be given by

$$\zeta(s) = \sigma_1(s)T(s) + \sigma_2(s)N(s) + \sum_{j=1}^{n-4} \sigma_{j+2}(s)B_j(s) + \sigma_n(s)B_{n-2}(s) \tag{2}$$

with differentiable real functions $\sigma_1, \sigma_2, \sigma_3, ..., \sigma_{n-2}, \sigma_n$.

Let $\zeta : I \subset R \to E_1^n$ be an arclength parametrized non-null osculating curve of type (n-3) in E_1^n. The position vector of the curve satisfies Eq. (2) for smooth functions $\sigma_1, \sigma_2, \sigma_3, ..., \sigma_{n-2}, \sigma_n$. Differentiating Eq. (2) with respect to s and utilizing Eq. (1), we get

$$\begin{aligned} T(s) &= (\sigma_1'(s) - \varepsilon_1 k_1(s)\sigma_2(s))T(s) + (\varepsilon_2 k_1(s)\sigma_1(s) + \sigma_2'(s) - \varepsilon_2 k_2(s)\sigma_3(s))N(s) \\ &+ \sum_{j=2}^{n-4} (\varepsilon_{n-j-1} k_{n-j-2}(s)\sigma_{n-j-2}(s) + \sigma_{n-j-1}'(s) - \varepsilon_{n-j-1} k_{n-j-1}(s)\sigma_{n-j}(s))B_{n-j-3}(s) \\ &+ (\varepsilon_{n-2} k_{n-3}(s)\sigma_{n-3}(s) + \sigma_{n-2}'(s))B_{n-4}(s) \\ &+ (\varepsilon_{n-1} k_{n-2}(s)\sigma_{n-2}(s) - \varepsilon_{n-1} k_{n-1}(s)\sigma_n(s))B_{n-3}(s) + \sigma_n'(s)B_{n-2}(s). \end{aligned} \tag{3}$$

From Eq. (3), we obtain

$$\sigma_1' - \varepsilon_1 k_1 \sigma_2 = 1, \tag{4}$$

$$\varepsilon_2 k_1 \sigma_1 + \sigma_2' - \varepsilon_2 k_2 \sigma_3 = 0, \tag{5}$$

$$\varepsilon_{n-j-1} k_{n-j-2} \sigma_{n-j-2} + \sigma_{n-j-1}' - \varepsilon_{n-j-1} k_{n-j-1} \sigma_{n-j} = 0, \tag{6}$$

$$\varepsilon_{n-2} k_{n-3} \sigma_{n-3} + \sigma_{n-2}' = 0\,, \tag{7}$$

$$\varepsilon_{n-1} k_{n-2} \sigma_{n-2} - \varepsilon_{n-1} k_{n-1} \sigma_n = 0\,, \tag{8}$$

$$\sigma_n' = 0\,, \tag{9}$$

where $j \in \{2, 3, ..., n-4\}$. Thus, the coefficient functions σ_j are expressed in terms of the curvature functions and their derivatives. From Eq. (9), we have

$$\sigma_n = c = \text{constant}, \qquad c \in R. \tag{10}$$

Using Eqs. (6), (7), (8) and Eq. (10), the functions $\sigma_{n-2,0}$, $\sigma_{n-3,0}, \sigma_{n-3,1}, \sigma_{n-4,0}$, $\sigma_{n-4,1}$ and $\sigma_{n-4,2}$ can be obtained as:

$$\sigma_{n-2} = c\left(\frac{k_{n-1}}{k_{n-2}}\right) = \sigma_{n-2,0}\left(\frac{k_{n-1}}{k_{n-2}}\right), \qquad \sigma_{n-2,0} = c, \tag{11}$$

$$\sigma_{n-3} = -c\frac{\varepsilon_{n-2}}{k_{n-3}}\left(\frac{k_{n-1}}{k_{n-2}}\right)' = \sigma_{n-3,1}\left(\frac{k_{n-1}}{k_{n-2}}\right)', \quad \sigma_{n-3,0} = 0, \qquad \sigma_{n-3,1} = -c\frac{\varepsilon_{n-2}}{k_{n-3}}, \tag{12}$$

$$\sigma_{n-4} = \sigma_{n-4,0}\left(\frac{k_{n-1}}{k_{n-2}}\right) + \sigma_{n-4,1}\left(\frac{k_{n-1}}{k_{n-2}}\right)' + \sigma_{n-4,2}\left(\frac{k_{n-1}}{k_{n-2}}\right)'', \tag{13}$$

$$\sigma_{n-4,0} = c\frac{k_{n-3}}{k_{n-4}}, \qquad \sigma_{n-4,1} = c\frac{\varepsilon_{n-2}\varepsilon_{n-3}}{k_{n-4}}\left(\frac{1}{k_{n-3}}\right)', \qquad \sigma_{n-4,2} = c\frac{\varepsilon_{n-2}\varepsilon_{n-3}}{k_{n-4}}\left(\frac{1}{k_{n-3}}\right). \tag{14}$$

Then, from Eq. (6), we determine

$$\sigma_j = \sum_{t=0}^{n-j-2} \sigma_{j,t} \frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right), \qquad 1 \le j \le n-2. \tag{15}$$

At once, the functions $\sigma_{j,t}$ can be given by

$$\begin{aligned}
&\sigma_{n-2,0} = c, \qquad \sigma_{n-3,0} = 0, \qquad \sigma_{n-3,1} = -c\frac{\varepsilon_{n-2}}{k_{n-3}},\\
&\sigma_{n-j,0} = \frac{\varepsilon_{n-j+1}}{k_{n-j}}\left(\varepsilon_{n-j+1}k_{n-j+1}\sigma_{n-j+2,0} - \sigma'_{n-j+1,0}\right),\\
&\sigma_{n-j,t} = \frac{\varepsilon_{n-j+1}}{k_{n-j}}\left(\varepsilon_{n-j+1}k_{n-j+1}\sigma_{n-j+2,t} - \sigma'_{n-j+1,t} - \sigma_{n-j+1,t-1}\right),\\
&\sigma_{n-j,j-3} = -\frac{\varepsilon_{n-j+1}}{k_{n-j}}\left(\sigma_{n-j+1,j-4} + \sigma'_{n-j+1,j-3}\right),\\
&\sigma_{n-j,j-2} = -\frac{\varepsilon_{n-j+1}}{k_{n-j}}\sigma_{n-j+1,j-3}
\end{aligned} \tag{16}$$

where $t \in \{1,2,...,j-4\}$ and $j \in \{4,...,n-1\}$. Then, using Eq. (15) in Eqs. (5) and (6), we obtain

$$\sigma_2 = \frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)', \tag{17}$$

$$\sigma_1 = \frac{k_2}{k_1}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_2}{k_1}\left(\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\right)', \tag{18}$$

by substituting Eqs. (10), (15), (17) and (18) into Eq. (2), the position vector of the non-null osculating curve of type (n-3) can be written as:

$$\begin{aligned}
\zeta = &\left[\frac{k_2}{k_1}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_2}{k_1}\left(\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\right)'\right]T\\
&+\left(\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\right)N + \sum_{j=3}^{n-2}\left(\sum_{t=0}^{n-j-2}\sigma_{j,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)B_{j-2} + cB_{n-2}.
\end{aligned} \tag{19}$$

According to Eqs. (4)-(9), we state the next theorem.

Theorem 3.1. Let $\zeta : I \subset R \to E_1^n$ be an arc-length parametrized non-null curve in E_1^n with non-zero curvatures. Then, $\zeta(s)$ is congruent to the non-null osculating curve of type (n-3) if and only if

$$\left[\frac{k_2}{k_1}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)-\frac{\varepsilon_2}{k_1}\begin{pmatrix}\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)\\-\frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\end{pmatrix}'\right]'-\varepsilon_1k_1\begin{pmatrix}\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)\\-\frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\end{pmatrix}=1 \tag{20}$$

with $\sigma_{j,t}$ represented by Eqs. (16).

Proof: Let $\zeta(s)$ be a non-null osculating curve of type (n-3). Then, substituting Eqs. (17) and (18) in Eq. (4), we obtain Eq. (20). Conversely, suppose that Eq. (20) holds. Then, we define the vector $g(s)\in E_1^n$ which is specified by

$$g(s)=\zeta(s)-\sigma_1(s)T(s)-\sigma_2(s)N(s)-\sum_{j=1}^{n-4}\sigma_{j+2}(s)B_j(s)-\sigma_n(s)B_{n-2}(s) \tag{21}$$

with $\sigma_n(s)=c=\text{constant}$ and $\sigma_j(s),\ 1\le j\le n-2$ given by Eqs. (15), (17) and (18). By taking the derivative of Eq. (21) with respect to s and applying Eq. (1)

$$g'=\begin{pmatrix}1-\left[\frac{k_2}{k_1}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)-\frac{\varepsilon_2}{k_1}\left(\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)-\frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\right)'\right]'\\+\varepsilon_1k_1\left(\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)-\frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\right)\end{pmatrix}T \tag{22}$$

gives $g'(s)=0$. Then, $g(s)$ is a constant vector and $\zeta(s)$ is congruent to the non-null osculating curve of type (n-3).

Now, assume that all the curvature functions $k_1,k_2,...,k_{n-1}$ of non-null osculating curve of type (n-3) are non-zero constants. Then, we present the following theorem.

Theorem 3.2. For odd n, there exists no non-null osculating curve of type (n-3) with non-zero constant curvatures and for even n, every curve with non-zero constant curvatures is an osculating curve of type (n-3) in E_1^n.

Proof: Consider that there exists a non-null osculating curve of type (n-3) with its non-zero constant curvatures. The Eqs. (11), (12) and Eq. (13) give

$$\sigma_{n-2}=c\left(\frac{k_{n-1}}{k_{n-2}}\right),\qquad \sigma_{n-3}=0,\qquad \sigma_{n-4}=c\frac{k_{n-3}k_{n-1}}{k_{n-4}k_{n-2}}.$$

For $j\in\{4,...,n-1\}$ Eq. (6) becomes

$$\sigma_{n-j}=\frac{\varepsilon_{n-j+1}}{k_{n-j}}\left(\varepsilon_{n-j+1}k_{n-j+1}\sigma_{n-j+2}-\sigma'_{n-j+1}\right).$$

By induction, we conclude that

$$\sigma_{n-2m} = c\frac{\prod_{j=1}^{m} k_{n-2j+1}}{\prod_{j=1}^{m} k_{n-2j}}, \qquad \sigma_{n-2m-1} = 0\,. \tag{23}$$

If Eq. (23) is correct for $m \in \{1,2,...,M\}$, then Eqs. (23) for $m = M+1$ can be generalized by

$$\begin{aligned}\sigma_{n-2M-2} &= \frac{\varepsilon_{n-2M-1}}{k_{n-2M-2}}\left(\varepsilon_{n-2M-1}k_{n-2M-1}\sigma_{n-2M} - \sigma'_{n-2M-1}\right)\\ &= c\frac{k_{n-1}k_{n-3}...k_{n-2M+1}k_{n-2M-1}}{k_{n-2}k_{n-4}...k_{n-2M}k_{n-2M-2}},\\ \sigma_{n-2M-3} &= \frac{\varepsilon_{n-2M-2}}{k_{n-2M-3}}\left(\varepsilon_{n-2M-2}k_{n-2M-2}\sigma_{n-2M-1} - \sigma'_{n-2M-2}\right)\\ &= \frac{\varepsilon_{n-2M-2}}{k_{n-2M-3}}\left(c\frac{k_{n-1}k_{n-3}...k_{n-2M+1}k_{n-2M-1}}{k_{n-2}k_{n-4}...k_{n-2M}k_{n-2M-2}}\right)' = 0.\end{aligned}$$

When n is odd, with the aid of (23), Eqs. (5) and (6) give

$$\sigma_2 = 0, \qquad \sigma_1 = c\frac{k_{n-1}k_{n-3}...k_2}{k_{n-2}k_{n-4}...k_1}.. \tag{24}$$

Then, using Eqs. (24), Eq (4) takes the following form $\left(c\frac{k_{n-1}k_{n-3}...k_2}{k_{n-2}k_{n-4}...k_1}\right)' = 1.$ However, if all curvatures are assumed to be non-zero constants, we get a contradiction.

When n is even, using (23), Eqs. (5), and Eq. (6) give

$$\sigma_2 = c\frac{k_{n-1}k_{n-3}...k_5k_3}{k_{n-2}k_{n-4}...k_4k_2}, \qquad \sigma_1 = 0.. \tag{25}$$

Then, using Eqs (25), Eq. (4) can be rearranged as $\frac{k_{n-1}k_{n-3}...k_5k_3k_1}{k_{n-2}k_{n-4}...k_4k_2} = -\frac{\varepsilon_1}{c}$. Since the curvature functions satisfy Theorem 3.1, $\zeta(s)$ is congruent to the non-null osculating curve of type (n-3).

We now present examples verifying that Theorem 3.2 holds for the special cases $n=5$ and $n=6$.

Example 3.1. Let us consider the unit speed timelike curve in E_1^5 given in [25] parametrized as

$$\alpha(s) = (\sqrt{3}\sinh s, \sqrt{3}\cosh s, \sin s, s, \cos s).$$

Then, the tangent, normal, first binormal, second binormal and third binormal Frenet vectors of the curve are given, respectively, by

$$T(s)=\left(\sqrt{3}\cosh s,\sqrt{3}\sinh s,\cos s,1,-\sin s\right),$$
$$N(s)=\varepsilon_2\left(\frac{\sqrt{3}}{2}\sinh s,\frac{\sqrt{3}}{2}\cosh s,-\frac{1}{2}\sin s,0,-\frac{1}{2}\cos s\right),$$
$$B_1(s)=\frac{1}{\sqrt{14}}\left(-3\sqrt{3}\cosh s,-3\sqrt{3}\sinh s,-5\cos s,-4,5\sin s\right),$$
$$B_2(s)=\lambda\left(-\frac{1}{2}\sinh s,-\frac{1}{2}\cosh s,-\frac{3}{2}\sin s,0,-\frac{3}{2}\cos s\right),$$
$$B_3(s)=\lambda\left(-\frac{1}{\sqrt{14}}\cosh s,-\frac{1}{\sqrt{14}}\sinh s,\sqrt{\frac{3}{14}}\cos s,-\sqrt{\frac{2}{7}},-\sqrt{\frac{3}{14}}\sin s\right)$$

where $\lambda=\pm1$ and the curvature functions are given by

$$k_1(s)=2,\quad k_2(s)=\sqrt{14},\quad k_3(s)=\sqrt{\frac{3}{14}},\ \ k_4(s)=\sqrt{\frac{2}{7}}\ .$$

It can be easily verified that $\langle\alpha,B_2\rangle\neq0$, which means that α is not an osculating curve in E_1^5.

Example 3.2. Let us consider the unit speed timelike curve in E_1^6 given in [26] parametrized as

$$\gamma(s)=\left(2\sinh s,2\cosh s,2\sin\frac{s}{2},-2cos\frac{s}{2},2\sin\frac{s}{\sqrt{2}},-2\cos\frac{s}{\sqrt{2}}\right).$$

Then, the tangent, normal, first binormal, second binormal, third binormal and fourth binormal Frenet vectors of the curve are given, respectively, by

$$T(s)=\left(2\cosh s,2\sinh s,\cos\frac{s}{2},\sin\frac{s}{2},\sqrt{2}\cos\frac{s}{\sqrt{2}},\sqrt{2}\sin\frac{s}{\sqrt{2}}\right),$$
$$N(s)=\left(\frac{4\sinh s}{\sqrt{21}},\frac{4\cosh s}{\sqrt{21}},-\frac{\sin\frac{s}{2}}{\sqrt{21}},\frac{\cos\frac{s}{2}}{\sqrt{21}},-\frac{2\sin\frac{s}{\sqrt{2}}}{\sqrt{21}},\frac{2\cos\frac{s}{\sqrt{2}}}{\sqrt{21}}\right),$$
$$B_1(s)=\frac{1}{\sqrt{193}}\left(-17\sqrt{2}\cosh s,-17\sqrt{2}\sinh s,-11\sqrt{2}\cos\frac{s}{2},-11\sqrt{2}\sin\frac{s}{2},-23\cos\frac{s}{\sqrt{2}},-23\sin\frac{s}{\sqrt{2}}\right),$$
$$B_2(s)=\frac{1}{\sqrt{4217}}\left(58\sinh s,58\cosh s,38\sin\frac{s}{2},-38\cos\frac{s}{2},97\sin\frac{s}{\sqrt{2}},97\cos\frac{s}{\sqrt{2}}\right),$$
$$B_3(s)=\frac{1}{\sqrt{193}}\left(-\cosh s,-\sinh s,-12\cos\frac{s}{2},-12\sin\frac{s}{2},5\sqrt{2}\cos\frac{s}{\sqrt{2}},5\sqrt{2}\sin\frac{s}{\sqrt{2}}\right),$$
$$B_4(s)=\frac{1}{\sqrt{677}}\left(\sinh s,\cosh s,24\sin\frac{s}{2},-24\cos\frac{s}{2},-10\sin\frac{s}{\sqrt{2}},10\cos\frac{s}{\sqrt{2}}\right)$$

where the curvature functions are given by

$$k_1(s)=\frac{\sqrt{21}}{2},\quad k_2(s)=\sqrt{\frac{193}{42}},\quad k_3(s)=\sqrt{\frac{677}{8106}},\ k_4(s)=15\sqrt{\frac{21}{130661}},\ k_5(s)=\sqrt{\frac{193}{677}}\ .$$

It can be easily verified that $\langle \gamma, B_3\rangle = 0$, which means that γ is an osculating curve in E_1^6.

Theorem 3.3. Let $\zeta : I \subset R \to E_1^n$ be an arclength parametrized non-null osculating curve of type (n-3) in E_1^n with non-zero curvatures. Then, the following expressions are obtained:

(i) The $(n-2)$ th binormal component of the position vector of the curve is given by $\left\langle \zeta, B_{n-2}\right\rangle = c\varepsilon_n = \text{constant}.$

(ii) The $(n-4)$ th, $(n-5)$ th, j-th, second, first binormal, normal and tangential components of the position vector of the curve are given, respectively, by

$$\left\langle \zeta, B_{n-4}\right\rangle = c\varepsilon_{n-2}\left(\frac{k_{n-1}}{k_{n-2}}\right),$$

$$\left\langle \zeta, B_{n-5}\right\rangle = -c\frac{\varepsilon_{n-3}\varepsilon_{n-2}}{k_{n-3}}\left(\frac{k_{n-1}}{k_{n-2}}\right)',$$

$$\left\langle \zeta, B_j\right\rangle = \varepsilon_{j+2}\sum_{t=0}^{n-j-4}\sigma_{j+2,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right),$$

$$\left\langle \zeta, N\right\rangle = \varepsilon_2\left(\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\right),$$

$$\left\langle \zeta, T\right\rangle = \varepsilon_1\left(\frac{k_2}{k_1}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_2}{k_1}\left(\frac{k_3}{k_2}\left(\sum_{t=0}^{n-6}\sigma_{4,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \frac{\varepsilon_3}{k_2}\left(\sum_{t=0}^{n-5}\sigma_{3,t}\frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\right)'\right)$$

where $1 \le j \le n-6$ and σ_{jt} are introduced by Eqs. (16).

Conversely, if $\zeta(s)$ is an arc-length parametrized non-null curve in E_1^n with non-zero curvatures and the situations (i) and (ii) hold, then $\zeta(s)$ is congruent to the non-null osculating curve of type (n-3).

Proof: Suppose that $\zeta(s)$ is an arclength parametrized non-null osculating curve of type (n-3) in E_1^n. Taking the inner product of the two sides of Eq. (19) with $B_{n-2}, B_{n-4}, B_{n-5}, ..., B_2, B_1, N, T$, we obtain the situations in (i) and (ii).

Conversely, suppose that (i) and (ii) are given. Differentiating $\left\langle \zeta, B_{n-2}\right\rangle = c\varepsilon_n = \text{constant}$ with respect to s and by utilizing Eq. (1), we obtain $\left\langle \zeta, B_{n-3}\right\rangle = 0,$ which implies that $\zeta(s)$ is congruent to the osculating curve of type (n-3). Similarly, differentiating $\left\langle \zeta, B_{n-4}\right\rangle = c\varepsilon_{n-2}\left(\frac{k_{n-1}}{k_{n-2}}\right)$ with respect to s, we get $\left\langle \zeta, B_{n-3}\right\rangle = 0,$ then $\zeta(s)$ is a non-null osculating curve of type (n-3).

3.2. NON-NULL OSCULATING CURVES OF TYPE (n-3) ACCORDING TO HARMONIC CURVATURE FUNCTIONS

In this part, we reformulate the characterizations of non-null osculating curves of type (n-3) in terms of harmonic curvature functions.

Definition 3.2. Let $\zeta : I \subset R \to E_1^n$ be an arclength parametrized non-null curve with non-zero curvatures k_j $(j = 1,2,...,n-1)$ in E_1^n. The harmonic curvature functions $H_j : I \to R$ in terms of B_{n-2} for the non-null osculating curve of type (n-3) are defined by

$$H_j = \begin{cases} 0, & \text{for} \quad j = 0, \\ c\varepsilon_{n-2}\varepsilon_n \dfrac{k_{n-1}}{k_{n-2}}, & \text{for} \quad j = 1, \quad c \in R, \\ \dfrac{\varepsilon_{n-j-1}}{k_{n-j-1}}\left\{\varepsilon_{n-j+1}k_{n-j}H_{j-2} - H'_{j-1}\right\}, & \text{for} \quad j = 2,3,...,n-2. \end{cases} . \tag{26}$$

Corollary 3.1. Let $\zeta : I \subset R \to E_1^n$ be an arc-length parametrized non-null osculating curve of type (n-3) with non-zero curvatures. Then, the coefficient functions $\sigma_j(s)$, $1 \le j \le n-2$ and $\sigma_n(s)$ according to the harmonic curvatures of $\zeta(s)$ are calculated as follows:

(i) $\sigma'_n = 0 \Rightarrow \sigma_n(s) = c, \; c \in R,$

(ii) $\sigma_{n-j} = \dfrac{\varepsilon_{n-j+1}}{k_{n-j}}\left(\varepsilon_{n-j+1}k_{n-j+1}\sigma_{n-j+2} - \sigma'_{n-j+1}\right),$

$$= \frac{\varepsilon_n}{k_{n-j}}\left(\varepsilon_{n-j+2}k_{n-j+1}H_{j-3} - H'_{j-2}\right) = \varepsilon_{n-j}\varepsilon_n H_{j-1}, \qquad 2 \le j \le n-1.$$

Theorem 3.4. Let $\zeta : I \subset R \to E_1^n$ be an arc-length parametrized non-null curve with non-zero curvatures. Then, $\zeta(s)$ is congruent to the non-null osculating curve of type (n-3) in E_1^n if and only if

$$H'_{n-2}(s) - \varepsilon_2 k_1(s) H_{n-3}(s) = \varepsilon_1\varepsilon_n \,. \tag{27}$$

where H_j, $j \in \{1,2,...,n-2\}$ are differentiable harmonic curvature functions of $\zeta(s)$.

Proof: Let $\zeta : I \subset R \to E_1^n$ be an arclength parametrized non-null osculating curve of type (n-3) with non-zero curvatures. Then, from Corollary 3.1, we have

$$\sigma_2 = \varepsilon_2\varepsilon_n H_{n-3}, \qquad \sigma_1 = \varepsilon_1\varepsilon_n H_{n-2}. \tag{28}$$

Substituting Eq. (28) in Eq. (4), we obtain Eq. (27). Conversely, suppose that Eq. (27) is satisfied. Let us define the curve $\eta(s)$ with

$$\eta(s) = \zeta(s) - \sigma_1(s)T(s) - \sigma_2(s)N(s) - \sigma_3(s)B_1(s) - ... - \sigma_{n-2}(s)B_{n-4}(s) - \sigma_n(s)B_{n-2}(s) \tag{29}$$

where $\sigma_1, \sigma_2, \sigma_3, ...\sigma_{n-2}, \sigma_n$ are differentiable functions. Differentiating Eq. (29) with respect to s and utilizing Eq. (1) and (i-ii) in Corollary 3.1, we get $\eta'(s)=0$. Hence, $\eta(s)$ is constant and $\zeta(s)$ is congruent to the non-null osculating curve of type (n-3) in E_1^n.

Corollary 3.2. Let $\zeta : I \subset R \to E_1^n$ be an arclength parametrized non-null osculating curve of type (n-3) with non-zero curvatures. Then, the position vector of $\zeta(s)$ is

$$\zeta(s) = \varepsilon_n \left(\varepsilon_1 H_{n-2}(s)T(s) + \varepsilon_2 H_{n-3}(s)N(s) + \sum_{j=1}^{n-4} \varepsilon_{j+2} H_{n-j-3}(s) B_j(s) \right) + cB_{n-2}(s) \tag{30}$$

where H_j, are differentiable harmonic curvature functions of $\zeta(s)$.

Corollary 3.3. Let $\zeta : I \subset R \to E_1^n$ be an arc-length parametrized non-null curve with non-zero curvatures. Then, the following statements are equivalent.

(i) $\zeta(s)$ is the osculating curve of type (n-3).

(ii) $H'_{n-2}(s) - \varepsilon_2 k_1(s) H_{n-3}(s) = \varepsilon_1 \varepsilon_n$.

(iii)
$$\left[\begin{array}{l} \dfrac{k_2}{k_1}\left(\displaystyle\sum_{t=0}^{n-5} \sigma_{3,t} \frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) \\ -\dfrac{\varepsilon_2}{k_1}\left(\dfrac{k_3}{k_2}\left(\displaystyle\sum_{t=0}^{n-6} \sigma_{4,t} \frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) - \dfrac{\varepsilon_3}{k_2}\left(\displaystyle\sum_{t=0}^{n-5} \sigma_{3,t} \frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)'\right)' \end{array} \right]' - \varepsilon_1 k_1 \left(\begin{array}{l} \dfrac{k_3}{k_2}\left(\displaystyle\sum_{t=0}^{n-6} \sigma_{4,t} \frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right) \\ -\dfrac{\varepsilon_3}{k_2}\left(\displaystyle\sum_{t=0}^{n-5} \sigma_{3,t} \frac{\partial^t}{\partial s^t}\left(\frac{k_{n-1}}{k_{n-2}}\right)\right)' \end{array} \right) = 1$$

where H_j, $j \in \{1,2,...,n-2\}$ are differentiable harmonic curvature functions of non-null osculating curve of type (n-3) and σ_{jt} are introduced by Eqs. (16).

Theorem 3.5. Let $\zeta : I \subset R \to E_1^n$ be an arc-length parametrized non-null curve with non-zero curvatures and the harmonic curvature function $H_{n-2}(s) \neq 0$. Then, $\zeta(s)$ is a non-null osculating curve of type (n-3) if and only if

$$\sum_{j=1}^{n-2} \varepsilon_{n-j-1} H_j^2(s) = 2\varepsilon_n \int H_{n-2}(s)ds. \tag{31}$$

Proof: Let $\zeta(s)$ be a non-null osculating curve of type (n-3) in E_1^n. Then, from Theorem 3.4, we have

$$H'_{n-2} - \varepsilon_2 k_1 H_{n-3} = \varepsilon_1 \varepsilon_n. \tag{32}$$

Since $\varepsilon_1 H_{n-2}(s) \neq 0$, multiplying Eq. (32) with $\varepsilon_1 H_{n-2}(s)$, we obtain

$$\varepsilon_1 H_{n-2} H'_{n-2} = \varepsilon_n H_{n-2} + \varepsilon_1 \varepsilon_2 k_1 H_{n-3} H_{n-2}. \tag{33}$$

Also, using Eq. (26), we get

$$\varepsilon_{n-j-1} k_{n-j-1} H_j = \varepsilon_{n-j+1} k_{n-j} H_{j-2} - H'_{j-1}, \qquad 2 \leq j \leq n-2. \tag{34}$$

If we write $j+1$ instead of j in Eq. (34) and putting $j=1$, we obtain

$$H_j' = \varepsilon_{n-j}k_{n-j-1}H_{j-1} - \varepsilon_{n-j-2}k_{n-j-2}H_{j+1} \qquad 1 \le j \le n-3 \tag{35}$$

$$H_1' = -\varepsilon_{n-3}k_{n-3}H_2. \tag{36}$$

Then multiplying Eqs. (35) and (36) with $\varepsilon_{n-j-1}H_j(s)$ and $\varepsilon_{n-2}H_1(s)$ respectively, we have

$$\varepsilon_{n-j-1}H_jH_j' = \varepsilon_{n-j}\varepsilon_{n-j-1}k_{n-j-1}(s)H_{j-1}H_j - \varepsilon_{n-j-1}\varepsilon_{n-j-2}k_{n-j-2}H_jH_{j+1}, \tag{37}$$

$$\varepsilon_{n-2}H_1H_1' = -\varepsilon_{n-2}\varepsilon_{n-3}k_{n-3}H_1H_2 \tag{38}$$

and for

$$\begin{aligned} &j=2, \quad \varepsilon_{n-3}H_2H_2' = \varepsilon_{n-2}\varepsilon_{n-3}k_{n-3}H_1H_2 - \varepsilon_{n-3}\varepsilon_{n-4}k_{n-4}H_2H_3, \\ &j=3, \quad \varepsilon_{n-4}H_3H_3' = \varepsilon_{n-3}\varepsilon_{n-4}k_{n-4}H_2H_3 - \varepsilon_{n-4}\varepsilon_{n-5}k_{n-5}H_3H_4, \\ &\vdots \\ &j=n-3, \quad \varepsilon_2H_{n-3}H_{n-3}' = \varepsilon_2\varepsilon_3k_2H_{n-4}H_{n-3} - \varepsilon_2\varepsilon_1k_1H_{n-3}H_{n-2} \end{aligned} \tag{39}$$

Then, using Eqs. (33), (38) and (39), we conclude that

$$\varepsilon_{n-2}H_1H_1' + \varepsilon_{n-3}H_2H' + \ldots + \varepsilon_2H_{n-3}H_{n-3}' + \varepsilon_1H_{n-2}H_{n-2}' = \varepsilon_nH_{n-2}. \tag{40}$$

By integrating Eq. (40), we can easily see that

$$\sum_{j=1}^{n-2}\varepsilon_{n-j-1}H_j^2 = 2\varepsilon_n\int H_{n-2}ds. \tag{41}$$

Conversely, suppose that Eq. (31) is satisfied. Taking the derivative of Eq. (31), we get

$$\varepsilon_{n-2}H_1H_1' + \varepsilon_{n-3}H_2H_2' + \ldots + \varepsilon_2H_{n-3}H_{n-3}' + \varepsilon_1H_{n-2}H_{n-2}' = \varepsilon_nH_{n-2}. \tag{42}$$

Substituting Eqs. (38) and (39) into Eq. (42) and using $H_{n-2}(s) \neq 0$, we get $H_{n-2}' - \varepsilon_2k_1H_{n-3} = \varepsilon_1\varepsilon_n$. Thus, Theorem 3.4 implies that $\zeta(s)$ is congruent to the non-null osculating curve of type (n-3).

Special case for $n=4$

In this part, we will see that the results obtained for the general theory confirm the characterizations for $n=4$ which is examined in [12] for the non-null second kind osculating curves, and we will characterize these curves according to the harmonic curvature functions of the curve in E_1^4.

The arclength parametrized non-null second kind osculating curve $\zeta(s)$ in E_1^4 is expressed as follows:

$$\zeta(s) = \lambda(s)T(s) + \mu(s)N(s) + \upsilon(s)B_2(s). \tag{43}$$

where $\lambda(s), \mu(s), \upsilon(s)$ are differentiable functions. Differentiating Eq. (43) with respect to s, we have

$$T = (\lambda' - \varepsilon_1 k_1 \mu)T + (\varepsilon_2 k_1 \lambda + \mu')N + \varepsilon_3 (k_2 \mu - k_3 \upsilon)B_1 + \upsilon' B_2 . \tag{44}$$

Then, by using the harmonic curvature functions of the curve $\zeta(s)$ given in Eq. (26), the functions
(i) $\upsilon' = 0 \Rightarrow \upsilon = c,$
(ii) $k_2 \mu - k_3 \upsilon = 0 \Rightarrow \mu = c \dfrac{k_3}{k_2} = \varepsilon_2 \varepsilon_4 H_1,$
(iii) $\varepsilon_2 k_1 \lambda + \mu' = 0 \Rightarrow \lambda = -\varepsilon_2 \dfrac{\mu'}{k_1} = -\dfrac{\varepsilon_4 H_1'}{k_1} = \varepsilon_1 \varepsilon_4 H_2,$
(iv) $\lambda' - \varepsilon_1 k_1 \mu = 1 \Rightarrow H_2' - \varepsilon_2 k_1 H_1 = \varepsilon_1 \varepsilon_4$
are easily obtained.

Corollary 3.4. Let $\zeta(s)$ be an arc-length parametrized non-null curve in E_1^4 with non-zero curvatures. Then, the following relations are equivalent.
(i) $\zeta(s)$ is a non-null second kind osculating curve.
(ii) $H_2'(s) - \varepsilon_2 k_1(s) H_1(s) = \varepsilon_1 \varepsilon_4.$
(iii) $\dfrac{\varepsilon_2}{k_1}\left(\dfrac{k_3}{k_2}\right)' + \varepsilon_1 k_1 \dfrac{k_3}{k_2} = -\dfrac{1}{c}, \quad c \in R.$

Corollary 3.5. Let $\zeta(s)$ be an arclength parametrized non-null second kind osculating curve in E_1^4 with non-zero curvatures. Then, the position vector of $\zeta(s)$ is

$$\zeta(s) = \varepsilon_4 \left(\varepsilon_1 H_2(s) T(s) + \varepsilon_2 H_1(s) N(s)\right) + c B_2(s)$$

where H_j, $j \in \{1,2\}$ are differentiable harmonic curvature functions of $\zeta(s)$.

4. CONCLUSIONS

This study examines some characterizations of non-null osculating curves of type (n-3) in terms of their curvature functions and harmonic curvature functions in Lorentzian n-space E_1^n**.** We have determined necessary and sufficient conditions for non-null curves to be congruent to non-null osculating curves of type (n-3) according to their curvature functions in E_1^n**.** On the other hand, examining the osculating curves is considerably more complex in high-dimensional spaces. For this reason, we have used harmonic curvature functions to recharacterize the non-null osculating curves of type (n-3), thereby allowing us to express our calculations and characterizations more simply. We have characterized non-null osculating curves of type (n-3) according to the harmonic curvature functions in terms of B_{n-2} in E_1^n. Our results provide insights into the geometry of non-null osculating curves in n-dimensional Lorentzian space. The results of this study may be extended to other alternative spaces, and the generalized ruled surfaces can be constructed using these curves. Moreover, the mathematical characterizations developed in this study could provide a theoretical foundation for future applied research.

REFERENCES

[1] Chen, B. Y., *American Mathematical Monthly*, **110**, 147, 2003.
[2] İlarslan. K., Nešovic, E., Torgasev-Petrovic, M., *Novi Sad Journal of Mathematics*, **33**(2), 23, 2003.
[3] Ali, A.T., Önder, M., *Global Journal of Science Frontier Research*, **12**(1), 57, 2012.
[4] Cambie, S., Goemans, W., Bussche, I.V.D., *Turkish Journal of Mathematics*, **40**, 210, 2016.
[5] Kalkan Boyacıoğlu, Ö., Öztürk, H., *Compt. Rend. Acad. Bulg. Sci*, **72**(2), 158, 2019.
[6] İlarslan, K, Nesovic, E., *Indian Journal of Pure and Applied Mathematics*, **35**(7), 881, 2004.
[7] İlarslan, K., *Turkish Journal of Mathematics*, **29**(1), 53, 2005.
[8] İlarslan, K, Nesovic, E., *Publications de l'Institut Mathématique*, **105**, 111, 2009.
[9] Bektaş, Ö., Advances in Difference Equations, **456**, 1, 2018.
[10] Kalkan Boyacıoğlu, Ö., *AIMS Mathematics*, **5**(4), 3510, 2020.
[11] İlarslan, K., Nešovic, E., *Demonstratio Mathematica*, **16**(4), 931, 2008.
[12] İlarslan, K., Nešovic, E., *Compt. Rend. Acad. Bulg. Sci*, **62**(6), 685, 2009.
[13] Bektaş, Ö., Bekiryazıcı, Z., *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, **71**(1), 212, 2022.
[14] Bektaş, Ö., *Journal of Science and Arts*, **23**(1), 51, 2023.
[15] Kalkan Boyacıoğlu, Ö., *Bitlis Eren University Journal of Science*, **13**(2), 467, 2024.
[16] Cheng, Y., Li, Y., Badyal, P., Singh, K., Sharma, S., *Mathematics*, **13**(5), 881, 2025.
[17] Erdem, E., Yılmaz, M.Y., *Journal of Science and Arts*, **25**(1), 133, 2025.
[18] Özdamar, E., Hacısalihoğlu, H. H., *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, **24**, 15, 1975.
[19] Gök, İ., Camcı, Ç., Hacısalihoğlu, H. H., *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics.*, **58**(1), 29, 2009.
[20] Ekmekçi, N., Hacısalihoğlu, H. H., İlarslan, K., *Bulletin of the Malaysian Mathematical Sciences Society*, **23**, 173, 2000.
[21] İyigün, E., Arslan, K., *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, **54**(1), 29, 2005.
[22] Kalkan Boyacıoğlu, Ö., *Comptes Rendus de l'Académie Bulgare des Sciences*, **73**(6), 776, 2020.
[23] Yılmaz, B., Gök, İ., Yaylı, Y., *Journal of Mathematical Sciences and Modelling*, **25**(1), 8, 2022.
[24] O'Neill B., *Semi-Riemannian geometry with applications to relativity*, Academic Press, London, 1983.
[25] İyigün, E., *Ukrainian Mathematical Journal*, **70**(5), 731, 2018.
[26] Yilmaz, S., Ali, A.T., López, B., *Middle-East Journal of Scientific Research*, **28**(2), 123, 2020.